Differential Categories I

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Introduction

- Motivating example of linear logic

$$A \Rightarrow B \;=\; !A \multimap B$$

- coKleisli category of cotriple ! (comonad)

stable domains & coherence spaces

- Differential λ-calculus of Ehrhard & Regnier

Köthe spaces

Finiteness spaces

Our aim:

Categorically "reconstruct" the E-R differential structure

Basic setting: (symmetric) monoidal category with comonad on it

Intuition: The "base category" maps are "linear"

CoKleisli maps are "smooth"

An illustration of how this works

A smooth map $f : \mathbb{R}^3 \to \mathbb{R}^2$ $\quad f\langle x,y,z\rangle = \langle x^2 + xyz,\ z^3 - xy\rangle$

Its Jacobian $\begin{pmatrix} 2x+yz & xz & xy \\ -y & -x & 3z^2 \end{pmatrix}$

For chosen $\langle x,y,z\rangle$ this is a linear map $\mathbb{R}^3 \to \mathbb{R}^2$

ie from $\dfrac{f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m}{D[f] : A \longrightarrow Lin(\mathbb{R}^n, \mathbb{R}^m)}$ we get

These are both smooth ie CoKleisli maps

So: in our setting we would have this:

$$\frac{f : !A \longrightarrow B}{D[f] : !A \longrightarrow (A \multimap B)}$$

all maps in base cat ✗

Linear Hom

To avoid the need for closed structure, we shall take

$$D[f] : A \otimes !A \longrightarrow B$$

An atlas

$$\begin{tikzcd}
\text{CoKleisli cat} & & \text{Base cat}, \times & \text{Base cat}, \otimes, \times \\
\text{Abstract Storage Diff Cat} \arrow[rr, shift left] & & \text{Storage Pre-Diff Cat} \arrow[ll, shift left] & \text{Storage Diff Cat} \arrow[l, phantom, "\supset"] \\
& & & \cap \\
\text{Strong Abstract Diff Cat} \arrow[rr, shift left] & & \text{Strong Pre-diff cat} \arrow[ll, shift left] & \text{Strong Diff Cat} \arrow[l, phantom, "\supset"] \\
& & & \cap \\
& & & \text{Bialgebra modality} + D + \nabla\text{-rule} \\
& & & \cap \\
\text{Abstract Diff Cat} \arrow[rr, shift left, dotted, "\text{(+ exactness condition)}"'] & & \text{Pre-diff cat} \arrow[ll, shift left] & \text{Diff Cat} \arrow[l, phantom, "\supset"]
\end{tikzcd}$$

storage transform iso

Diff Cat $\equiv$ coalgebra modality $+ D$

$\equiv$ storage transform $+ D$

What we don't suppose:

- we don't require the underlying cat to be $*$-autonomous - nor even monoidal closed
- we don't require (initially) biproducts
- we don't require (initially) all the properties of ! that the Linear logic ! has
  - in particular "storage"... $\{!A \otimes !B \cong !(A \times B)\}$

(But we do get some nice structures if we do ~~have~~ have biproducts & storage
- a "not-nec-closed" version of E-R's notion.)

Why? Because we want some simple examples of standard "differentiation" which don't have closed structure, nor "storage modalities"...

Outline of the talk:

- Basic notions

Comm monoid enriched

Differential Category
- coalgebra modality on a (semi)additive symmetric monoidal category
- differential combinator (we'll show this in 2 presentations)

- Examples
  - sets & relations (with "bag" functor)
  - suplattices (with dual of free algebra functor)
  - Commutative polynomials + "ordinary" derivatives (on Vec^{op})
  - S_∞ construction (generalises previous eg)

- Extending the theory
  - storage
  - $\longrightarrow$ Ehrhard & Regnier (a not-necessarily-closed version of their structure)

Basic context

- (semi) additive symmetric monoidal category $\mathbb{X}$

commutative monoid enriched
no assumption of biproducts - yet!

