Multivalued Functors

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Adjoints

Lambek - Category theory course 1967

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Free objects

$$\begin{tikzcd}
 & FA \arrow[drr, dashed, "\exists! g"] & & \\
UFA \arrow[dr, "Ug"] & & & B \\
 & UB & & \\
A \arrow[uu, "\eta A"] \arrow[ur, "\forall f"'] & & &
\end{tikzcd}$$

Bijection $\mathbf{B}(FA, B) \cong \mathbf{A}(A, UB)$

I.e. FA represents the functor $\mathbf{A}(A, U-)$

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I.e. FA represents the functor $\mathbf{A}(A, U-)$

F becomes a functor in a unique way so that η is natural

$$
\begin{tikzcd}
FA \arrow[r, dashed, "\exists! Fa"] & FA' \\
UFA \arrow[r, "UFa"] & UFA' \\
A \arrow[u, "\eta A"] \arrow[r, "a"'] & A' \arrow[u, "\eta A'"']
\end{tikzcd}
$$

Example

Free commutative ring

$$
\begin{tikzcd}
& \mathbb{Z}[A] \arrow[rr, dashed, "\exists ! g"] & & R \\
U\mathbb{Z}[A] \arrow[rr, "Ug"] & & UR & \\
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Free field? $\mathbb{Q}(A)$?
No! Not even a functor!

Multi-adjoints

Diers - *Familles Universelles de Morphismes*
Ann. Soc. Sci. Bruxelles 93 # 3 (1979)

$U : \mathbf{B} \longrightarrow \mathbf{A}$ has a *left multi-adjoint at* A if there is a family $\langle \eta_i A : A \longrightarrow UF_i A \rangle_{i \in I}$ such that $\forall f : A \longrightarrow UB$, $\exists! i$, g such that

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\begin{tikzcd}
 & F_i A \arrow[dr, dashed, "\exists! g"] & \\
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Bijection $\mathbf{A}(A, UB) \cong \sum_{i \in I} \mathbf{B}(F_i A, B)$
i.e. $\mathbf{A}(A, U-)$ is a sum of representables
what Diers calls *multi-representable*

Example: Fields

$$
\begin{tikzcd}
U(\overline{\mathbb{Z}[A]/\ker g}) \arrow[ddr, tail, "Uk"] & \\
U(\mathbb{Z}[A]/\ker g) \arrow[u, tail] \arrow[dr, tail, "Uh"] & \\
U\mathbb{Z}[A] \arrow[u, two heads] \arrow[r, "Ug"] & UF \\
A \arrow[u] \arrow[ur, "f"'] &
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$$

Take $I = \{P \lhd \mathbb{Z}[A] | P \text{ prime}\}$

Universal family $\langle A \longrightarrow \overline{\mathbb{Z}[A]/P} \rangle_{P \in I}$

Different Example

Any groupoid has multi-products:
For $A, B \in \mathbf{G}$, take all spans

$$\begin{tikzcd} & A \arrow[dl, "1_A"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

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Works for arbitrary (non-empty) products.

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Multi-limits, multi Lawvere theories, multi locally presentable categories, etc.

But stops short of making multi-adjoints into honest adjoints in a bicategory.

Will give three quite different, though biequivalent, bicategories in which they live.

MVFam - multivalued functors
ParFun - partial functors
DetProf - deterministic profunctors

Functoriality

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For $a : A \longrightarrow A'$, we have $\forall j \in S_U A' \;\; \exists! i \in S_U A$ such that

$$\begin{tikzcd}
F_i A \arrow[r, dashed] & F_j A' \\
UF_i A' \arrow[r] & UF_j A' \\
A \arrow[u, "\eta_i A"] \arrow[r, "a"'] & A' \arrow[u, "\eta_j A'"']
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$$\alpha : S_U A' \longrightarrow S_U A, \quad F_j a : F_{\alpha(j)} A \longrightarrow F_j A'$$

MVFun

Let $\mathrm{Fam}^*\mathbf{B}$ be the category whose objects are families $(I, \langle B_i \rangle_{i \in I})$ for I a set and the A_i objects of $\mathbf{A}$.

