MAT 1341, INTRODUCTION TO LINEAR ALGEBRA, WINTER 2003

Answers to the Midterm, March 7

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Question 1. Under what condition can a point (a, b, c) be written as a linear combination of $(1, 2, 0)$ and $(1, 1, 1)$?

- ☐ $3a - b - 2c = 0.$
- ☐ $a + b - 2c = 0.$
- ☒ $2a - b - c = 0.$
- ☐ $2a - b + 2c = 0.$
- ☐ $a - b = 0.$
- ☐ $a - 3b + 2c = 0.$

Question 2. Suppose that a given matrix A satisfies $A^2 - 2A - I = 0$. Give a formula for A^{-1}:

- ☐ $A^{-1} = 2A + I.$
- ☐ $A^{-1} = A + I.$
- ☒ $A^{-1} = A - 2I.$
- ☐ $A^{-1} = 2A - I.$
- ☐ $A^{-1} = A + 2I.$
- ☐ $A^{-1} = A - I.$

Question 3. In the vector space $\mathbf{M}_{2,2}$ of real 2×2-matrices, express M as a linear combination of the matrices A, B, and C, where

$$M = \begin{pmatrix} 4 & 7 \\ 7 & 9 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

Answer: $\boxed{M = \underline{\ \mathbf{-1}\ } A + \underline{\ \mathbf{3}\ } B + \underline{\ \mathbf{2}\ } C}$

Question 4. Write the polynomial $p(t) = t^2 - 2t + 1$ as a linear combination of the three polynomials $q_1(t) = t^2 - 1$, $q_2(t) = t^2 + t$ and $q_3(t) = t^2 + t + 1$.

Answer: $p = \underline{\quad \mathbf{3} \quad} q_1 + \underline{\quad \mathbf{-6} \quad} q_2 + \underline{\quad \mathbf{4} \quad} q_3$

Question 5. Determine whether or not W is a subspace of $\mathbb{R}^3$ where W consists of all vectors (a, b, c) in $\mathbb{R}^3$ such that:

(a)	$b = a^2$	☐ yes	☒ no
(b)	$a = 2b = 3c$	☒ yes	☐ no
(c)	$a = 3b$	☒ yes	☐ no
(d)	$ab = 0$	☐ yes	☒ no
(e)	$a \leqslant b \leqslant c$	☐ yes	☒ no
(f)	$a + b + c = 0$	☒ yes	☐ no

Question 6. Let V be a vector space over a field K. Which of the following statements are always valid:

(a) Every subset of V is a subspace of V ☐ true ☒ false

(b) Every subspace of V is a subset of V ☒ true ☐ false

(c) $\{0\}$ is a subspace of V ☒ true ☐ false

(d) Let $u, v \in V$ be vectors, and let W be a subspace of V. If W contains the vectors u and v, then W also contains the sum $u + v$. ☒ true ☐ false

(e) Let $u, v \in V$ be vectors, and let W be a subspace of V. If W contains the sum $u + v$, then W also contains u and v. ☐ true ☒ false

Question 7. Let W be a subset of a vector space V over a field K.

(a) Write down the three conditions which W must satisfy in order to be a subspace of V:

(1) $0 \in W$

(2) $\forall v, u (v, u \in W \Rightarrow v + u \in W)$.

(3) $\forall v, k (v \in W, k \in K \Rightarrow kv \in W)$.

(b) Show that $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y \geqslant z\}$ is not a subspace of $\mathbb{R}^3$ by showing, in a specific example, that one of the above three conditions is violated.

Answer: The condition (3) is violated. Specific example: $v = (1, 1, 1)$, $k = -1$. Then $v \in W$, $k \in \mathbb{R}$, but $kv = (-1, -1, -1) \notin W$.

Question 8. (a) Let $V = \mathbf{P}(t)$ be the vector space of polynomials in the variable t, with real coefficients. Let $W = \{p(t) \in V \mid p(2) = p(-2)\}$. Show that W is a subspace of V.

Answer:

(1) The constant zero polynomial $p(t) = 0$ is in W, because $p(2) = 0 = p(-2)$.

(2) Suppose $p(t) \in W$, $q(t) \in W$. Then $p(2) = p(-2)$ and $q(2) = q(-2)$. Let $r(t) = p(t)+q(t)$. Then $r(2) = p(2) + q(2) = p(-2) + q(-2) = r(-2)$, hence $r(t) \in W$.

(3) Suppose $p(t) \in W$, $k \in \mathbb{R}$. Then $p(2) = p(-2)$. Thus, $kp(2) = kp(-2)$, hence $kp(t) \in W$.

(b) Recall that an $n \times n$-matrix A is called *symmetric* if $A = A^T$. Prove that the set of all symmetric $n \times n$-matrices is a subspace of $\mathbf{M}_{n,n}$.

Answer: Let W be the subset of symmetric matrices.

(1) $0 = 0^T$, hence $0 \in W$.

(2) Suppose $A, B \in W$. Then $A = A^T$ and $B = B^T$. Hence $(A + B)^T = A^T + B^T = A + B$, hence $A + B \in W$.

(3) Suppose $A \in W$ and $k \in K$. Then $A = A^T$. Thus, $(kA)^T = k(A^T) = kA$, hence $kA \in W$.