Eg Sets & relations is (semi) additive but not AbGrp-enriched

- coalgebra modality $!$
  - a cotriple (comonad)
  - $\top \xleftarrow{e} !X \xrightarrow{\Delta} !X \otimes !X$ natural coalgebra str.
  - $(!X, \Delta, e)$ is a comonoid

$$\begin{tikzcd} !X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & !X \otimes !X \arrow[d, "\Delta \otimes 1"] \\ !X \otimes !X \arrow[r, "1 \otimes \Delta"'] & !X \otimes !X \otimes !X \end{tikzcd} \qquad \begin{tikzcd} & !X \arrow[dl, "1"'] \arrow[d, "\Delta"] \arrow[dr, "1"] & \\ !X & !X \otimes !X \arrow[l, "1 \otimes e"] \arrow[r, "e \otimes 1"'] & !X \end{tikzcd}$$

commute

  - $\delta : !X \to !!X$ is a comonoid morphism

$$\begin{tikzcd} !X \arrow[rr, "\delta"] \arrow[dr, "e"'] & & !!X \arrow[dl, "e"] \\ & \top & \end{tikzcd} \qquad \begin{tikzcd} !X \arrow[r, "\delta"] \arrow[d, "\Delta"] & !!X \arrow[d, "\Delta"] \\ !X \otimes !X \arrow[r, "\delta \otimes \delta"'] & !!X \otimes !!X \end{tikzcd}$$

commute

[we don't assume that δ, or any of these transformations are monoidal - yet]

Intuition : $!A \to B$ is "a differentiable map $A \to B$"

(but we need more structure to realize this)

Examples

- id on any cat with finite products
- ! in linear logic
- Dual of "algebra modality"
  - The free algebra $T(X) = \bigoplus_{n=0}^{\infty} X^{\otimes n}$
  - The free symmetric algebra $Sym(X) = \bigoplus_{0}^{\infty} X^{\otimes n}/S_n$
  - The "exterior algebra" $\Lambda(X) = \bigoplus_{0}^{\infty} X^{\otimes n}/A$

so $xy = -yx$

Differential Combinators

$$D_{AB} : \mathbb{X}(!A, B) \longrightarrow \mathbb{X}(A \otimes !A, B)$$

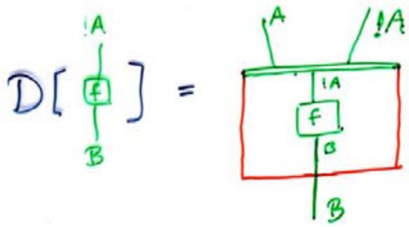
$$\frac{!A \xrightarrow{f} B}{A \otimes !A \xrightarrow[D[f]]{} B}$$

Think $!A \to (A \multimap B)$

This must satisfy :

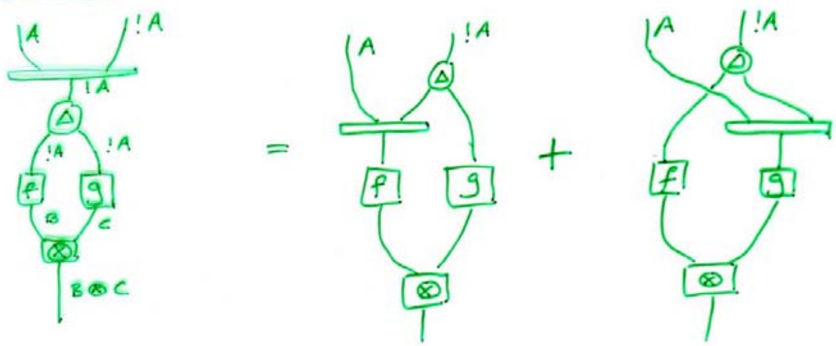
- naturality (for combinators) , additivity
- "constants have deriv = 0" $D[e] = 0$
- product rule $(1 \otimes \Delta)(D[f] \otimes g) + (1 \otimes \Delta)(c \otimes 1)(f \otimes D[g]) = D[\Delta(f \otimes g)]$
- "Linear maps have constant deriv" $D[\epsilon f] = (1 \otimes e) f$
- chain rule $D[\delta !f g] = (1 \otimes \Delta)(D[f] \otimes \delta !f) D[g]$

There is a "circuit calculus" for all this ...

