An update on effort locales and quasi-uniform locales (part 1)

Notes for a talk by JME

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Introduction

Classically, one derives topology from analytic structures as follows.

$$\begin{tikzcd}
\text{inner product} \arrow[d] & \\
\text{norm} \arrow[d] & \\
\text{metric} \arrow[r, tail] \arrow[d] & \text{effort} \arrow[d] \\
\text{uniformity} \arrow[r, tail] \arrow[d] & \text{quasi-uniformity} \arrow[d] \\
\text{completely regular topology} \arrow[r, tail] & \text{topology}
\end{tikzcd}$$

(One need not start all the way at the top.)

But, when working constructively, it seems better to impose analytic structures on a pre-existing topology; indeed, it seems to me that, before imposing an analytic structure, one should pre-suppose not merely a topology, but all of the intermediate analytic structures, so that each of the arrows above simply forgets structure.

In particular, we intend to replace the notion of *effort space* with the notion of *effort (quasi-uniform) locale*—*i.e.*, a locale (=constructively appropriate notion of topology) with an added quasi-uniform structure, and (then) an added effort structure. To the best of my knowledge, this has not been attempted except in the symmetric case (metrics and uniformities). The "update" comprises reconciling my previous work with the symmetric case.

Review

Effort spaces and entourages (old)

By an *effort space*, I mean a "generalised metric space in the sense of Lawvere"—*i.e.*, a $(([0,\infty],\geq),+,0)$-enriched category—*i.e.*, a set E together with a map $\varphi\colon E\times E\to[0,\infty]$ satisfying

$$\begin{array}{rcl} 0 & \geq & \varphi(\sigma,\sigma) \\ \varphi(\pi,\sigma)+\varphi(\sigma,\tau) & \geq & \varphi(\pi,\tau) \end{array}$$

for all $\pi,\sigma,\tau\colon E$.

[Elements of E should be thought of as "states", rather than "points"; $\varphi(\sigma,\tau)$ represents the effort which would be required to transition from state σ to state τ—note the use of subjunctive mood; $\varphi(\sigma,\tau)=\infty$ represents the idea that it is impossible to transition from state σ to state τ (there is no such thing as an infinite amount of effort); note that it is reasonable that $\varphi(\sigma,\tau)\neq\varphi(\tau,\sigma)$ in general; in particular, the situation $\varphi(\sigma,\tau) = \infty \neq \varphi(\tau,\sigma)$ is especially reasonable—some changes of state are irreversible; similarly, it is possible that $\varphi(\sigma,\tau) = 0 \neq \varphi(\tau,\sigma)$; an effort space is *skeletal* if "$\varphi(\sigma,\tau) = 0$ and $\varphi(\tau,\sigma) = 0$" implies $\sigma = \tau$; we do not insist that effort spaces be skeletal, though it would not be entirely unreasonable to do so.]

Given an effort space (E,φ), and a positive rational ε, a set of the form

$$N_\varepsilon = \varphi^{\leftarrow}[0,\varepsilon) = \{(\sigma,\tau) : E \times E \mid \varphi(\sigma,\tau) < \varepsilon\}$$

is called a *basic entourage*; an *entourage* is any $N \subseteq E \times E$ which contains a basic entourage.

The set E together with the collection of all entourages $N \subseteq E \times E$ is an example of a quasi-uniform space, according to the Bourbaki definition—indeed, the motivating example. A map $\omega\colon E \to F$ is called *uniformly continuous* if $(\omega \times \omega)^{\leftarrow}$ preserves entourages.

