MAT 3343, APPLIED ALGEBRA, FALL 2003

Answers to Problem Set 1 (due Sept. 16)

Problem 1, Handout 1

(a) (2) Suppose $a \in I \cap J$ and $k \in \mathbb{Z}$. We want to show that $ka \in I \cap J$. By assumption, we know that $a \in I$ and $a \in J$. Therefore, by property (2) of I and J, $ka \in I$ and $ka \in J$. Therefore $ka \in I \cap J$.

(3) Since I is an ideal, we know $0 \in I$ by property (3) of I. Similarly $0 \in J$ by property (3) of J. It follows that $0 \in I \cap J$.

(b) (1) Suppose $a, b \in I + J$. We want to show that $a + b \in I + J$. We know that $a = a_1 + a_2$, for some $a_1 \in I$ and $a_2 \in J$. Similarly, $b = b_1 + b_2$ for some $b_1 \in I$ and $b_2 \in J$. It follows that $a + b = (a_1 + a_2) + (b_1 + b_2) = (a_1 + b_1) + (a_2 + b_2) \in I + J$ by property (1) of I and J.

(2) Suppose $a \in I + J$ and $k \in \mathbb{Z}$. We want to show that $ka \in I + J$. But $a = a_1 + a_2$ for some $a_1 \in I$ and $a_2 \in J$, therefore $ka_1 \in I$ and $ka_2 \in J$ by property (2) of I and J. Thus $ka = ka_1 + ka_2 \in I + J$, as desired.

(3) We want to show that $0 \in I + J$. But $0 \in I$ and $0 \in J$, thus $0 = 0 + 0 \in I + J$.

Problem 2, Handout 1 Suppose $n|m$ and $m|p$. By definition of divisibility, there are integers k and l such that $nk = m$ and $ml = p$. It follows that $n(kl) = (nk)l = ml = p$, and thus $n|p$.

Problem 3, Handout 1 "$\Rightarrow$": Suppose $a\mathbb{Z} \subseteq b\mathbb{Z}$. Because $a = a \cdot 1$, we have $a \in a\mathbb{Z}$, hence $a \in b\mathbb{Z}$. This means that $a = bk$ for some $k \in Z$, hence $b|a$.

"$\Leftarrow$": Suppose $b|a$. By definition of divisibility, we have $a = bl$ for some $l \in \mathbb{Z}$. We want to show that $a\mathbb{Z} \subseteq b\mathbb{Z}$. So take any element $x \in a\mathbb{Z}$. Then $x = ak$ for some $k \in \mathbb{Z}$. It follows that $x = ak = blk$, hence $x \in b\mathbb{Z}$.

Problem 4, Handout 1 (a) If we drop the word "non-empty", the principle is not valid. Counterexample: the empty set $\emptyset$ has no least element. (b) If we drop the word "positive", the principle is not valid. Counterexample: the set $\mathbb{Z}$ of all integers has no least element. (c) If we replace "integer" by "rational number",

the principle is not valid. Counterexample: the set $A = \{x \in \mathbb{Q} \mid x > 0\}$ has no least element. Proof: take any $x \in A$. Then $x/2 \in A$ as well. Since $x/2 < x$, x is not a least element. (d) If we replace the word "positive" by "non-negative", the principle is valid. Proof: Let A be any non-empty set of non-negative integers. Then either $0 \in A$, in which case 0 is the least element of A. Otherwise, $0 \notin A$, in which case A is a non-empty set of positive integers and the original well-foundedness principle applies.

Problem 1.2 # 7 (a) Suppose $d > 0$, $d|(11k+4)$, and $d|(10k+3)$ for some integer k. Then $d|10(11k + 4) + (-11)(10k + 3)$ by Theorem 2(4), p.37. Therefore $d|110k + 40 - 110k - 33$, hence $d|7$. Since 7 is prime, we have $d = 1$ or $d = 7$.

(b) Suppose $d > 0$, $d|(35k + 26)$, and $d|(7k + 3)$ for some integer k. Then $d|1(35k + 26) + (-5)(7k + 3)$ by Theorem 2(4), p.37. Therefore $d|35k + 26 - 35k - 15$, hence $d|11$. Since 11 is prime, it follows that $d = 1$ or $d = 11$.

Problem 1.2 #16 Suppose $n|k(n + 1)$. Then there exists $a \in \mathbb{Z}$ such that $na = k(n + 1)$, hence $na = kn + k$, hence $n(a - k) = k$, hence $n|k$.