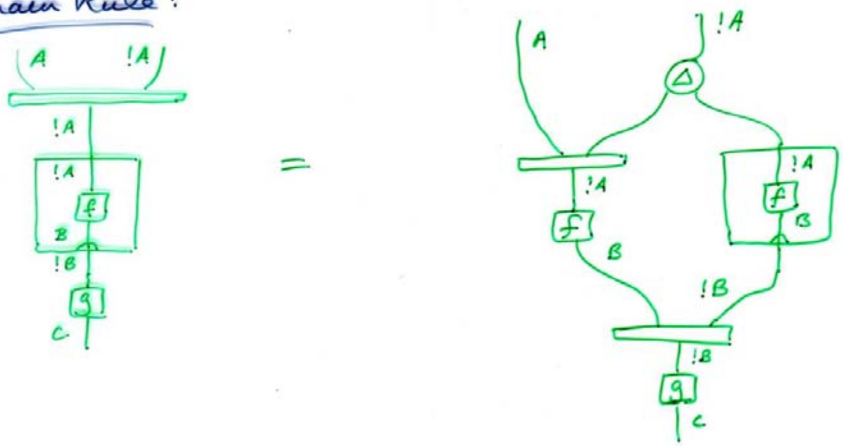
$D[\ $

scope is "changeable" - so "irrelevant"

Product rule:

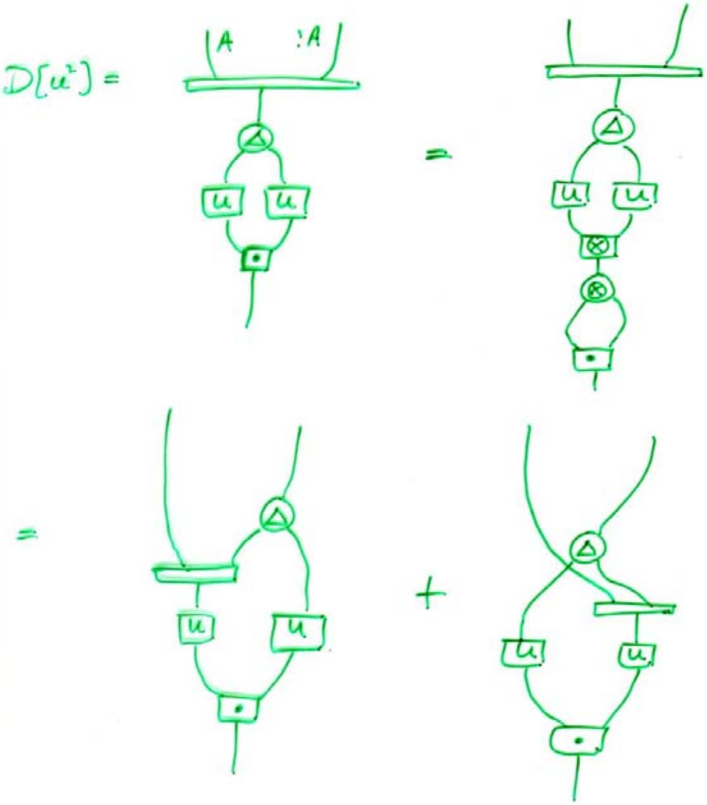


Chain Rule:



"$D[u^2] = 2u \cdot u'$" (assume $\bullet : A \otimes A \to A$) ($u^2 = u \bullet u$)

$D[u^2] =$



$= u' \bullet u + u \bullet u' = 2u \bullet u'$

[Def A differential category is a (semi) additive s m cat with a coalg. modality & a differential combinator.]

Deriving transformations

[An alternate presentation of differential combinators]

Note that $D[1_{!A}]$ is "special": $\left(d_A \overset{DEF}{=} D[1_{!A}] \right)$

$$
\begin{tikzcd}
!A \arrow[r, "1"] \arrow[d, "!A"'] & !A \arrow[d, "f"] \\
!A \arrow[r, "f"] & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A \otimes !A \arrow[r, "d_A"] \arrow[d, "1"'] & !A \arrow[d, "f"] \\
A \otimes !A \arrow[r, "{D[f]}"'] & B
\end{tikzcd}
$$

So $D[f] = d_A ; f$

"Evident" axioms for d_A

So (equiv) a diff cat is a coalg modality + "deriving trans^{fn}"

(The circuit axioms are a bit simpler with d_A)

Examples

• Sets & Relations

$!X$ = "bag functor" ... converse of the free commutative monoid monad

$d_X : X \otimes !X \longrightarrow !X$

$x_0, \{x_1 \cdots x_n\} \longmapsto \{x_0, x_1 \ldots x_n\}$

Examples

• Sup lattices (with maps preserving all joins) {This is a $*$-autonomous category}

$!X$ is deMorgan dual of free $\oplus$-algebra; equiv free commutative algebra

$$!X = \bigoplus_{n=0}^{\infty} X^{\otimes n} / S_n$$ ... This has a bialgebra structure

$d : X \otimes !X \longrightarrow !X$

"multiply by the new elt" (in sense of symm alg str.)

