MATH 582, INTRODUCTION TO SET THEORY, WINTER 1999

Construction of the Real Numbers via Cauchy Sequences

1 Definition of the real numbers

Definition. On the rational numbers, we define the ***absolute value function*** $|-| : \mathbb{Q} \to \mathbb{Q}$ as follows:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$

Lemma 1. *The following hold for all $x, y \in \mathbb{Q}$:*

(a) $|x| \geqslant 0$.

(b) $x \leqslant |x|$ and $|x| = |-x|$.

(c) $|xy| = |x| \cdot |y|$ and, if $x \neq 0$, $|x^{-1}| = |x|^{-1}$.

(d) $|x + y| \leqslant |x| + |y|$ (triangle inequality).

Proof. (a) through (c) are proved by easy case distinction. For (d), there are two cases: if $x + y \geqslant 0$, then $|x + y| = x + y \leqslant |x| + |y|$ by (b), and if $x + y < 0$, then $|x + y| = -(x + y) = (-x) + (-y) \leqslant |-x| + |-y| = |x| + |y|$ by (b). □

In the following, when we say $\epsilon > 0$, we always mean that ϵ is a rational number.

Definition. A ***Cauchy sequence*** is a sequence of rational numbers $s : \omega \to \mathbb{Q}$, such that

$$(\forall \epsilon > 0)(\exists k \in \omega)(\forall m, n > k)\, |s_m - s_n| < \epsilon.$$

Two Cauchy sequences s, r are said to be ***equivalent***, in symbols $s \sim r$, if

$$(\forall \epsilon > 0)(\exists k \in \omega)(\forall n > k)\, |s_n - r_n| < \epsilon.$$

Lemma 2. *The relation $\sim$ is an equivalence relation on the set of Cauchy sequences.*

Proof. Reflexivity and symmetry are straightforward. To show transitivity, assume $s \sim r$ and $r \sim t$. Consider arbitrary $\epsilon > 0$. Since $s \sim r$, there exists $k_0 \in \omega$ such that for all $n > k_0$, we have $|s_n - r_n| < \epsilon/2$. Similarly, since $r \sim t$, there exists $k_1 \in \omega$ such that for all $n > k_1$, we have $|r_n - t_n| < \epsilon/2$. Let $k = \max\{k_0, k_1\}$. Then for all $n > k$, one has

$$\begin{aligned} |s_n - t_n| &= |s_n - r_n + r_n - t_n| \\ &\leqslant |s_n - r_n| + |r_n - t_n| \quad \text{by the triangle inequality} \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Since ϵ was arbitrary, this shows that $s \sim t$. □

Let C be the set of Cauchy sequences. We define the set of ***real numbers*** to be $\mathbb{R} = C/\sim$.

The following lemma provides two facts about Cauchy sequences that we will use repeatedly: any Cauchy sequence is bounded, and if $s \not\sim 0$, then s is bounded away from 0.

Lemma 3. *1. If s is a Cauchy sequence, then there exists $M \in \mathbb{Q}$ such that for all $n \in \omega$, $|s_n| < M$.*

2. If s is a Cauchy sequence and $s \not\sim 0$, then there exists $\epsilon_0 > 0$ and $k_0 \in \omega$ such that for all $n > k_0$, $|s_n| > \epsilon_0$.

Proof. 1.: Because s is a Cauchy sequence, there exists $k \in \omega$ such that for all $m, n > k$, $|s_m - s_n| < 1$. Let $B = \{|s_0|, |s_1|, \ldots, |s_k|, |s_{k+1}|\}$, and define $M = (\max B) + 1$; notice that since $B \subseteq \mathbb{Q}$ is finite, this maximum exists. We claim that for all $n \in \omega$, $|s_n| < M$. There are two cases: if $n \leqslant k$, then $|s_n| \in B$, and thus $|s_n| \leqslant \max B < M$. If $k < n$, then

$$|s_n| = |s_n - s_{k+1} + s_{k+1}| \leqslant |s_n - s_{k+1}| + |s_{k+1}| < 1 + |s_{k+1}| \leqslant M.$$

2.: Because $s \not\sim 0$, it follows from the definition of "$\sim$" that there exists $\epsilon > 0$ such that

$$(\forall k \in \omega)(\exists m > k)\, |s_m - 0| \geqslant \epsilon. \tag{1}$$

