Nova Scotia Math League

2018–2019

Game One

Problems and Solutions

Team Questions

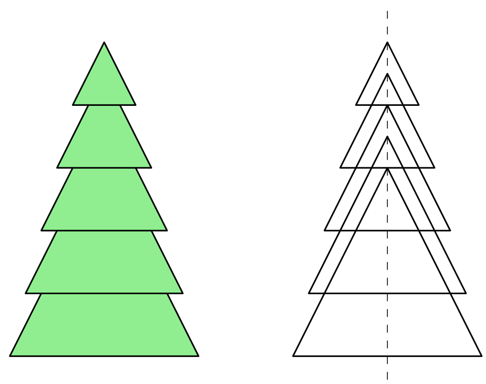
1. Let

$$S = 1 + 2 + 3 + \cdots + 300$$
$$T = 4 + 8 + 12 + \cdots + 600.$$

Compute $T - S$.

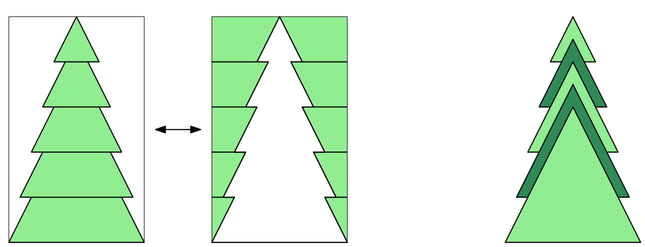
Solution: Compute directly using $1 + 2 + \cdots + 300 = \frac{1}{2}(300)(301)$ and $T = 4(1 + 2 + \cdots + 150) = 2(150)(151)$, or notice that $T = 2(2 + 4 + 6 + \cdots + 300)$ so $T - S = -1 + 2 - 3 + 4 - \cdots - 299 + 300 = 1 + 1 + \cdots + 1 = 150$. $\square$

2. Paper Christmas trees can made by cutting out several similar isosceles triangles, each with base equal to its height, and then laying them down in decreasing order of size so that their altitudes align along a common axis and the top of each triangle lies above that of the last.



Find the area of the tree formed from five triangles of heights 2, 3, 4, 5, and 6 cm with their tops uniformly spaced 1 cm apart.

Solution: By symmetry we see that the tree takes up exactly half the of the surrounding 6×10 box . Thus it has area $\frac{1}{2}(60) = 30$.



Alternatively, one can split the tree into a triangle plus trapezoidal segments or triangular "shells". Using shells (shown above, right) leads to a nice telescoping sum, $\frac{1}{2}[6^2 + (5^2 - 4^2) + (4^2 - 3^2) + (3^2 - 2^2) + (2^2 - 1^2)] = \frac{1}{2}[6^2 + 5^2 - 1^2] = 30$. In general, a tree composed of triangles of heights $2, \ldots, n$ will have area $n(n-1)$. $\square$

3. Six friends have a race. Bob places second. Jim finishes ahead of Andy and Nancy but behind Ray. Nancy finishes behind Todd. Assuming there were no ties, in how many different orders could they have finished?

Solution: Jim, Andy, Nancy and Ray must place in one of the two orders $RJNA$ or $RJAN$, listed fastest-to-slowest from left-to-right. In the first case Todd can be inserted in 3 places, and in the second he can appear in 4 places. Bob must then be squeezed iinto the second position. So there are $3 + 4 = 7$ possible orderingsl. $\square$

4. A bus begins its journey at the mall. At its first stop, a third of the passengers get off while 8 new passengers get on. At the next stop, a third of the passengers again get off while 10 get on. At this point there are 25% fewer people on the bus than there were at the beginning of the journey.

   How many people were on the bus as it left the mall?

   (*Hint:* Don't forget the driver!)

   Solution: If x is the original number of passengers, then we have $\frac{2}{3}(\frac{2}{3}x + 8) + 10 + 1 = \frac{3}{4}(x + 1)$. Solve to get $x = 51$. So there were $x + 1 = 52$ people onboard initially. $\square$

5. Alan and Beth run around an oval track, starting at noon from the same position and each running at constant speed in the same direction. Alan takes 2 minutes to complete a lap, while Beth takes 15 seconds less.

   If they run for 1 hour, when does Beth overtake Alan for the last time?

   (Give your answer in HH:MM:SS format.)

   Solution: Measured in laps per minute, their speeds are $\frac{1}{2}$ and $1/\frac{7}{4} = \frac{4}{7}$, so the relative speed is $\frac{4}{7} - \frac{1}{2} = \frac{1}{14}$. Thus Beth will overtake Alan every 14 minutes and the last time will be at 56 minutes, i.e. 12:56pm.

   $\square$

6. Three six-sided dice are rolled. Find the probability that the sum of the two smallest numbers showing is less than the third.