• Commutative polynomials & (standard) derivatives

(we'll generalize this in a moment)

Idea: to "fix" (the dual of) Mod_R ... modules over a rig R

with free non-commutative algebra monad and "usual" derivative

$$df = \sum x \otimes \frac{\partial f}{\partial x}$$

(which fails the chain rule)

$\longrightarrow$ use the free commutative algebra monad S.

Consider (dual of)

$$
\begin{tikzcd}
cPoly_K \; (= Vec_S) \arrow[rr, tail] \arrow[dr, shift left] & & cAlg_K \; (= Vec^S) \arrow[dl, shift left] \\
& Vec_K \arrow[ul, shift left] \arrow[ur, shift left] \arrow[ul, phantom, "\dashv"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}
$$

• S is a monad & an algebra modality

• $cPoly_K^{op}$ 'is' the cat of polynomial functions: [$f : W \to S(V)$ det^d by its basis, so may be seen as a collection of poly^s in the basis of V.]

Generalization: S_∞

Start with a rig R (eventually we'll really want a field)

- construct a monad on Mod_R
- if this "supports" 'partial derivatives', we get a co-deriving transformation [Then $(\ )^{op}$ everything]

First: suppose $U(R)$ is the initial alg for an alg theory $\mathbb{T}$ which includes the theory of commutative polynomials over R {$U: \mathrm{Mod}_R \to \mathrm{Set}$; R is unit of Mod_R (as smcat)}

- $\mathbb{T}[0,1]$ contains exactly elts of R
- $\mathbb{T}[2,1]$ contains (at least) $\cdot, +$
- $\mathbb{T}[n,1]$ contains (at least) $R[x_1 \cdots x_n]$ with usual interpretation of $\cdot, +$

(call $\mathbb{T}$ a "polynomial theory over R")

[Eg: $\mathbb{T}$ = "smooth theory" of ∞'ly diff'bl cont real functions (or complex): $T[n,1] = C^\infty(R^n, R)$]

Monad?

$$S_{\mathbb{T}}(V) = \left\{ h : V^* \xrightarrow{\text{set map}} R \;\middle|\; \begin{array}{l} \exists v_1 \ldots v_n \in V,\ \alpha \in T[n,1] \\ \text{st } h(u) = \alpha(u(v_1), \ldots, u(v_n)) \end{array} \right\} \ldots$$

$V^* = V \multimap R$

Regard h as "instantiation of α": the v_i determine a fin dim subspace where h "is" α.

Eg: $\mathbb{T}$ = "pure theory" $\mathbb{T}(n,1) = R[x_1, \ldots, x_n]$

Then $S_{\mathbb{T}}(V)$ is the symm. algebra monad $\mathrm{Sym}(V)$ and $\mathrm{Lin}(\mathbb{R}^m, S_{\mathbb{T}}(\mathbb{R}^n)) \approx \mathrm{Poly}(n,m)$

Monad structure:

$$\frac{f : V \to S_{\mathbb{T}}(W)}{f^\# : S_{\mathbb{T}}(V) \to S_{\mathbb{T}}(W)} \quad \left\{ \frac{h : u \mapsto \alpha(u(v_1), \ldots, \alpha(v_n))}{h' : u' \mapsto \alpha(f(v_1)(u'), \ldots, f(v_n)(u'))} \right.$$

$$\eta : V \to S_{\mathbb{T}}(V) : \quad v \mapsto [u \mapsto u(v)]$$

(This is / becomes an algebra homomorphism)

So we have a coalgebra modality on Mod_R^{op} — what of diff?

