MATH 316, DIFFERENTIAL EQUATIONS, WINTER 2000

Answers to the Second Midterm

Problem 1 (12 points)

(a) Find all (real or complex) solutions of the form $y = e^{rt}$ which satisfy the following equation.

$$y'' - 6y' + 10y = 0.$$

Answer: Plug $y = e^{rt}$ into the equation, we get $(r^2 - 6r + 10)e^{rt} = 0$. Thus, e^{rt} will be a solution if and only if $r^2 - 6r + 10 = 0$. The quadratic formula gives us two answers for r, namely $r = 3 + i$ and $r = 3 - i$. Thus, the desired solutions are $y_1 = e^{(3+i)t}$ and $y_2 = e^{(3-i)t}$.

(b) Find two linearly independent *real* solutions of the equation.

Answer: Let us write the two solutions from part (a) in terms of their real and imaginary parts:

$$y_1 = e^{3t}(\cos t + i \sin t),$$
$$y_2 = e^{3t}(\cos t - i \sin t).$$

Since the system of equations is linear, any linear combination of solutions is again a solution. In particular, by adding $y_1 + y_2$ (and dividing the result by 2), we get a real solution, $y_3 = e^{3t}\cos t$. By subtracting $y_1 - y_2$ (and dividing the result by $2i$), we get another real solution, $y_4 = e^{3t}\sin t$. These are two linearly independent real solutions.

Problem 2 (12 points)

(a) Carefully sketch the phase plane for the system of equations

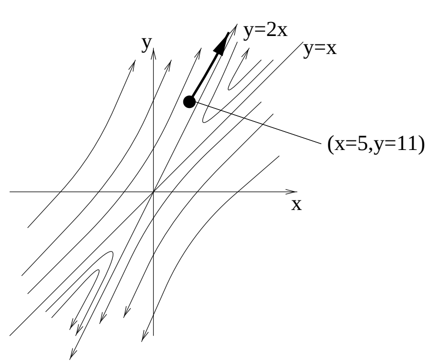
$$\frac{dx}{dt} = -8x + 5y, \qquad \frac{dy}{dt} = -10x + 7y.$$

Indicate clearly in which direction t increases, and make sure that you mark any asymptotes. From your sketch, what can you say about the short-term behavior of the solution with initial condition $x(0) = 5$, $y(0) = 11$? What can you say about its long-term behavior?

Answer: We write the equations in matrix form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8 & 5 \\ -10 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

To prepare the sketch, we find the eigenvalues and eigenvectors. The eigenvalues are given by $(-8 - \lambda)(7 - \lambda) + 50 = \lambda^2 + \lambda - 6 = 0$, with solutions $\lambda_1 = 2$ and $\lambda_2 = -3$. The corresponding eigenvectors are $v_1 = (1, 2)$ and $v_2 = (1, 1)$. Since the eigenvalues have opposite signs, we have the following situation for the phase plane:



The solution with initial condition $x(0) = 5$, $y(0) = 11$ is highlighted. In the short term, x and y increase. In the long term, they both increase towards infinity, asymptotically along the line $y = 2x$.

(a) Find the general solution to the above system of differential equations.

Answer: Since we have already found a basis of eigenvectors, we can simply write out the general solution, where c_1 and c_2 are arbitrary constants:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

(c) Find the general solution of the non-homogeneous system of equations

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8 & 5 \\ -10 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Answer: We first need to find a particular solution. We try for a constant solution: Setting $x' = y' = 0$, we get

$$\begin{pmatrix} -8 & 5 \\ -10 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}.$$

Using Gaussian elimination, we find that $(x, y) = (1, 1)$ is a solution. Thus, the general solution to the non-homogeneous system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Problem 3 (12 points) Consider the system of linear differential equations

$$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = \frac{x - ty}{1 - t}.$$

(a) Find a solution $(x(t), y(t))$ for which $y(t) = 1$ is constant.

Answer: If $y = 1$ is constant, then $y' = 0$, and thus $x - ty = 0$ from the second equation. Hence $x = ty = t$ will satisfy the second equation. We need to check that the first equation is also satisfied; but this is the case since $x' = 1$. Thus $(x, y) = (t, 1)$ is a solution.

(b) Verify that $(x, y) = (e^t, e^t)$ is another solution.

Answer: We need to check that each equation is satisfied. For the first equation, left and right-hand sides are equal to e^t. For the second equation, the left-hand side is e^t, while the right-hand side is $(e^t - te^t)/(1 - t) = e^t(1 - t)/(1 - t) = e^t$ (unless $t = 1$, in which case the equation is undefined).

(c) Find a solution which satisfies the initial conditions $x(0) = 1$ and $y(0) = 0$.

Answer: Since we are dealing with a linear system here, linear combinations of solutions are again solutions. Thus, the general solution is $(x, y) = c_1(t, 1) + c_2(e^t, e^t)$, where c_1 and c_2 are arbitrary constants. We plug in the initial condition $t = 0$, $x = 1$, $y = 0$ to determine c_1 and c_2: $(1, 0) = c_1(0, 1) + c_2(1, 1)$. This system is easily solved, and we find that $c_2 = 1$ and $c_1 = -1$. Thus, the desired solution is $(x, y) = -1 \cdot (t, 1) + 1 \cdot (e^t, e^t) = (e^t - t, e^t - 1)$.