MAT 3343, APPLIED ALGEBRA, FALL 2003

Problem Set 5, due November 21, 2003

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Problem 1. How many roots does the polynomial $p(x) = x^2 + x + 8$ have in $\mathbb{Z}_{10}$? Why does this not contradict the root theorem?

Problem 2. (a) Let F be a field, and let $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ be a third-degree polynomial in $F[x]$, with $a_3 \neq 0$. Prove that $p(x)$ is irreducible if any only if $p(x)$ has no roots.

(b) Show that the analogous statement is not true for fourth-degree polynomials, i.e., give an example of a field F and a fourth-degree polynomial $q(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ such that $q(x)$ is reducible but has no roots.

Problem 3. Find the gcd $d(x)$ of $p(x) = x^8 + x^7 + x^5 + x^3 + x^2 + x + 1$ and $q(x) = x^7 + x^6 + x^4 + x^3 + 1$ in $\mathbb{Z}_2[x]$. Find $s(x)$ and $t(x)$ such that $p(x)s(x) + q(x)t(x) = d(x)$.

Problem 4. (a) Find all irreducible polynomials of degree up to 5 in $\mathbb{Z}_2[x]$.

(b) Factor $p(x) = x^{12} + x^{10} + x^7 + x^6 + 1$ into irreducible factor in $\mathbb{Z}_2[x]$.

Problem 5. Factor the following polynomials into irreducible factors in the given ring. In each case, give a reason why you know that the factors you found are irreducible.

(a) $x^5 - 1$ in $\mathbb{Q}[x]$.

(b) $x^5 + 1$ in $\mathbb{Z}_2[x]$.

(c) $x^4 + 1$ in $\mathbb{Z}_5[x]$.

(d) $2x^3 + x^2 + 4x + 2$ in $\mathbb{Q}[x]$.

(e) $x^4 - 9x + 3$ in $\mathbb{Q}[x]$.

(f) $x^8 - 16$ in $\mathbb{Q}[x]$.

Problem 6. Find all irreducible polynomials of the form $x^2 + ax + b$ over $\mathbb{Z}_5$ (i.e., all irreducible polynomials of degree 2, with leading coefficient 1).

Problem 7. Are the following polynomials irreducible in the given ring? Give reasons.

(a) $x^3 + x^2 + x + 1$ in $\mathbb{Q}[x]$.

(b) $3x^8 - 4x^6 + 8x^5 - 10x + 6$ in $\mathbb{Q}[x]$.

(c) $x^4 + x^2 - 6$ in $\mathbb{Q}[x]$.

(d) $4x^3 + 3x^2 + x + 1$ in $\mathbb{Z}_5[x]$.

Problem 8. In each case, find a polynomial in $\mathbb{Q}[x]$ with a as a root. Then prove that a is irrational.

(a) $a = \sqrt{2}/\sqrt[3]{5}$.

(b) $a = \sqrt{2} + \sqrt{3}$.

Problem 9. Find all rational roots of $3x^3 + 4x^2 - x - 2$.