As discussed in the summer, a quasi-uniformity on E induces more than one underlying topology on E, but we prioritise what I call the *welling topology*, which is generated by vertical slices of entourages. [In the case of the quasi-uniformity generated by an effort, the vertical slices of N_ε are the basic open *wells* $W_{\sigma,\varepsilon} = \{\tau\colon E \mid \varphi(\sigma,\tau) < \varepsilon\}$—as opposed to the *sinks* $S_{\tau,\varepsilon} = \{\sigma\colon E \mid \varphi(\sigma,\tau) < \varepsilon\}$, which are the horizontal slices of N_ε. For an arbitrary quasi-uniformity, we define the *sinking* topology to be that generated by the horizontal slices of entourages—equivalently, the welling topology associated to the *opposite* quasi-uniformity. The least common refinement of the welling and sinking topologies is called the *regular* topology. A uniformly continuous map is continuous wrt all three of these topologies.]

Locales (very old)

A *complete Heyting algebra* (CHA) is a complete lattice which (when regarded as a category) is cartesian closed; equivalently, it is a complete lattice satisfying the following distributivity law.

$$\alpha \wedge \bigvee_k \beta_k = \bigvee_k \alpha \wedge \beta_k$$

[The right adjoint of $\alpha \wedge -$ is denoted $\alpha \Rightarrow -$; $\neg\alpha$ abbreviates $\alpha \Rightarrow \bot$. If ℓ is a CHA, $\wedge$ defines a sup-homomorphism $\ell \otimes \ell \to \ell$; $\Rightarrow$ defines a sup-homomorphism $\ell \otimes \ell^{\mathsf{op}} \to \ell^{\mathsf{op}}$; $\neg$ defines a sup-homomorphism $\ell \to \ell^{\mathsf{op}}$; $\top$ defines a sup-homomorphism $\mathbb{L} \to \ell$. (Here $\otimes$ denotes the tensor product of complete lattices in the category of complete lattices and sup-homomorphisms, and $\mathbb{L}$ denotes its unit, which is the complete lattice of "truth-values".) A CHA is called *boolean* if $\neg$ is invertible.]

A *frame homomorphism* is a map between CHAs that preserves finite meets and arbitrary joins. If E is a topological space, then its open subsets form a CHA which we denote $\mathcal{O}(E)$; if $\omega\colon E \to F$ is a continuous map, then its inverse image defines a frame homomorphism $\omega^{\leftarrow}\colon \mathcal{O}(F) \to \mathcal{O}(E)$.

We define the category of *locales* and *continuous maps* to be the opposite of the category of CHAs and frame homomorphisms. Nevertheless, it is notationally convenient to pretend that a locale is not the same thing as a CHA, but rather a kind of topological object; thus, if E is a locale, $\mathcal{O}(E)$ denotes the "corresponding CHA", which is to say, *itself*, now regarded as a CHA. Similarly, if $\omega\colon E \to F$ is a continuous map, then $\omega^{\leftarrow}\colon \mathcal{O}(F) \to \mathcal{O}(E)$ denotes the "corresponding frame homomorphism".

The (cartesian) product of locales is defined by

$$\mathcal{O}(E \times F) = \mathcal{O}(E) \otimes \mathcal{O}(F)$$

(this does have the correct universal property), and the terminal locale is defined by $\mathcal{O}(1) = \mathbb{L}$.

[More generally, for any set E, the powerset $\mathcal{P}(E) = \mathbb{L}^E$ is a CHA; so there is a *discrete locale* E_{d} with $\mathcal{O}(E_{\mathsf{d}}) = \mathcal{P}(E)$; moreover, a continuous map $E_{\mathsf{d}} \to F_{\mathsf{d}}$ is the same thing as a map $E \to F$; so in future we shall elide the distinction between the set E and the discrete locale E_{d}.]

Open locales (very old)

A continuous map of locales $\omega\colon E \to F$ is called *open* if $\omega^{\leftarrow}\colon \mathcal{O}(F) \to \mathcal{O}(E)$ has a left adjoint $\exists_\omega\colon \mathcal{O}(E) \to \mathcal{O}(F)$ satisfying one of the three equivalent conditions below.

$$\exists_\omega(\alpha) \Rightarrow \gamma \quad = \quad \forall_\omega(\alpha \Rightarrow \omega^{\leftarrow}(\gamma))$$

$$\begin{aligned} \exists_\omega(\alpha) \wedge \beta &= \exists_\omega(\alpha \wedge \omega^\leftarrow(\beta)) \\ \omega^\leftarrow(\beta \Rightarrow \gamma) &= \omega^\leftarrow(\beta) \Rightarrow \omega^\leftarrow(\gamma) \end{aligned}$$

(Here $\alpha\colon \mathcal{O}(E)$ and $\beta, \gamma\colon \mathcal{O}(F)$.)

