Arrow Categories

Michael Winter
Department of Computer Science
Brock University
St. Catharines, Canada
mwinter@brocku.ca

Content

1. Binary (Boolean valued), fuzzy and *L*-fuzzy relations
2. Dedekind categories (Boolean valued relations)
3. Goguen categories (Fuzzy/*L*-fuzzy relations)
4. Arrow categories

Binary (Boolean valued) relation (Category Rel)

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Fuzzy relation (Category Rel([0,1]))

$$\begin{pmatrix} 0.1 & 0.8 & 0.0 \\ 1.0 & 0.4 & 0.9 \\ 0.0 & 0.2 & 0.1 \end{pmatrix}$$

L-fuzzy relation (L a complete distributive lattice, Category Rel(L))

$$\begin{pmatrix} 1 & k & l \\ 0 & k & m \\ 0 & 1 & l \end{pmatrix}$$

$$L = \begin{tikzcd}[arrows=dash] & 1 & \\ l \arrow[ur] & & m \arrow[ul] \\ & k \arrow[ul] \arrow[ur] & \\ & 0 \arrow[u] & \end{tikzcd}$$

Dedekind categories

Definition: A Dedekind category $\mathcal{R}$ is a category satisfying the following:

1. For all objects A and B the collection $\mathcal{R}[A,B]$ is a complete distributive lattice (complete Heyting algebra). Meet, join, the induced ordering, the least and the greatest element are denoted by $\sqcap, \sqcup, \sqsubseteq, \perp\!\!\!\perp_{AB}, \top\!\!\!\top_{AB}$, respectively.

2. There is a monotone operation $^{\smile}$ (called converse) such that for all relations $Q : A \longrightarrow B$ and $R : B \longrightarrow C$ the following holds

$$(Q;R)^{\smile} = R^{\smile};Q^{\smile}, \quad (Q^{\smile})^{\smile} = Q.$$

3. For all relations $Q : A \longrightarrow B, R : B \longrightarrow C$ and $S : A \longrightarrow C$ the

modular law holds:

$$Q;R \sqcap S \sqsubseteq Q;(R \sqcap Q^{\smile};S).$$

4. For all relations $R : B \rightarrow C$ and $S : A \rightarrow C$ there is a relation $S/R : A \rightarrow B$ (called the left residual of S and R) such that for all $Q : A \rightarrow B$ the following holds

$$Q;R \sqsubseteq S \iff Q \sqsubseteq S/R.$$

Definition (Matrix category)

Let $\mathcal{R}$ be a Dedekind category. The category $\mathcal{R}^+$ of matrices with coefficients from $\mathcal{R}$ is defined by:

1. The class of objects of $\mathcal{R}^+$ is the collection of all functions from an arbitrary set I into the class of objects $\mathrm{Obj}_{\mathcal{R}}$ of $\mathcal{R}$.

2. For every pair $f : I \longrightarrow \mathrm{Obj}_{\mathcal{R}}, g : J \longrightarrow \mathrm{Obj}_{\mathcal{R}}$ of objects from $\mathcal{R}^+$, a morphism $R : f \longrightarrow g$ is a function from $I \times J$ into the class of all morphisms $\mathrm{Mor}_{\mathcal{R}}$ of $\mathcal{R}$ such that $R(i, j) : f(i) \longrightarrow g(j)$ holds.

3. For $R : f \longrightarrow g$ and $S : g \longrightarrow h$ composition is defined by

$$(R;S)(i,k) := \bigsqcup_{j \in J} R(i, j);S(j,k).$$

4. For $R : f \to g$ conversion defined by

$$R^{\smile}(j,i) := (R(i,j))^{\smile}.$$

5. For $R,S : f \to g$ join and meet are defined by

$$\begin{aligned}(R \sqcup S)(i,j) &:= R(i,j) \sqcup S(i,j),\\ (R \sqcap S)(i,j) &:= R(i,j) \sqcap S(i,j).\end{aligned}$$