   Solution: Consider triples (a, b, c) with $1 \le a \le b \le c \le 6$ and $a + b < c$. The only possibilities with $a = b$ are $(1, 1, \{3, 4, 5, 6\})$, and $(2, 2, \{5, 6\})$. Each can be ordered in 3 ways, accounting for $3(4 + 2) = 18$ permissible rolls of the dice. When $a < b$ we have only the triples $(1, 2, \{4, 5, 6\})$, $(1, 3, \{5, 6\})$, $(1, 4, 6)$, and $(2, 3, 6)$. Each of these can be ordered in 6 ways, resulting in $6(3 + 2 + 1 + 1) = 42$ possible rolls. There are $18 + 42 = 60$ good rolls in total, so the desired probability is $\frac{60}{6^3} = \frac{5}{18}$.

   Note: It's a good exercise to show more generally that when n-sided dice are rolled, the probability is $(n-1)(n-2)/2n^2$.

   $\square$

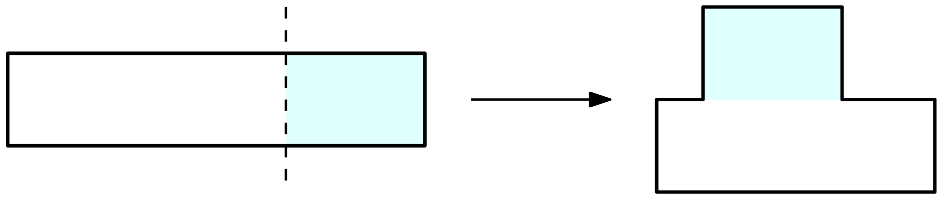
7. The quadratic $3x^2 + 12x + 7 = 0$ has roots p and q. Find the unique pair (a, b) such that $x^2 + ax + b$ has roots $2p - q$ and $2q - p$.

   Solution: The Viete formulae (for sum and product of the roots) applied to $3x^2 + 12x + 7$ give $p + q = -\frac{12}{3} = -4$ and $pq = \frac{7}{3}$., and applied to $x^2 + ax + b$ give $-a = (2p - q) + (2q - p) = p + q = -4$ and $b = (2p - q)(2q - p) = -2(p + q)^2 + 9pq = -2(-4)^2 + 9(\frac{7}{3}) = -11$. Thus $(a, b) = (4, -11)$.

   Alternatively, one can solve the quadratic to get roots $p, q = -2 \pm \frac{1}{3}\sqrt{15}$. Then $2p - q$ and $2q - p$ are $-2 \pm \sqrt{15}$, so we must have $x^2 + ax + b = (x + 2 - \sqrt{15})(x + 2 + \sqrt{15})$. Expanding and comparing coefficients gives $(a, b) = (4, -11)$.

   $\square$

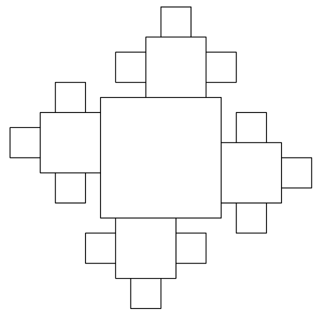
8. A rectangle with perimeter 20 and area 16 is cut parallel to its shortest side to create two unequal rectangles. The smaller of these rectangles is then translated so that it abuts the top of the larger, as shown below. If the perimeter of this new figure is 19, what is the area of the smaller rectangle?



Solution: Let the smaller larger rectangles have sides a and b, respectively, both with height h. Then $16 = (a+b)h$, $20 = 2(a+b)+2h$, and $19 = 2b+4h$. From the first two equations get $10 = 16/h + h \implies h^2 - 10h + 16 = 0 \implies (h-2)(h-8) = 0$, so $h = 2$ or $h = 8$. From $19 = 2b+4h$ we find that $h = 2$ gives $b = 11/2$, while $h = 8$ gives $b < 0$, which is impossible. Thus $h = 2$, $b = 11/2$, and $16 = (a+b)h \implies a = 5/2$. The area of the smaller rectangle is $ah = 5$. □

Pairs Relay

P-A. The figure below was created from several squares having **perimeters** 1, 2, and 5.



Let A be the overall perimeter of the figure. $\boxed{\text{Pass on A}}$

Solution: Let s and S be the sides of the small and medium squares. Then each "knob" of the figure contributes perimeter $3S + 6s$, and the remaining perimeter around the large square is $5 - 4S$. So $\mathsf{A} = 4(3S + 6s) + 5 - 4S = 2(4S) + 6(4s) + 5 = 2(2) + 6(1) + 5 = 15$. □

P-B. You will receive A.