We need another assumption: that the theory $\mathbb{T}$ "admits partial derivatives"

Combinators on $T[n,1]$: $\dfrac{x_1, \ldots, x_n \vdash t}{x_1, \ldots, x_n \vdash \partial_i t}$

(with "obvious axioms") a differential theory over R

inducing $d : S_{\mathbb{T}}(V) \longrightarrow V \otimes S_{\mathbb{T}}(V)$

$d : [u \mapsto \alpha(u(v_1), \ldots, u(v_n))]$

$$\mapsto \sum_i v_i \otimes [u \mapsto \partial_i(\alpha)(u(v_1), \ldots, u(v_n))]$$

Need to verify this is well defined & satisfies appropriate axioms

- this needs another condition on R, which is automatic if R is a field. (so for now, think of R as a field!)

Then:

If $\mathbb{T}$ is a differential theory over suitable R, then Mod_R^{op} is a differential category (wrt S modality & d above)

Storage

Given a s.m.cat with products and a comonad !

a comonoidal transformation $s : ! \to !$

from $(\mathcal{X}, \times, 1)$ to $(\mathcal{X}, \otimes, \top)$ amounts to

$$s_0 : !(1) \longrightarrow \top \quad \text{and} \quad s_2 : !(X \times Y) \to !X \otimes !Y$$

st

$$\begin{tikzcd}
!((X\times Y)\times Z) \arrow[r, "s_2"] \arrow[d, "!(a_\times)"'] & !(X\times Y)\otimes !Z \arrow[r, "s_2\otimes 1"] & (!X\otimes !Y)\otimes !Z \arrow[d, "a_\otimes"] \\
!(X\times(Y\times Z)) \arrow[r, "s_2"'] & !X\otimes !(Y\times Z) \arrow[r, "1\otimes s_2"'] & !X\otimes(!Y\otimes !Z)
\end{tikzcd}$$

$$\begin{tikzcd}
!(1\times X) \arrow[r, "s_2"] \arrow[d, "!\pi_1"'] & !(1)\otimes !X \arrow[d, "s_0\otimes 1"] \\
!X & \top\otimes !X \arrow[l, "u_\otimes"]
\end{tikzcd}
\qquad
\begin{tikzcd}
!(X\times 1) \arrow[r, "s_2"] \arrow[d, "!\pi_1"'] & !X\otimes !(1) \arrow[d, "1\otimes s_0"] \\
!X & !X\otimes\top \arrow[l, "u_\otimes"]
\end{tikzcd}$$

(+ diagram for symmetry if appropriate)

In our setting, requiring that ! be comonoidal (as a comonad) is too strong - we'd want δ to be so, but not ϵ (The Id functor is not comonoidal)

$$\begin{tikzcd}
F(X\times Y) \arrow[r, "\alpha"] \arrow[d, "s^F"'] & G(X\times Y) \arrow[d, "s^G"] \\
FX\otimes FY \arrow[r, "\alpha\otimes\alpha"'] & GX\otimes GY
\end{tikzcd}
\qquad
\begin{tikzcd}
F(1) \arrow[rr, "\alpha"] \arrow[dr, "s^F"'] & & G(1) \arrow[dl, "s^G"] \\
& \top &
\end{tikzcd}$$

no such s^G for $G = Id$

A (s)m cat $\mathbb{X}$ with comonad $!$, products has a <u>storage transformation</u> if there is a comonoidal transformation

$$s : ! \to ! : (\mathbb{X}, \times, 1) \to (\mathbb{X}, \otimes, \top)$$

so that δ is comonoidal

using the canonical comonoidal trans $(\mathbb{X}, \times, 1)$ ie $!(X \times Y) \to !X \times !Y$, $!(1) \to 1$

Key Fact:

For a (symm) monoidal cat with products: to have a comonad with (symm) storage trans is equiv. to having a (cocommutative) coalgebra modality.

($\Downarrow$) Define $\Delta : !X \xrightarrow{!(\Delta_X)} !(X \times X) \xrightarrow{s_2} !X \otimes !X$

$e : !X \xrightarrow{!(\pi)} !(1) \xrightarrow{s_0} \top$

($\Uparrow$) Define $s_2 : !(X \times Y) \xrightarrow{\Delta} !(X \times Y) \otimes !(X \times Y) \xrightarrow{!\pi_0 \otimes !\pi_1} !X \otimes !Y$

$s_0 : !(1) \xrightarrow{e} \top$

This works!