Also, since s is a Cauchy sequence, there is k_0 such that for all $m, n > k_0$, $|s_m - s_n| < \epsilon/2$. Now let $\epsilon_0 = \epsilon/2$. We claim that ϵ_0 and k_0 satisfy the desired conclusion, namely, that $|s_n| > \epsilon_0$ for all $n > k_0$. To prove this, take an arbitrary $n > k_0$. By (1), there exists $m > k_0$ such that $|s_m| \geqslant \epsilon$. Then we have

$$\epsilon \leqslant |s_m| = |s_m - s_n + s_n| \leqslant |s_m - s_n| + |s_n| < \epsilon/2 + |s_n|,$$

and thus, $\epsilon_0 = \epsilon/2 < |s_n|$, as desired. □

2 Arithmetic operations

Next, we want to equip $\mathbb{R}$ with some operations to make it into a field. We first define the operations on Cauchy sequences, and later we show that they are compatible with $\sim$. We sometimes use the notation $(s_n)_{n\in\omega}$ to denote the sequence s.

Definition. Let $0 = (0)_{n\in\omega}$ and $1 = (1)_{n\in\omega}$ be the constant zero, respectively one, sequence. On the set of Cauchy sequences, we define the following operations:

$$\begin{aligned} s + r &= (s_n + r_n)_{n\in\omega}, \\ -s &= (-s_n)_{n\in\omega}, \\ s \cdot r &= (s_n \cdot r_n)_{n\in\omega}. \end{aligned}$$

Moreover, if $s \not\sim 0$, then we define

$$s^{-1} = (t_n)_{n\in\omega}, \quad \text{where } t_n = \begin{cases} {s_n}^{-1} & \text{if } s_n \neq 0, \\ 0 & \text{otherwise.} \end{cases}.$$

Lemma 4. *The four operations in the previous definition are well-defined operations on Cauchy sequences, i.e., if s and r are Cauchy sequences, then so are $s + r$, $-s$, and $s \cdot r$; moreover, if $s \not\sim 0$, then s^{-1} is a Cauchy sequence.*

Proof. Suppose that s, r are Cauchy sequences. To show that $s + r$ is a Cauchy sequence, consider arbitrary $\epsilon > 0$. Since s is a Cauchy sequence, there exists $k_0 \in \omega$ such that for all $m, n > k_0$, $|s_m - s_n| < \epsilon/2$. Also, since r is a Cauchy sequence, there exists $k_1 \in \omega$ such that for all $m, n > k_1$, $|r_m - r_n| < \epsilon/2$. Now let $k = \max\{k_0, k_1\}$. Then for all $m, n > k$, one has

$$\begin{aligned} |(s_m + r_m) - (s_n + r_n)| &= |s_m - s_n + r_m - r_n| \\ &\leqslant |s_m - s_n| + |r_m - r_n| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Since ϵ was arbitrary, this shows that $(s_n + r_n)_{n\in\omega}$ is a Cauchy sequence.

To show that $-s$ is a Cauchy sequence, consider arbitrary $\epsilon > 0$. Since s is a Cauchy sequence, there exists $k \in \omega$ such that for all $m, n > k$, $|s_m - s_n| < \epsilon$. Thus, for all $m, n > k$, one also has $|(-s_m) - (-s_n)| < \epsilon$, and thus $(-s_n)_{n\in\omega}$ is a Cauchy sequence.

To show that $s \cdot r$ is a Cauchy sequence, we first notice that by Lemma 3, s and r are bounded. Let M be a bound for both s and r, i.e., such that $|s_n| < M$ and $|r_n| < M$ for all $n \in \omega$. Since s is a Cauchy sequence, there exists $k_0 \in \omega$ such that for all $m, n > k_0$, $|s_m - s_n| < \epsilon/2M$. Also, since r is a Cauchy sequence, there exists $k_1 \in \omega$ such that for all $m, n > k_1$, $|r_m - r_n| < \epsilon/2M$. Now let $k = \max\{k_0, k_1\}$. Then for all $m, n > k$, one has

$$\begin{aligned} |s_m \cdot r_m - s_n \cdot r_n| &= |s_m \cdot r_m - s_m \cdot r_n + s_m \cdot r_n - s_n \cdot r_n| \\ &\leqslant |s_m \cdot r_m - s_m \cdot r_n| + |s_m \cdot r_n - s_n \cdot r_n| \\ &= |s_m| \cdot |r_m - r_n| + |s_m - s_n| \cdot |r_n| \\ &< M \cdot \epsilon/2M + \epsilon/2M \cdot M \\ &= \epsilon. \end{aligned}$$

Since ϵ was arbitrary, this shows that $(s_n \cdot r_n)_{n\in\omega}$ is a Cauchy sequence.

