7. Consider the linear system

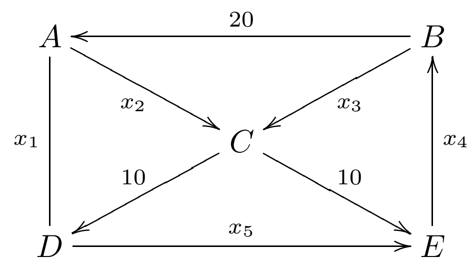
$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 0 \\ -x & + & y & - & z & = & 0 \\ x & + & ky & + & z & = & 0 \end{array}$$

a) If $[A\,|\,0]$ is the augmented matrix of the system above, find $\operatorname{rank} A$ and $\operatorname{rank}[A\,|\,0]$ for all values of k.

b) Find all k so that this system has
   i) a unique solution,
   ii) infinitely many solutions, and
   iii) no solutions.

c) In case (ii) above, give a complete geometric description of the set of solutions.

8. Consider the closed network of streets and intersections below. The arrows indicate the direction of traffic flow along the one–way streets, and the numbers are the flows observed during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Using Kirchoff's law, write down the linear system which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 5$. (Do not perform any operations on your equations: this is done for you in (b)!)

b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints at this point.)

c) Using (b) and the constraints from (a), find all possible traffic flows. (You do not need to list them all individually: simply give the possible vales of parameter(s).). How many different network flows are there?

d) Are there any network flows with both AD and BC closed for roadwork? What are the possible flows on the other streets in this case?

9. Let $W = \text{span}\{(-1,1,0),(1,1,-2)\}$ and $L = \text{span}\{(1,1,1)\}$.

For $v \in \mathbf{R}^3$, let $\text{proj}_W(v)$ and $\text{proj}_L(v)$ denote the orthogonal projections of v onto W and L respectively.

a) Give complete geometric descriptions of W and L.

b) Find a basis of $W^\perp = \{v \in \mathbf{R}^3 \mid v \cdot w = 0 \text{ for all } w \in W\}$ and use this to show that $L = W^\perp$.

c) If $(x,y,z) \in \mathbf{R}^3$, find a formula for $\text{proj}_W(x,y,z)$.

d) If $(x,y,z) \in \mathbf{R}^3$, show that $\text{proj}_L(x,y,z) = (\frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3})$.

e) Use (c) and (d) to show that $v = \text{proj}_W(v) + \text{proj}_L(v)$ for all $v \in \mathbf{R}^3$.

10. Let $A = \frac{1}{6}\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ and $v_1 = (1, 1, 1)$ and $v_2 = (1, -1, 0)$ be two vectors in $\mathbf{R}^3$.

a) Show that $Av_1 = v_1$ and $Av_2 = v_2$, and hence show that $\{v_1, v_2\} \subseteq \operatorname{col}(A)$, where $\operatorname{col}(A)$ denotes the column space of P.

b) Show that rank $A = 2$, and use this to deduce that $\dim \operatorname{col}(A) < 3$.

c) Use (a) and (b) to show that $\{v_1, v_2\}$ is a basis of $\operatorname{col}(A)$, and give a complete geometric description of $\operatorname{col}(A)$.

d) Show that $\ker A = \operatorname{span}\{v_1 \times v_2\}$.

11. Let $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

a) Show that the eigenvalues of A are 0 and 3.

b) Find a basis B_0 of $E_0 = \{x \in \mathbf{R}^3 \mid Ax = 0\}$.

c) Find a basis of $E_3 = \{x \in \mathbf{R}^3 \mid (A - 3I)x = 0\}$.

d) Show that the set consisting of all vectors from the bases for E_0 and E_3 is a basis for $\mathbf{R}^3$.

e) If possible, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

12. a) Let A be a real $n \times n$ matrix. Give 3 different statements which are equivalent to:

The columns of A are linearly dependent.

b) State whether the following are always true or could be false. If true, explain why, if false, give a numerical example to illustrate.

(i) Suppose an $n \times n$ matrix A satisfies $A^2 = A$. If v is an eigenvector of A with eigenvalue λ, then $\lambda = 0$ or $\lambda = 1$.

(ii) If an $n \times n$ matrix A satisfies $A^2 = 0$, then $A = 0$.

(iii) Let $\mathbf{F}[0,1] = \{f \mid f : [0,1] \to \mathbf{R}\}$ be the vector space of real-valued funtions defined on $[0,1]$. If f, g and h are any three different functions in $\mathbf{F}[0,1]$, then $\{f, g, h\}$ must be linearly independent.