6. The identity, zero and universal elements are defined by

$$\begin{aligned}\mathbb{I}_f(i_1,i_2) &:= \begin{cases} \perp\!\!\!\perp_{f(i_1)f(i_2)} & : i_1 \neq i_2 \\ \mathbb{I}_{f(i_1)} & : i_1 = i_2, \end{cases}\\ \perp\!\!\!\perp_{fg}(i,j) &:= \perp\!\!\!\perp_{f(i)g(j)},\\ \top\!\!\!\top_{fg}(i,j) &:= \top\!\!\!\top_{f(i)g(j)}.\end{aligned}$$

Some results

Lemma: $\mathscr{R}^+$ is a Dedekind category.

Corollary: Let $L = (L, \vee, \wedge, 0, 1)$ be a complete distributive lattice with least element 0 and greatest element 1. Then L is an one-object Dedekind category with identity 1 and composition $\wedge$ (the residual is given by the pseudo-complement). Consequently, L^+ is a Dedekind category, called the full category of L-relations.

Lemma: The collection of scalar relations on A, i.e., the relations $k : A \to A$ with $k \sqsubseteq \mathbb{I}_A$ and $\mathbb{T}_{AA};k = k;\mathbb{T}_{AA}$, constitutes a complete distributive lattice.

Example:

$$\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

Theorem: There is no formula φ in the language of Dedekind categories such that for all lattices L and L-relations $R : A \to B$ we have

$$L^+ \models \varphi[R] \iff R \text{ is 0-1 crisp.}$$

Goguen categories

Definition: A Goguen category $\mathcal{G}$ is a Dedekind category with $\mathbb{\perp}_{AB} \neq \mathbb{\top}_{AB}$ for all objects A and B together with two operations $^{\uparrow}$ and $^{\downarrow}$ satisfying the following:

1. $R^{\uparrow}, R^{\downarrow} : A \to B$ for all $R : A \to B$.
2. $(^{\uparrow}, ^{\downarrow})$ is a Galois correspondence, i.e., $R^{\uparrow} \sqsubseteq S \iff R \sqsubseteq S^{\downarrow}$ for all $R, S : A \to B$.
3. $(R^{\smile}; S^{\downarrow})^{\uparrow} = R^{\uparrow\smile}; S^{\downarrow}$ for all $R : B \to A$ and $S : B \to C$.
4. If $\alpha \neq \mathbb{\perp}_{AA}$ is a nonzero scalar then $\alpha^{\uparrow} = \mathbb{I}_A$.

$$\begin{pmatrix} 1 & k & l \\ 0 & k & m \\ 0 & 1 & l \end{pmatrix}$$

$$L = \begin{tikzcd}[column sep=small] & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ l \arrow[dr, dash] & & m \arrow[dl, dash] \\ & k \arrow[d, dash] & \\ & 0 & \end{tikzcd}$$

$$\begin{pmatrix} 1 & k & l \\ 0 & k & m \\ 0 & 1 & l \end{pmatrix}^{\uparrow} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & k & l \\ 0 & k & m \\ 0 & 1 & l \end{pmatrix}^{\downarrow} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

4. For all functions f so that $f(\bigsqcup M) = \bigsqcap_{\alpha \in M} f(\alpha)$ for all sets of scalars and $f(\alpha)^{\uparrow} = f(\alpha)$ for all scalars the following equivalence holds

$$R \sqsubseteq \bigsqcup_{\substack{\alpha : A \to A \\ \alpha \text{ scalar}}} \alpha ; f(\alpha) \quad \Longleftrightarrow \quad (\alpha \backslash R)^{\downarrow} \sqsubseteq f(\alpha) \text{ for all scalars } \alpha.$$

$$\left(\begin{pmatrix} l & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & l \end{pmatrix} \backslash \begin{pmatrix} 1 & k & l \\ 0 & k & m \\ 0 & 1 & l \end{pmatrix} \right)^{\downarrow} = \begin{pmatrix} 1 & k & 1 \\ 0 & k & m \\ 0 & 1 & 1 \end{pmatrix}^{\downarrow} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Some results

Theorem: Let L be a complete distributive lattice. Then L^+ together with the operations

$$R^{\uparrow}(x,y) := \begin{cases} 1 \text{ iff } R(x,y) \neq 0 \\ 0 \text{ iff } R(x,y) = 0 \end{cases},$$

$$R^{\downarrow}(x,y) := \begin{cases} 1 \text{ iff } R(x,y) = 1 \\ 0 \text{ iff } R(x,y) \neq 1 \end{cases},$$

is a Goguen category. Furthermore, for a relation R in L^+ we have $R^{\uparrow} = R$ iff R 0-1 crisp.