Now assume that $s \not\sim 0$, and let $t = s^{-1}$. To show that t is a Cauchy sequence, consider an arbitrary $\epsilon > 0$. By Lemma 3, there exist $\epsilon_0 > 0$ and $k_0 \in \omega$ such that for all $n > k_0$, $|s_n| > \epsilon_0$. Also, since s is a Cauchy sequence, there exists $k_1 \in \omega$ such that for all $m, n > k_1$, $|s_m - s_n| < \epsilon\, \epsilon_0^2$. Let $k = \max\{k_0, k_1\}$. Then for all $m, n > k$, one has $|s_n| > \epsilon_0$ and $|s_m| > \epsilon_0$, which in particular implies $s_n, s_m \neq 0$. We calculate

$$|t_m - t_n| = |{s_m}^{-1} - {s_n}^{-1}| = |\frac{s_n - s_m}{s_n s_m}| = \frac{|s_n - s_m|}{|s_n| \cdot |s_m|} < \frac{\epsilon\, \epsilon_0^2}{\epsilon_0 \cdot \epsilon_0} = \epsilon.$$

This shows that s^{-1} is a Cauchy sequence, which finishes the proof of Lemma 4.□

Lemma 5. *The four operations, plus, minus, times, and inverse, are compatible with equivalence of Cauchy sequences. Thus, they give rise to operations on $\mathbb{R}$.*

Proof. The show that addition is a compatible operation, suppose s, s', r, r' are Cauchy sequences such that $s \sim s'$ and $r \sim r'$. We need to show that $s + r \sim s' + r'$. Consider arbitrary $\epsilon > 0$. Since $s \sim s'$, there exists $k_0 \in \omega$ such that for all $n > k_0$, $|s_n - s'_n| < \epsilon/2$. Also, since $r \sim r'$, there exists $k_1 \in \omega$ such that for all $n > k_1$, $|r_n - r'_n| < \epsilon/2$. Now let $k = \max\{k_0, k_1\}$. Then for all $n > k$, one has

$$\begin{aligned} |(s_n + r_n) - (s'_n + r'_n)| &= |s_n - s'_n + r_n - r'_n| \\ &\leqslant |s_n - s'_n| + |r_n - r'_n| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Since ϵ was arbitrary, this shows that $s + r \sim s' + r'$.

To show that negation is a compatible operation, assume $s \sim s'$. We want to show $-s \sim -s'$. Consider arbitrary $\epsilon > 0$. Since $s \sim s'$, there exists $k \in \omega$ such that for all $n > k$, $|s_n - s'_n| < \epsilon$. Thus, for all $n > k$, one also has $|(-s_n) - (-s'_n)| < \epsilon$, and thus $-s \sim -s'$.

To show that multiplication is a compatible operation, assume $s \sim s'$ and $r \sim r'$. We need to show $s \cdot r \sim s' \cdot r'$. First notice that since s, s', r, r' are Cauchy sequences, they are bounded by Lemma 3. Let M be a bound for all of them, i.e., let $M \in \mathbb{Q}$ such that $|s_n|, |s'_n|, |r_n|, |r'_n| < M$ for all $n \in \omega$. Since $s \sim s'$, there exists $k_0 \in \omega$ such that for all $n > k_0$, $|s_n - s'_n| < \epsilon/2M$. Also, since $r \sim r'$, there exists $k_1 \in \omega$ such that for all $n > k_1$, $|r_n - r'_n| < \epsilon/2M$. Now let $k = \max\{k_0, k_1\}$. Then for all $n > k$, one has

$$\begin{aligned} |s_n \cdot r_n - s'_n \cdot r'_n| &= |s_n \cdot r_n - s_n \cdot r'_n + s_n \cdot r'_n - s'_n \cdot r'_n| \\ &\leqslant |s_n \cdot r_n - s_n \cdot r'_n| + |s_n \cdot r'_n - s'_n \cdot r'_n| \\ &= |s_n| \cdot |r_n - r'_n| + |s_n - s'_n| \cdot |r'_n| \\ &< M \cdot \epsilon/2M + \epsilon/2M \cdot M \\ &= \epsilon. \end{aligned}$$

This shows that $s \cdot r \sim s' \cdot r'$, as desired.