An *open locale* is a locale E such that the unique continuous map $!\colon E \to 1$ is open. Classically, every locale is open because, in that case,

$$\exists_!(\omega) = \text{“}\omega \text{ is non-empty”} = \begin{cases} \bot & \text{if } \omega = \bot \\ \top & \text{otherwise} \end{cases}$$

does the trick.

In an arbitrary topos, we can define a similar-looking map $\llcorner\bot\lrcorner\colon \mathcal{O}(E) \to \mathbb{L}^{\mathsf{op}}$, satisfying

$$\llcorner\bot\lrcorner(\omega) = \bot_{\mathbb{L}^{\mathsf{op}}} = \top_{\mathbb{L}} \iff \omega = \bot$$

for all $\omega\colon \mathcal{O}(E)$. If E is open, then the diagram

$$\begin{tikzcd} \mathcal{O}(E) \arrow[r, "\exists_!"] \arrow[dr, "\llcorner\bot\lrcorner"'] & \mathbb{L} \arrow[d, "\neg"] \arrow[dl, phantom, "(*)"] \\ & \mathbb{L}^{\mathsf{op}} \end{tikzcd}$$

holds. (But $\neg$ is not invertible, in general.)

We can define the word *positive* in such a way that the truth-value of the statement “ω is positive” equals $\exists_!(\omega)$. Then the diagram $(*)$ asserts that

“ω is empty” iff “ω is non-positive”

from which we deduce

“ω is non-empty” iff “ω is non-non-positive”.

Entourages revisited (partly new)

For any open locale E, the composite

$$\mathcal{O}(E) \oslash \mathcal{O}(E) \xrightarrow{\ \wedge\ } \mathcal{O}(E) \xrightarrow{\ \exists_!\ } \mathbb{L}$$

(which we denote γ_E) makes $\mathcal{O}(E)$ into a *Hermitian complete lattice* (*i.e.*, a Hermitian object in $(\mathsf{Sup}, \oslash, \mathbb{L})$); we say that E is *quasi-discrete* if γ_E's transpose

$$\mathcal{O}(E) \longrightarrow \mathcal{O}(E) \multimap \mathbb{L}$$

is invertible—*i.e.*, if $(\mathcal{O}(E), \gamma_E)$ is *weakly definite* as a Hermitian object of $(\mathsf{Sup}, \oslash, \mathbb{L})$. ($\multimap$ denotes the closed structure of Sup.)

Classically, a locale E is quasi-discrete iff $\mathcal{O}(E)$ is boolean. This is because, in general, $(*)$ entails

$$\begin{tikzcd} \mathcal{O}(E) \arrow[r] \arrow[d, "\neg"'] & \mathcal{O}(E) \multimap \mathbb{L} \arrow[d, "\mathsf{id} \multimap \neg"] \\ \mathcal{O}(E)^{\mathsf{op}} \arrow[r, equal, "\sim"] & \mathcal{O}(E) \multimap (\mathbb{L}^{\mathsf{op}}) \end{tikzcd}$$

and classical logic means that the right-hand arrow is invertible; so, in this case, the top arrow is invertible iff the left-hand arrow is.

