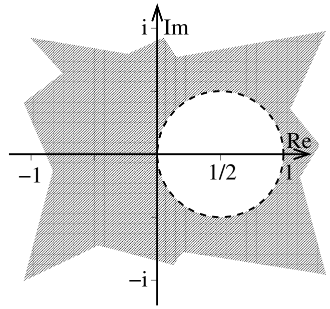
MAT 3321, COMPLEX ANALYSIS AND INTEGRAL TRANSFORMS, WINTER 2005

Answers to Homework 3
12.3 #6,8,22; 12.4 #2,6

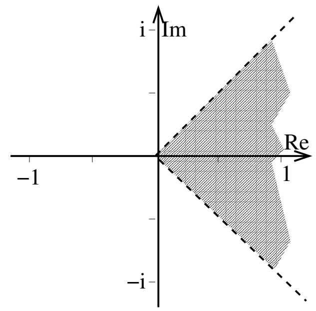
Problem 12.3 #6 We have

$$\mathrm{Re}(1/z) = \mathrm{Re}\,\frac{1}{x+iy} = \mathrm{Re}\,\frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} < 1.$$

The describes a region bounded by the curve $x/(x^2+y^2) = 1$, or equivalently, $x = x^2+y^2$. As in Problem 12.5 #10, we can rewrite this as $(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$, and the curve is therefore a circle of radius $1/2$ centered at $(1/2, 0)$. By trying some values for z inside and outside the circle, we find that the desired region is the open outside of the circle, as shown in the illustration.



Problem 12.3 #8 $|\arg z| < \pi/4$ means that $-\pi/4 < \arg z < \pi/4$. This describes the region of argument strictly between $-45°$ and $45°$, as shown in the illustration.



Problem 12.3 #22 We have $f(z) = (iz^3+3z^2)^3$, hence

$$f'(z) = 3(iz^3+3z^2)^2(i3z^2+6z).$$

Therefore

$$\begin{aligned}
f'(2i) &= 3(i(2i)^3+3(2i)^2)^2(i3(2i)^2+6(2i)) \\
&= 3(8i^4+12i^2)^2(12i^3+12i) \\
&= 3(8i^4+12i^2)^2(12i^3+12i) \\
&= 3(8i^4+12i^2)^2(-12i+12i) \\
&= 3(8i^4+12i^2)^2\cdot 0 \\
&= 0
\end{aligned}$$

Problem 12.4 #2 We have

$$f(z) = i|z|^3 = i\sqrt{x^2+y^2}^{\,3} = i(x^2+y^2)^{\frac{3}{2}},$$

where $z = x+iy$. Therefore

$$\begin{aligned}
u(x,y) &= 0 \\
v(x,y) &= (x^2+y^2)^{\frac{3}{2}}.
\end{aligned}$$

We calculate the partial derivatives:

$$\begin{aligned}
u_x &= 0 & v_x &= \tfrac{3}{2}(x^2+y^2)^{\frac{1}{2}}2x \\
u_y &= 0 & v_y &= \tfrac{3}{2}(x^2+y^2)^{\frac{1}{2}}2y
\end{aligned}$$

Clearly, neither $u_x = v_y$ nor $u_y = -v_x$, thus the Cauchy-Riemann equations are not satisfied; the function f is not analytic.

Problem 12.3 #6 We have $f(z) = z + 1/z$. Of course, we already know that this is analytic, because of the differentiation rules. However, we are asked to check this using the Cauchy-Riemann equations. We first express f in terms of x and y, where $z = x+iy$:

$$f(z) = z + \frac{1}{z} = x+iy+\frac{1}{x+iy} = x+iy+\frac{x-iy}{x^2+y^2}.$$

Therefore,

$$\begin{aligned}
u(x,y) &= x + \frac{x}{x^2+y^2} \\
v(x,y) &= y - \frac{y}{x^2+y^2}
\end{aligned}$$

We calculate the partial derivatives:

$$\begin{aligned}
u_x &= 1 + \frac{1(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = 1 + \frac{y^2-x^2}{(x^2+y^2)^2} \\
v_x &= \frac{2xy}{(x^2+y^2)^2} \\
u_y &= \frac{-2xy}{(x^2+y^2)^2} \\
v_y &= 1 - \frac{1(x^2+y^2)-y(2y)}{(x^2+y^2)^2} = 1 + \frac{y^2-x^2}{(x^2+y^2)^2}
\end{aligned}$$

So therefore the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are indeed satisfied, and the function f is analytic.