To show that $(-)^{-1}$ is a compatible operation, assume $s \not\sim 0$ and $s \sim s'$. First notice that this implies $s' \not\sim 0$, for otherwise one would have $s \sim 0$ by transitivity. Let $t = s^{-1}$ and $t' s'^{-1}$. We need to show that $t \sim t'$. So consider an arbitrary $\epsilon > 0$. By Lemma 3, there exist $\epsilon_0 > 0$ and $k_0 \in \omega$ such that for all $n > k_0$, $|s_n| > \epsilon_0$, and there exist $\epsilon_1 > 0$ and $k_1 \in \omega$ such that for all $n > k_1$, $|s'_n| > \epsilon_1$. Also, since $s \sim s'$, there exists $k_2 \in \omega$ such that for all $n > k_2$, $|s_n - s'_n| < \epsilon\, \epsilon_0\, \epsilon_1$. Let $k = \max\{k_0, k_1, k_2\}$. Then for all $n > k$, one has $|s_n| > \epsilon_0$ and $|s'_n| > \epsilon_1$, which in particular implies $s_n, s'_n \neq 0$. We calculate

$$|t_n - t'_n| = |{s_n}^{-1} - {s'_n}^{-1}| = |\frac{s'_n - s_n}{s'_n s_n}| = \frac{|s'_n - s_n|}{|s'_n| \cdot |s_n|} < \frac{\epsilon\, \epsilon_0\, \epsilon_1}{\epsilon_1 \cdot \epsilon_0} = \epsilon.$$

Since ϵ was arbitrary, this shows that $t \sim t'$ as desired, which finishes the proof of Lemma 5. □

As a consequence of Lemma 5 and a version of Theorem 3Q, there exist well-defined operations on real numbers $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $- : \mathbb{R} \to \mathbb{R}$, $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $(-)^{-1} : \mathbb{R}\backslash\{0\} \to \mathbb{R}$, satisfying

$$\begin{aligned} [s] + [r] &= [s + r], \\ -[s] &= [-s], \\ [s] \cdot [r] &= [s \cdot r], \text{ and} \\ [s]^{-1} &= [s^{-1}], \end{aligned}$$

where $[s] \neq [0]$ in the last equation.

Proposition 6. $\langle \mathbb{R}, [0], +, -, [1], \cdot, (-)^{-1} \rangle$ *is a field.*

Proof. Each of the defining equations of a field follows easily from the corresponding equation for the rational numbers. For example, to show distributivity, consider any $x, y, z \in \mathbb{R}$. Let s, r, t be Cauchy sequences such that $x = [s]$, $y = [r]$, and $z = [t]$. Then

$$(s + r)t = ((s_n + r_n)t_n)_{n \in \omega} = (s_n t_n + r_n t_n)_{n \in \omega} = st + rt.$$

This implies $(x + y)z = xz + yz$.

For multiplicative inverses, consider any $x \in \mathbb{R}\backslash\{0\}$. Let s be a representative Cauchy sequence for x, i.e., $x = [s]$. Then $x^{-1} = [s^{-1}]$. Let $t = s^{-1}$. Notice that $s \not\sim 0$, so by Lemma 3(2), there exists $k_0 \in \omega$ such that for all $n > k_0$, $s_n \neq 0$, and thus $t_n = {s_n}^{-1}$. Then, for all $n > k$, we have

$$s_n \cdot t_n = 1,$$

and hence $s \cdot s^{-1} = (s_n \cdot t_n)_{n \in \omega} \sim 1$, which implies $x \cdot x^{-1} = [1]$.