The parabolas $y = x^2 + 3x + \mathsf{A}$ and $y = 2x^2 + 4x + 9$ cross at two points. Let B be the y-intercept of the line joining these points.

$\boxed{\text{Pass on B}}$

Solution: Let (x, y) be point of intersection, so that $y = x^2 + 3x + \mathsf{A}$ and $y = 2x^2 + 4x + 9$ hold simultaneously. Twice the first equation minus the second gives $y = 2x + (2\mathsf{A} - 9)$. This must be the equation of the desired line, and its y-intercept is $\mathsf{B} = 2\mathsf{A} - 9$. With $\mathsf{A} = 15$ get $\mathsf{B} = 21$. □

P-C. You will receive B.

The elements of the set $\{\mathsf{B} + 20,\ 3\mathsf{B} - 50,\ 75 - 2\mathsf{B}\}$ are taken in pairs and the average of each pair is calculated.

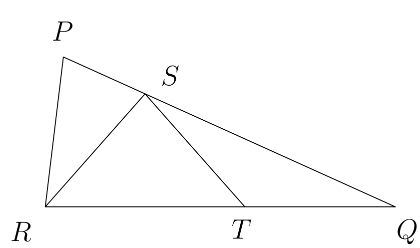
The results are $\mathsf{C} - 2$, $\mathsf{C} + 2$, and $\mathsf{C} + 12$, in no particular order.

Determine C. $\boxed{\text{Pass on C}}$

Solution: We must have $(\mathsf{B} + 20) + (3\mathsf{B} - 50) + (75 - 2\mathsf{B}) = (\mathsf{C} - 2) + (\mathsf{C} + 2) + (\mathsf{C} + 12)$, which simplifies to $\mathsf{C} = \frac{2}{3}\mathsf{B} + 11$. With $\mathsf{B} = 21$ get $\mathsf{C} = 25$. □

P-D. You will receive C.

In the figure below, $\angle PQR = \mathsf{C}^\circ$ and $|PR| = |RS| = |ST| = |TQ|$.



Let D be the degree measure of $\angle PRS$. $\boxed{\text{Done!}}$

Solution: Chase angles through the isosceles trianges to get $\angle PRS = 180 - 6\mathsf{C}$. With $\mathsf{C} = 25$ get $\mathsf{D} = 30$. □

Individual Relay

I-A. The *CVV code* on a credit card is a sequence of three digits (0 permitted) that is used for security purposes. The CVV on my card happens to have **distinct** digits, two of which are 4 and 7.

Let A be the number of possible CVV codes on my card. $\boxed{\text{Pass on A}}$

Solution: The unknown digit is any one of 8 possibilities, and there are $3! = 6$ ways to arrange order 3 digits. So $\mathsf{A} = 8 \cdot 6 = 48$. □

I-B. You will receive A.

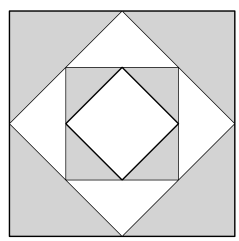
John is struggling with math. His average score on the first three tests was A%. Then he did so horribly on the next test that his average over the first four sunk to 40%.

Let B be John's percentage score on the fourth test. $\boxed{\text{Pass on B}}$

Solution: We have $40 = (3\mathsf{A} + \mathsf{B})/4$, so $\mathsf{B} = 4 \cdot 40 - 3\mathsf{A}$ With $\mathsf{A} = 48$ get $\mathsf{B} = 16$. □

I-C. You will receive B.

The figure below was created by joining the midpoints of each side of a square to create a smaller square, and then repeating this process twice more. The largest square has area B.



Let C be the area of the shaded region. $\boxed{\text{Pass on C}}$

Solution: Each square has half the area of the square in which it is inscribed. So the shaded area is $\mathsf{C} = (1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8})\mathsf{B} = \frac{5}{8}\mathsf{B}$. With $\mathsf{B} = 16$ get $\mathsf{C} = 10$. □

I-D. You will receive C.

Jane lives C km from work and usually bikes to and from the office. Occasionally she instead rides her bike to work and runs home. She can bike twice as fast as she runs, so doing this extends her overall commute time by 25 minutes.

Let D be the speed (in km/h) at which Jane bikes. $\boxed{\text{Done!}}$

Solution: Jane's biking and running speeds are D and $\frac{\mathsf{D}}{2}$, and 25 minutes is $\frac{5}{12}$ of an hour. So we have $\frac{5}{12} = 2\frac{\mathsf{C}}{\mathsf{D}} - \frac{\mathsf{C}}{\mathsf{D}} = \frac{\mathsf{C}}{\mathsf{D}}$. Thus $\mathsf{D} = \frac{12}{5}\mathsf{C}$. With $\mathsf{C} = 10$ get $\mathsf{D} = 24$. □