

Multivalued Functors

Robert Paré

Dalhousie University

pare@mathstat.dal.ca

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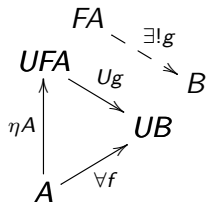
Adjoints

Lambek - Category theory course 1967

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Free objects



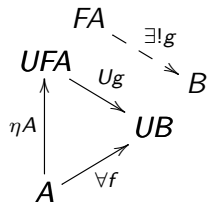
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I.e. FA represents the functor $\mathbf{A}(A, U-)$

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I.e. FA represents the functor $\mathbf{A}(A, U-)$

F becomes a functor in a unique way so that η is natural

$$\begin{array}{ccc} FA & \xrightarrow{\exists! Fa} & FA' \\ UFA & \xrightarrow{UFa} & UFA' \\ \eta A \uparrow & & \uparrow \eta A' \\ A & \xrightarrow{a} & A' \end{array}$$

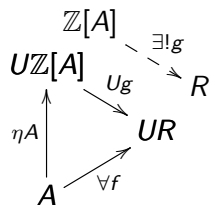
Example

Free commutative ring

$$\begin{array}{ccc} & \mathbb{Z}[A] & \\ & \swarrow \exists! g & \\ U\mathbb{Z}[A] & & R \\ \uparrow \eta_A & \searrow U_g & \\ A & \nearrow \forall f & \\ & UR & \end{array}$$

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Free field? $\mathbb{Q}(A)$?

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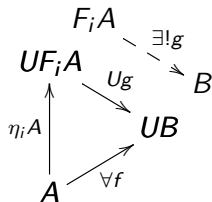
No! Not even a functor!

Multi-adjoints

Diers - *Familles Universelles de Morphismes*

Ann. Soc. Sci. Bruxelles 93 # 3 (1979)

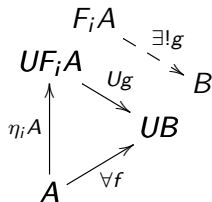
$U : \mathbf{B} \longrightarrow \mathbf{A}$ has a *left multi-adjoint* at A if there is a family $\langle \eta_i A : A \longrightarrow UF_i A \rangle_{i \in I}$ such that $\forall f : A \longrightarrow UB, \exists ! i, g$ such that



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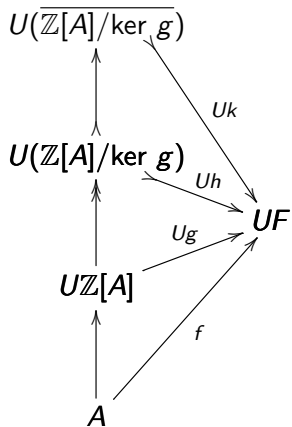
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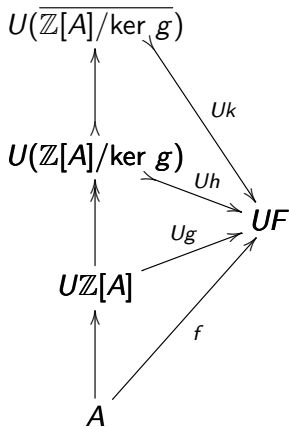


Bijection $\mathbf{A}(A, UB) \cong \sum_{i \in I} \mathbf{B}(F_i A, B)$
i.e. $\mathbf{A}(A, U-)$ is a sum of representables
what Diers calls *multi-representable*

Example: Fields



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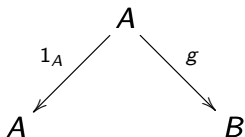
Take $I = \{P \triangleleft \mathbb{Z}[A] \mid P \text{ prime}\}$

Universal family $\langle A \longrightarrow \overline{\mathbb{Z}[A]/P} \rangle_{P \in I}$

Different Example

Any groupoid has multi-products:

For $A, B \in \mathbf{G}$, take all spans

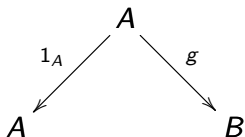


$$g \in \mathbf{G}(A, B)$$

Different Example

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Works for arbitrary (non-empty) products.

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But stops short of making multi-adjoints into honest adjoints in a bicategory.

Will give three quite different, though biequivalent, bicategories in which they live.

MVFam - multivalued functors

ParFun - partial functors

DetProf - deterministic profunctors

Functoriality

If $\langle \eta_i : A \rightarrow UF_i A \rangle_{i \in I}$ is a universal family, then the I , $F_i A$, $\eta_i A$ are unique up to isomorphism.