The other equations of a field follow similarly. □

3 Linear order

Definition. On the set of Cauchy sequences, we define the following relation: $s < r$ if and only if

$$(\exists \epsilon > 0)(\exists k \in \omega)(\forall n > k)\, s_n + \epsilon < r_n.$$

Lemma 7. *The relation "$<$" is compatible with equivalence of Cauchy sequences, which is to say, if $s < r$ and $s \sim s'$ and $r \sim r'$, then $s' < r'$.*

Proof. Suppose $s < r$, $s \sim s'$, and $r \sim r'$. By definition of $s < r$, there exists $\epsilon > 0$ and $k \in \omega$ such that for all $n > k$, $s_n + \epsilon < r_n$. Notice that this implies $s_n - r_n < -\epsilon$. Now since $s \sim s'$, there exists $k_0 \in \omega$ such that for all $n > k_0$, $|s_n - s'_n| < \epsilon/3$. Also, since $r \sim r'$, there exists $k_1 \in \omega$ such that for all $n > k_1$, $|r_n - r'_n| < \epsilon/3$. Now let $\epsilon' = \epsilon/3$, and let $k' = \max\{k, k_0, k_1\}$. We claim that this ϵ' and k' satisfy the definition of $s' < r'$. Indeed, for all $n > k'$, one has

$$\begin{aligned} s'_n + \epsilon' &= s'_n - s_n + s_n - r_n + r_n - r'_n + r'_n + \epsilon' \\ &\leqslant |s'_n - s_n| + s_n - r_n + |r_n - r'_n| + r'_n + \epsilon' \\ &< \epsilon/3 - \epsilon + \epsilon/3 + r'_n + \epsilon' \\ &= r'_n. \end{aligned}$$

This shows that $s' < r'$. □

As a consequence of the previous lemma, we can define a relation on the real numbers as follows:

$$[s] < [r] \qquad \text{iff} \qquad s < r.$$

Lemma 8. *The relation $<$ defines a linear order on $\mathbb{R}$.*

Proof. We show transitivity, irreflexivity, and connectedness. For transitivity, suppose $x, y, z \in \mathbb{R}$ such that $x < y$ and $y < z$. Let s, r, t be Cauchy sequences such that $x = [s]$, $y = [r]$, and $z = [t]$. Then $s < r$ and $r < t$. It follows, by definition of "$<$", that there exist $\epsilon_0, \epsilon_1 > 0$ and $k_0, k_1 \in \omega$ such that for all $n > k_0$, $s_n + \epsilon_0 < r_n$, and for all $n > k_1$, $r_n + \epsilon_1 < t_n$. Let $\epsilon = \epsilon_0 + \epsilon_1$ and let $k = \max\{k_0, k_1\}$. Then for all $n > k$, $s_n + \epsilon = s_n + \epsilon_0 + \epsilon_1 < r_n + \epsilon_1 < t_n$. It follows that $s < t$, hence $x < z$.

For irreflexivity, notice that there can never be an $\epsilon > 0$ and $n \in \omega$ such that $s_n + \epsilon < s_n$. Thus, for no Cauchy sequence, $s < s$. It follows that for no real number, $x < x$.

For connectedness, consider $x, y \in \mathbb{R}$ such that neither $x < y$ nor $y < x$. We will show that $x = y$. Let $x = [s]$ and $y = [r]$. It suffices to show that $s \sim r$. We know that $s \not< r$ and $r \not< s$. By using the definition of "$<$", it follows that

$$(\forall \epsilon > 0)(\forall k \in \omega)(\exists n > k)\, s_n + \epsilon \geqslant r_n, \tag{2}$$

$$(\forall \epsilon > 0)(\forall k \in \omega)(\exists n > k)\, r_n + \epsilon \geqslant s_n. \tag{3}$$

We claim that $s \sim r$, i.e.

$$(\forall \epsilon > 0)(\exists k \in \omega)(\forall n > k)\, |s_n - r_n| < \epsilon. \tag{4}$$

To prove this, take an arbitrary $\epsilon > 0$. Because s is a Cauchy sequence, there exists $k_0 \in \omega$ such that $|s_m - s_n| < \epsilon/3$ for all $m, n > k_0$. Similarly, there exists $k_1 \in \omega$ such that $|r_m - r_n| < \epsilon/3$ for all $m, n > k_1$. Let $k = \max\{k_0, k_1\}$. By (2), there exists $n_0 > k$ such that $s_{n_0} + \epsilon/3 \geqslant r_{n_0}$. Also, by (3), there exists