In our context, we want the storage transformatn to be an <u>iso</u>, with good coherence properties

we also consider the structure when we've the iso, but not all the "good coherence"

To guarantee this we may define a storage modality $!$

- symm monoidal cat $\mathbb{X}$
- symm monoidal comonad on $\mathbb{X}$ $!$
- cofree objects are naturally comm. comonoids
- comonoid str. given by $!$-<u>coalgebra</u> morphisms

<u>Then Note</u>:

- A s.m. cat has a storage modality iff the induced tensor on coalgebras for $!$ is a cartesian product. (Schalk)

- In a storage category ($\equiv$ s.m. cat with $\times$ and storage $!$)

$$!A \otimes !B \xrightarrow{\cong} !(A \times B) \quad \text{and} \quad \top \xrightarrow{\cong} !(1)$$

(whose inverses are the canonical s_2, s_0 and indeed, the iso's shown are also canonically given)

(In this context, the adjunction between $\mathbb{X}$ and $\mathbb{X}_!$ is monoidal) (Bierman)

Eg Mod_R^{op} is/has a storage modality (viz the dual of the symm alg. monad on Mod_R)

(for any rig R)

- $\mathbb{X}$, a storage modality $!$: $\mathbb{X}_!$ = free coalgebras (in $\mathbb{X}^!$)
  - $\mathbb{X}^!$ has products given by $\otimes$
  - the storage iso guarantees the tensor of 2 free obj is $\cong$ to a free one

So: $\mathbb{X}_!$ is closed under the induced tensor of $\mathbb{X}^!$

and $\mathbb{X}_!$ inherits products from $\mathbb{X}$.

All very "linear logic"

Bialgebra modalities

(a bit weaker than storage; seems not to have the storage iso's)

- comonad $!$ so each $!A$ is naturally a bialgebra

∂ a coalgebra homomorphism (not nec. an alg. one)

$\iota\epsilon = 0$

$\nabla\epsilon = \epsilon\otimes e + e\otimes\epsilon$

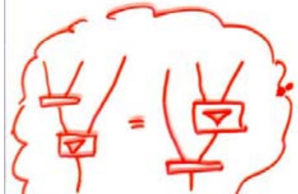
Storage modalities are bialgebra modalities

[In an additive storage cat, cofree objects are nat'ly comm. bialg's:
- transport bialg str on X to the tensor via storage iso.]

Differential Storage Cats

- (semi)additive storage cat
- deriving transformation
- (∇ rule)

$$\begin{tikzcd} A\otimes !A\otimes !A \arrow[r, "1\otimes\nabla"] \arrow[d, "d\otimes 1"'] \arrow[dr, phantom, "="] & A\otimes !A \arrow[d, "d_A"] \\ !A\otimes !A \arrow[r, "\nabla"'] & !A \end{tikzcd}$$



This is (our version of) a "not-necessarily-closed" version of Ehrhard & Regnier's structures

We can see this via an intermediate structure

Define a nat trans

$$\begin{tikzcd} A \arrow[rr, "\eta"] \arrow[dr, "1\otimes\iota"'] & \arrow[d, phantom, "\mathrm{Def}"] & !A \\ & A\otimes !A \arrow[ur, "d_A"'] & \end{tikzcd}$$

η is a primitive in E-R's sstm

η is essentially their differentiation

A categorical model of the differential calculus:

- (semi) additive cat with biproducts
- bialgebra modality : comonad $(!, \delta, \epsilon)$
  - each $!X$ has bialg str $(!X, \nabla, \iota, \Delta, e)$
  - natural $\eta_X : X \to !X$

+ 4 axioms:

$$\eta e = 0$$
$$\eta \Delta = \eta \otimes \iota + \iota \otimes \eta$$
$$\eta \epsilon = 1$$
$$(\eta \otimes 1) \nabla \delta = (\eta \otimes \Delta)((\nabla \eta) \otimes \delta) \nabla$$

Thm:

- A model of the diff calculus
  ≡ diff. cat with biproducts whose coalg modality is infact a bialg modality sat the ∇-rule
- Models of diff calculus on additive storage cats
  ≡ differential storage categories

$(\Downarrow)$ $d_X = (\eta_X \otimes 1) \nabla$

$(\Uparrow)$ $\eta_X = (1 \otimes \iota) d_X$

What's next?

- Eventually we hope to make connections with other notions of "differentiation" and "smoothness"
- More immediately: characterize those cats which are coKleisli cats of (several variants of) differential cats

Connection with Joyal "species" - link to work by Gambino & Fiore

sequel...