Lemma: For each pair of objects A and B the set of scalar elements on A resp. on B are isomorphic lattices.

Lemma: Let $\mathcal{G}$ be a Goguen category and $R : A \to B$ be a relation. Then we have

1. $$\bigsqcup_{\alpha \text{ scalar}} \alpha_A;(\alpha_A \backslash R)^{\downarrow} = R,$$
2. $$\bigsqcup_{\substack{\alpha_A \text{ scalar} \\ \alpha_A \neq \perp\!\!\perp_{AA}}} (\alpha_A \backslash R)^{\downarrow} = R^{\uparrow}.$$

Theorem (Pseudo-representation Theorem): Every Goguen category $\mathcal{G}$ is isomorphic to the category of antimorphisms mapping the scalars of $\mathcal{G}$ to the crisp relations of $\mathcal{G}$.

Corollary: A Goguen category is representable iff its subcategory of crisp relations is representable.

Further results/studies of Goguen categories

1. Definability of norm-based operations;
2. Validity of certain formulae in the subcategory of crisp relations;
3. Applications in computer science, e.g., fuzzy controller;
4. ...

Arrow categories

Definition: An arrow category $\mathcal{A}$ is a Dedekind category with $\mathbb{T}_{AB} \neq \perp\!\!\!\perp_{AB}$ for all objects A and B together with two operations $^{\uparrow}$ and $^{\downarrow}$ satisfying the following:

1. $R^{\uparrow}, R^{\downarrow} : A \longrightarrow B$ for all $R : A \longrightarrow B$.
2. $(^{\uparrow}, ^{\downarrow})$ is a Galois correspondence.
3. $(R^{\smile}; S^{\downarrow})^{\uparrow} = R^{\uparrow\smile}; S^{\downarrow}$ for all $R : B \longrightarrow A$ and $S : B \longrightarrow C$.
4. $(Q \sqcap R^{\downarrow})^{\uparrow} = Q^{\uparrow} \sqcap R^{\downarrow}$ for all $Q, R : A \longrightarrow B$.
5. If $\alpha_A \neq \perp\!\!\!\perp_{AA}$ is a non-zero scalar then $\alpha_A^{\uparrow} = \mathbb{I}_A$.

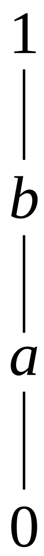
Lemma: For each pair of objects A and B the set of scalar elements on A resp. on B are isomorphic lattices.

Lemma: Let $\mathcal{A}$ be an arrow category and $R : A \to B$ be a relation. Then we have