$n_1 > k$ such that $r_{n_1} + \epsilon/3 \geqslant s_{n_1}$. Now we claim that (4) holds for this ϵ and k. So let $n > k$. Then

$$\begin{aligned} s_n - r_n &= s_n - s_{n_1} + s_{n_1} - r_{n_1} + r_{n_1} - r_n \\ &\leqslant |s_n - s_{n_1}| + s_{n_1} - r_{n_1} + |r_{n_1} - r_n| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon \\ r_n - s_n &= r_n - r_{n_0} + r_{n_0} - s_{n_0} + s_{n_0} - s_n \\ &\leqslant |r_n - r_{n_0}| + r_{n_0} - s_{n_0} + |s_{n_0} - s_n| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon \end{aligned}$$

The two inequalities together imply that $|s_n - r_n| < \epsilon$, as desired. Since ϵ was arbitrary, this implies $s \sim r$, and thus $x = y$. This finishes the proof that "$<$" is a linear order on $\mathbb{R}$. □

Proposition 9. $\langle \mathbb{R}, [0], +, -, [1], \cdot, (-)^{-1}, < \rangle$ *is an ordered field.*

Proof. We already know it is a field, and $<$ is a linear order. It remains to show that for all $x, y, z \in \mathbb{R}$, $x < y \Rightarrow x + z < y + z$, and if $z > 0$, then $x < y \Rightarrow xz < yz$. Let $x = [s]$, $y = [r]$, and $z = [t]$.

To prove the first property, assume $x < y$, i.e., $s < r$. Then there exist $\epsilon > 0$ and $k \in \omega$ such that for all $n > k$, $s_n + \epsilon < r_n$, and thus $s_n + t_n + \epsilon < r_n + t_n$. This implies $s + t < r + t$, and thus $x + z < y + z$.

To prove the second property, assume $x < y$ and $z > 0$. Then $s < r$, and thus there exist $\epsilon_0 > 0$ and $k_0 \in \omega$ such that for all $n > k_0$, $s_n + \epsilon_0 < r_n$. Also, $0 < t$, and thus there exist $\epsilon_1 > 0$ and $k_1 \in \omega$ such that for all $n > k_1$, $0 + \epsilon_1 < t_n$. Let $\epsilon = \epsilon_0\,\epsilon_1$ and $k = \max\{k_0, k_1\}$. Then for all $n > k$, $r_n - s_n > \epsilon_0$ and $t_n > \epsilon_1$, thus

$$\epsilon = \epsilon_0\,\epsilon_1 < (r_n - s_n)t_n = r_n t_n - s_n t_n,$$

which implies $s_n t_n + \epsilon < r_n t_n$. Since this holds for all $n > k$, we have $st < rt$, and thus $xz < yz$. □

4 Completeness

In this section, we show that the real numbers are ***complete***. By this we mean that any Cauchy sequence of real numbers has a limit. It is an interesting fact that the real numbers are, up to isomorphism, the only complete ordered field, but we will not prove this here.

First, notice that there is a natural function $\mathbb{Q} \to \mathbb{R}$ that maps each rational number q to the real number $[(q)_{n\in\omega}]$. This map is one-to-one, and it is a homomorphism of ordered fields. We write $\overline{q}$ for the image of q under this map; thus $\overline{q}$ is the number q regarded as a real number.

The absolute value function $|-|$ is defined on reals in the same way as on rationals, i.e., $|x| = x$ for $x \geqslant 0$, and $|x| = -x$ otherwise.

We begin with a useful lemma:

Lemma 10. *1. For any real number $x > 0$, there exists a rational number $q > 0$ such that $\overline{q} < x$.*

*2. The rational numbers are **dense** in $\mathbb{R}$, i.e., for any $x \in \mathbb{R}$ and any real $\epsilon > 0$, there exists $q \in \mathbb{Q}$ such that $|x - \overline{q}| < \epsilon$.*

Proof. 1.: Let s be a Cauchy sequence such that $x = [s]$. Since $0 < s$, there exists rational $\epsilon > 0$ and $k \in \omega$ such that $0 + \epsilon < s_n$ for all $n > k$. Let $q = \epsilon/2$. Then for all $n > k$, $q + \epsilon/2 = \epsilon < s_n$, and thus $(q)_{n\in\omega} < s$, which implies $\overline{q} < x$.