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For $a : A \longrightarrow A'$, we have $\forall j \in S_U A' \exists ! i \in S_U A$ such that

$$\begin{array}{ccc} F_i A & \dashrightarrow & F_j A' \\ UF_i A' & \longrightarrow & UF_j A' \\ \eta_i A \uparrow & & \uparrow \eta_j A' \\ A & \xrightarrow{a} & A' \end{array}$$

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$$\alpha : S_U A' \longrightarrow S_U A, \quad F_j a : F_{\alpha(j)} A \longrightarrow F_j A'$$

MVFun

Let $\text{Fam}^* \mathbf{B}$ be the category whose objects are families $(I, \langle B_i \rangle_{i \in I})$ for I a set and the B_i objects of \mathbf{A} .

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Get a Kleisli bicategory $\mathbf{MVFun} = \mathbf{Cat}_{\mathbf{Fam}^*}$.

(See Cheng, Hyland, Power - Electronic Notes in Th. Comp. Sci. 83 (2004).)

ParFun

If $\langle A \xrightarrow{\eta_i A} UF_i A \rangle_{i \in S_U A}$ is a universal family for $U : \mathbf{B} \rightarrow \mathbf{A}$, then S_U is a presheaf $\mathbf{A}^{op} \rightarrow \mathbf{Set}$

Thus we get

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Definition (Lawvere)

A *partial functor* $F : \mathbf{A} \rightarrow \mathbf{B}$ is a span

$$\begin{array}{ccc} & \mathbf{A}_F & \\ \bar{F} \swarrow & & \searrow \tilde{F} \\ \mathbf{A} & & \mathbf{B} \end{array}$$

with \bar{F} a discrete fibration.

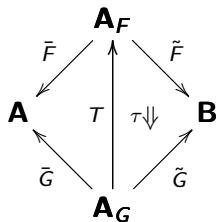
The bicategory structure

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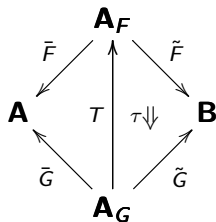
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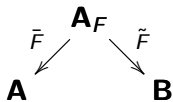


Theorem

ParFun is biequivalent to **MVFun**.

DetProf

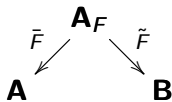
A partial functor



gives a profunctor $\tilde{F}_* \otimes \bar{F}^* : \mathbf{A} \dashrightarrow \mathbf{B}$.

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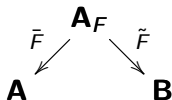
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$\mathbf{ParFun} \longrightarrow \mathbf{Prof}^{\text{co}}$, $F \mapsto \tilde{F}_* \otimes \bar{F}^*$ is a locally full and faithful strong morphism of bicategories.

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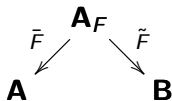
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Definition

$P : \mathbf{A} \dashrightarrow \mathbf{B}$ is *deterministic* if $\forall A, P(A, -) : \mathbf{B} \longrightarrow \mathbf{Set}$ is multi-representable, i.e. $P(A, -) \cong \sum_{i \in I} \mathbf{B}(B_i, -)$.

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Theorem

P is deterministic iff it is isomorphic to $\tilde{F}_* \otimes \bar{F}^*$ for a partial functor F . Thus we get a biequivalence of \mathbf{ParFun} with $\mathbf{DetProf}^{\text{co}}$.

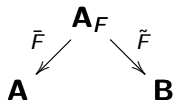
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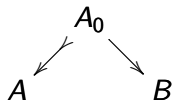
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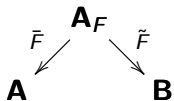
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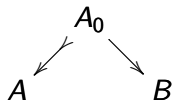
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Functors

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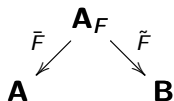
Functions

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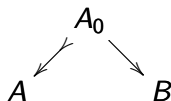
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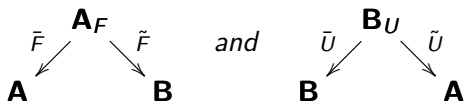
Single-valued relations

$$A \longrightarrow | \longrightarrow B$$

Multi-adjoints are Adjoints

Theorem

Given partial functors



$F \dashv U$ in **ParFun** iff \bar{U} is an isomorphism and F is left multi-adjoint to $\tilde{U}\bar{U}^{-1}$.