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Morphisms $(\alpha, \langle a_j \rangle) : (I, \langle B_i \rangle) \longrightarrow (J, \langle B'_j \rangle)$ are $\alpha : J \longrightarrow I$, $a_j : B_{\alpha(j)} \longrightarrow B'_j$.

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Get a Kleisli bicategory $\mathbf{MVFun} = \mathbf{Cat}_{\mathrm{Fam}^*}$.
(See Cheng, Hyland, Power - Electronic Notes in Th. Comp. Sci. 83 (2004).)

ParFun

If $\langle A \xrightarrow{\eta_i A} UF_iA\rangle_{i\in S_UA}$ ia a universal family for $U : \mathbf{B} \longrightarrow \mathbf{A}$, then S_U is a presheaf $\mathbf{A}^{op} \longrightarrow \mathbf{Set}$
Thus we get

$$\begin{tikzcd} & \mathrm{El}(S_U) \arrow[dl, "\bar{F}"'] \arrow[dr, "\tilde{F}"] & \\ \mathbf{A} & & \mathbf{B} \end{tikzcd}$$

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Definition (Lawvere)

A *partial functor* $F : \mathbf{A} \rightharpoonup \mathbf{B}$ is a span

$$\begin{tikzcd} & \mathbf{A}_F \arrow[dl, "\bar{F}"'] \arrow[dr, "\tilde{F}"] & \\ \mathbf{A} & & \mathbf{B} \end{tikzcd}$$

with $\bar{F}$ a discrete fibration.

The bicategory structure

Because discrete fibrations are closed under composition and stable under pullback, we can compose partial functors as spans, i.e. by pullback.

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We get a bicategory if we take as 2-cells $t : F \longrightarrow G$

$$\begin{tikzcd}
& \mathbf{A}_F \arrow[dl, "\bar{F}"'] \arrow[dr, "\tilde{F}"] & \\
\mathbf{A} & \arrow[r, phantom, "\tau\Downarrow"] & \mathbf{B} \\
& \mathbf{A}_G \arrow[ul, "\bar{G}"] \arrow[ur, "\tilde{G}"'] \arrow[uu, "T"] &
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Theorem

ParFun *is biequivalent to* **MVFun**.

DetProf

A partial functor

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gives a profunctor $\tilde{F}_* \otimes \bar{F}^* : \mathbf{A} \longrightarrow\!\!\!\!\!\!\bullet\!\!\!\!\!\!\longrightarrow \mathbf{B}$.

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Theorem

$\mathbf{ParFun} \longrightarrow \mathbf{Prof}^{co}$, $F \mapsto \tilde{F}_* \otimes \bar{F}^*$ *is a locally full and faithful strong morphism of bicategories.*

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Definition

$P : \mathbf{A} \longrightarrow\!\!\!\!\!\!\bullet\;\; \mathbf{B}$ is *deterministic* if $\forall A$, $P(A, -) : \mathbf{B} \longrightarrow \mathbf{Set}$ is multi-representable, i.e. $P(A, -) \cong \sum_{i \in I} \mathbf{B}(B_i, -)$.

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Theorem

P is deterministic iff it is isomorphic to $\tilde{F}_* \otimes \bar{F}^*$ for a partial funtor F. Thus we get a biequivalence of $\mathbf{ParFun}$ *with* $\mathbf{DetProf}^{co}$.

Partial Functors

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Spans with $\bar{F}$ a discrete fibration

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which correspond to

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$$A \longrightarrow \tilde{B}$$

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which correspond to

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Deterministic profunctors

$$\mathbf{A} \xrightarrow{\ \bullet\ } \mathbf{B}$$

which correspond to

Single-valued relations

$$A \xrightarrow{\ \shortmid\ } B$$

Multi-adjoints are Adjoints

Theorem

Given partial functors

$$\begin{tikzcd} & \mathbf{A}_F \arrow[dl, "\bar{F}"'] \arrow[dr, "\tilde{F}"] & \\ \mathbf{A} & & \mathbf{B} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & \mathbf{B}_U \arrow[dl, "\bar{U}"'] \arrow[dr, "\tilde{U}"] & \\ \mathbf{B} & & \mathbf{A} \end{tikzcd}$$

*$F \dashv U$ in **ParFun** iff $\bar{U}$ is an isomorphism and F is left multi-adjoint to $\tilde{U}\bar{U}^{-1}$.*