2.: Suppose $x, \epsilon \in \mathbb{R}$ are given and $\epsilon > 0$. By 1., we can assume without loss of generality that ϵ is rational. Let s be a Cauchy sequence such that $x = [s]$. Then there exists $k \in \omega$ such that for all $m, n > k$, $|s_m - s_n| < \epsilon/2$. Let $q = s_{k+1}$. Then for all $n > k$,

$$s_n - q + \epsilon/2 \leqslant |s_n - q| + \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon$$
$$q - s_n + \epsilon/2 \leqslant |q - s_n| + \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon$$

The first inequality shows that $x - \overline{q} < \epsilon$, and the second one shows that $\overline{q} - x < \epsilon$. Taken together, we get $|x - \overline{q}| < \epsilon$, as desired. □

A Cauchy sequence of real numbers is defined just like a Cauchy sequence of rational numbers. Notice that by Lemma 10(1), it does not make a difference in the following definition whether we take $\epsilon > 0$ to be a rational or a real number.

Definition. A ***real Cauchy sequence*** is a sequence of real numbers $x : \omega \to \mathbb{R}$, such that

$$(\forall \epsilon > 0)(\exists k \in \omega)(\forall m, n > k)\, |x_m - x_n| < \epsilon.$$

We say that a real Cauchy sequence x ***converges*** to $l \in R$ if

$$(\forall \epsilon > 0)(\exists k \in \omega)(\forall n > k)\, |x_n - l| < \epsilon.$$

In this case, we also say that l is a ***limit*** of the sequence.

Proposition 11 (Completeness of the real numbers). *Any real Cauchy sequence has a limit.*

Proof. Let $(x_n)_{n\in\omega}$ be a real Cauchy sequence. We define a (rational) Cauchy sequence $s : \omega \to \mathbb{Q}$ as follows: For any $n \in \omega$, let s_n be a rational number such that $|x_n - \overline{s_n}| < 1/n$. For each n, such a number exists by Lemma 10(2); we may apply the axiom of choice to obtain a sequence $(s_n)_{n\in\omega}$ of such numbers. We claim that s is a Cauchy sequence. So take arbitrary rational $\epsilon > 0$. Since $(x_n)_{n\in\omega}$ is a real Cauchy sequence, there exists $k_0 \in \omega$ such that for all $m, n > k_0$, $|x_m - x_n| < \epsilon/3$. Let $k_1 \in \omega$ be a natural number that is greater than $3/\epsilon$, and let $k = \max\{k_0, k_1\}$. Then for all $m, n > k$, we have $m > 3/\epsilon$ and $n > 3/\epsilon$, and thus

$$\begin{aligned}
|s_m - s_n| &= |\overline{s_m} - \overline{s_n}| \\
&= |\overline{s_m} - x_m + x_m - x_n + x_n - \overline{s_n}| \\
&\leqslant |\overline{s_m} - x_m| + |x_m - x_n| + |x_n - \overline{s_n}| \\
&< 1/m + \epsilon/3 + 1/n \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&= \epsilon.
\end{aligned}$$

This proves that s is a Cauchy sequence. Now let $l = [s]$. We claim that l is a limit of $(x_n)_{n\in\omega}$. So take any $\epsilon > 0$; by Lemma 10(1), we can assume without loss of generality that ϵ is rational. Since s is a Cauchy sequence, we can find $k_0 \in \omega$ such that for all $m, n > k_0$, $|s_m - s_n| < \epsilon/3$. Consider an arbitrary $n > k_0$. Then we have that for all $m > k_0$, $s_n - s_m + \epsilon/3 < 2\epsilon/3$, and $s_m - s_n + \epsilon/3 < 2\epsilon/3$. The first inequality implies that $\overline{s_n} - l < \overline{2\epsilon/3}$, and the second inequality implies that $l - \overline{s_n} < \overline{2\epsilon/3}$, so we get $|\overline{s_n} - l| < \overline{2\epsilon/3}$. Notice that this holds for arbitrary $n > k_0$. Now let k_1 be a natural number greater than $3/\epsilon$. Then for any $n > \max\{k_0, k_1\}$, we have

$$\begin{aligned}
|x_n - l| &\leqslant |x_n - \overline{s_n}| + |\overline{s_n} - l| \\
&< 1/n + 2\epsilon/3 \\
&< \epsilon/3 + 2\epsilon/3 \\
&= \epsilon.
\end{aligned}$$

Since ϵ was arbitrary, this shows that l is a limit of $(w_n)_{n\in\omega}$. □