But constructively, every discrete locale is quasi-discrete—hence the name! More generally, for any complete lattice ℓ, the transpose of

$$\mathcal{P}(E) \oslash \mathcal{P}(E) \oslash \ell \xrightarrow{\gamma \oslash \mathsf{id}} \mathbb{L} \oslash \ell \xrightarrow{\sim} \ell$$

as an arrow $\mathcal{P}(E) \oslash \ell \to \mathcal{P}(E) \multimap \ell$ is invertible. (Unless I'm much mistaken, this fact is also connected to the *constructive complete distributivity* of $\mathcal{P}(E)$, so we should not expect it to generalise to arbitrary quasi-discrete locales.)

In particular, if $\ell = \mathcal{P}(F)$, we get an isomorphism

$$\mathcal{P}(E \times F) \cong \mathcal{P}(E) \oslash \mathcal{P}(F) \cong \mathcal{P}(E) \multimap \mathcal{P}(F)$$

which should come as no surprise: the comparison functor $\mathsf{Rel} \to \mathsf{Sup}$ is fully faithful, and what we have here turns out to be the action of that functor on hom-sets.

All this is a convoluted way of introducing the notion that an entourage can—and perhaps should—be regarded, not as a "static" subset of $E \times E$, but as a "dynamic" operation on subsets of E. In the case of an effort space (E, φ), we write $\Phi_\varepsilon \colon \mathcal{P}(E) \to \mathcal{P}(E)$ for the operation corresponding to $N_\varepsilon \subseteq E \times E$. Then

$$\Phi_\varepsilon(A) = \{\tau \colon E \mid \exists \sigma \colon E . \varphi(\sigma, \tau) < \varepsilon\}$$

so, in particular, $\Phi_\varepsilon(\{\sigma\}) = W_{\sigma,\varepsilon}$.

Effort locales (old)

A *simple effort locale* is a locale E together with an order-preserving map $\Phi \colon \mathbb{Q}_+ \to \mathcal{O}(E) \multimap \mathcal{O}(E)$ satisfying the following axioms.

$$\begin{array}{rcl} \omega & \leq & \Phi_\varepsilon(\omega) \\ \Phi_\varepsilon(\Phi_\zeta(\omega)) & \leq & \Phi_{\varepsilon+\zeta}(\omega) \\ \Phi_\eta(\omega) & \leq & \bigvee_{\zeta < \eta} \Phi_\zeta(\omega) \end{array}$$

[The first two axioms correspond directly to the two axioms of an effort space; the third is a technical requirement that, for instance, guarantees, in the case E is discrete, that

$$\{\varepsilon \colon \mathbb{Q}_+ \mid \tau \in \Phi_\varepsilon(\{\sigma\})\}$$

is an *upper cut*—in which case, we recover $\varphi(\sigma, \tau)$ as the infimum of that set.]

The underlying quasi-uniformity of a simple effort locale is the upward closure of the range of Φ.

$$\mathcal{Q}(E) = \{\psi \colon \mathcal{O}(E) \multimap \mathcal{O}(E) \mid \exists \varepsilon \colon \mathbb{Q}_+ . \Phi_\varepsilon \leq \psi\}$$

[A *simple quasi-uniform locale* should mean a locale E together with a $\mathcal{Q}(E) \subseteq \mathcal{O}(E) \multimap \mathcal{O}(E)$ satisfying various axioms. A simple effort locale can then be regarded as a simple quasi-uniform locale together with a cofinal map $\Phi \colon \mathbb{Q}_+ \to \mathcal{Q}(E)$ satisfying further axioms.]

In the summer we showed that much of the usual theory of effort spaces generalises to simple effort locales. In particular, given a simple effort locale, one can easily define a *Kuratowski interior operator* on $\mathcal{O}(E)$ corresponding to the welling topology—in this way, we obtain a new locale E_w with $\mathcal{O}(E_\mathsf{w}) \subseteq \mathcal{O}(E)$; moreover, each Φ_ε restricts to a sup-homomorphism $\tilde{\Phi}_\varepsilon \colon \mathcal{O}(E_\mathsf{w}) \to \mathcal{O}(E_\mathsf{w})$. (Hence, $(E_\mathsf{w}, \tilde{\Phi})$ is again a simple effort locale; we call (E, Φ) *tight* if $E = E_\mathsf{w}$—*i.e.*, if the Kuratowski interior operator equals the identity.) Similarly, given a simple effort locale, we can define an appropriate notion of uniform cover.

Update

Symmetry

Given a simple effort locale (E, Φ), it is not clear whether one can define an "opposite" effort locale (E, Ψ).

Indeed, let (E, ϕ) be an effort space such that the welling topology is distinct from the sinking topology; then it is not clear that ϕ^{op} should restrict to E_{w}. (In other words, it is not clear that there should exist a Ψ, such that (E_{w}, Ψ) is the opposite of $(E_{\mathsf{w}}, \tilde{\Phi})$. It is also far from clear—indeed we expect it to be untrue—that every effort locale arises as $(E_{\mathsf{w}}, \tilde{\Phi})$ where E is discrete.)

The solution comes from the general theory of Hermitian objects in a closed involutive monoidal category $(\mathcal{V}, \otimes, \overline{(\ )}, I)$ with pullbacks: given Hermitian objects $(H, \gamma\colon \overline{H} \otimes H \to I)$ and $(K, \kappa\colon \overline{K} \otimes K \to I)$, consider the pullback below.

$$\begin{tikzcd}
\mathsf{Adjt}(H,\gamma;K,\kappa) \arrow[rr, dotted] \arrow[dd, dotted] & & K \multimapinv H \arrow[d] \\
& & (\overline{K} \multimap I) \multimapinv H \arrow[d, "\sim"] \\
\overline{H \multimapinv K} \arrow[r, "\sim"] & \overline{K} \multimap \overline{H} \arrow[r] & \overline{K} \multimap (I \multimapinv H)
\end{tikzcd}$$

If (H, γ) is weakly definite—*i.e.*, if the arrow $\overline{H} \to I \multimapinv H$ is invertible—then $\mathsf{Adjt}(H,\gamma;K,\kappa) \cong K \multimapinv H$; on the other hand, if (K, κ) is weakly definite, then $K \to \overline{K} \multimap I$ is also invertible, and hence $\mathsf{Adjt}(H,\gamma;K,\kappa) \cong \overline{H \multimapinv K}$. So if both (H, γ) and (K, κ) are weakly definite, we obtain an isomorphism

$$\overline{H \multimapinv K} \xrightarrow{\sim} \mathsf{Adjt}(H,\gamma;K,\kappa) \xrightarrow{\sim} K \multimapinv H$$

—which is the usual dagger operation on the category of weakly definite Hermitian objects.

But if neither (H, γ) nor (K, κ) are weakly definite, $\mathsf{Adjt}(H,\gamma;K,\kappa)$ yields a good notion of "adjoint pair of maps" between (H, γ) and (K, κ). (Indeed, if the ambient category is symmetric, then a Hermitian object is a species of Chu space, and $\mathsf{Adjt}(H,\gamma;K,\kappa)(H,\gamma;K,\kappa)$ is part of the definition of the internal hom of Chu spaces.) In particular, we can derive a dagger operation

$$\overline{\mathsf{Adjt}(K,\kappa;H,\gamma)} \to \mathsf{Adjt}(H,\gamma;K,\kappa)$$

which flips the two components of the pullback.

We therefore define an *adjointable effort locale* to be an open locale E together with an order-preserving map $\Phi\colon \mathbb{Q}_+ \to \mathsf{Adjt}(\mathcal{O}(E), \gamma_E; \mathcal{O}(E), \gamma_E)$ such that composition with either of the two projections to $\mathcal{O}(E) \multimap \mathcal{O}(E)$ results in a simple effort locale.

The *opposite* of an adjointable effort locale (E, Φ) is defined by composing Φ with the dagger operation on $\mathsf{Adjt}(\mathcal{O}(E), \gamma_E; \mathcal{O}(E), \gamma_E)$. A *symmetric effort locale* is then an adjointable effort locale equal to its opposite.