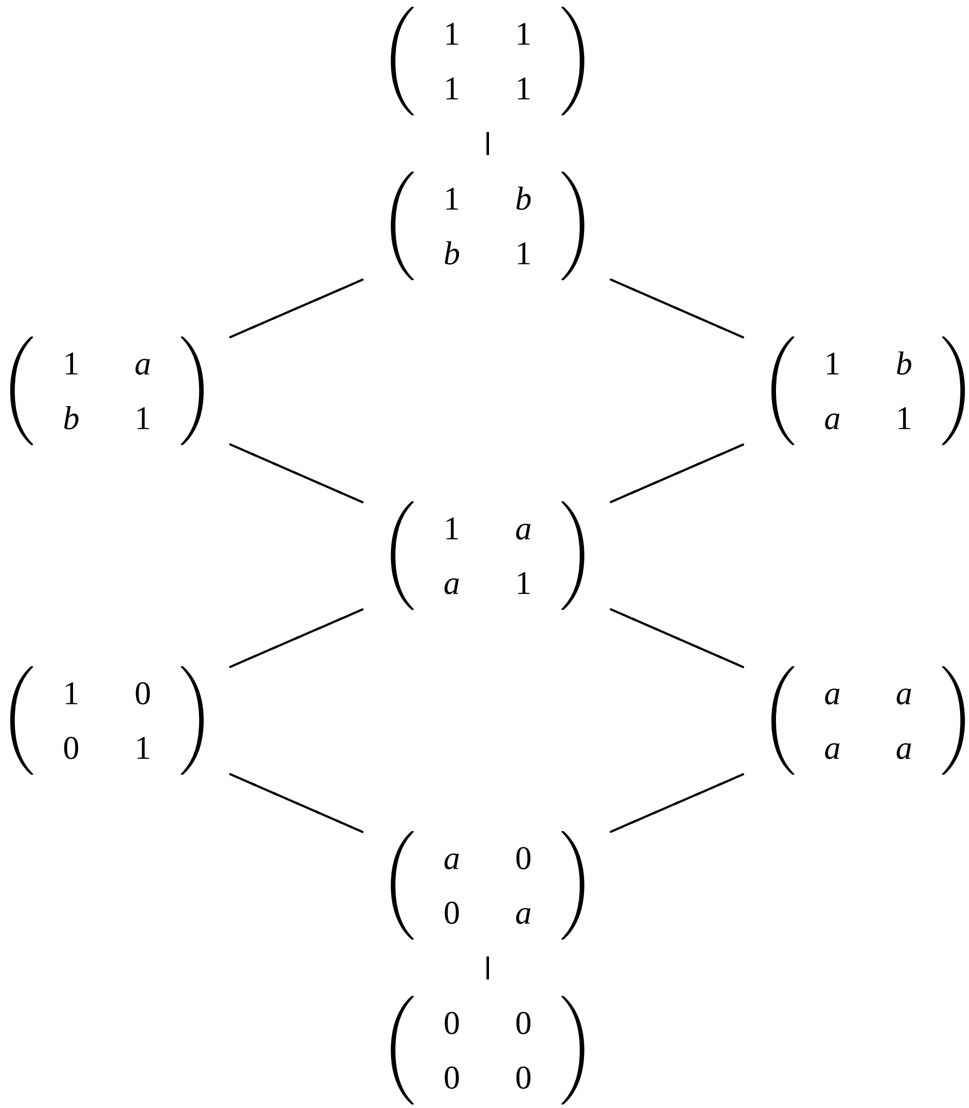
1. $\bigsqcup_{\alpha \text{ scalar}} \alpha_A;(\alpha_A \backslash R)^{\downarrow} \sqsubseteq R$,
2. $\bigsqcup_{\substack{\alpha_A \text{ scalar} \\ \alpha_A \neq \perp\!\!\!\perp_{AA}}} (\alpha_A \backslash R)^{\downarrow} \sqsubseteq R^{\uparrow}$.

Example 1:

L



1
b
a
0



$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$
$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Example 2:

$$
\begin{tikzcd}[arrows=-]
 & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \arrow[d] & & \\
 & \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix} \arrow[dl] \arrow[dr] & & \\
\begin{pmatrix} b & b \\ b & b \end{pmatrix} \arrow[dr] & & \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \arrow[dl] \arrow[dr] & \\
 & \begin{pmatrix} b & a \\ a & b \end{pmatrix} \arrow[dr] & & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \arrow[dl] \\
 & & \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \arrow[d] & \\
 & & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &
\end{tikzcd}
$$

Arrow categories with cuts

Definition: An arrow category with cuts $\mathcal{A}$ is an arrow category so that

$$R \sqsubseteq \bigsqcup_{\alpha \text{ scalar}} \alpha_A; (\alpha_A \backslash R)^{\downarrow}$$

for all relations $R : A \longrightarrow B$ holds.

Example

$$
L \qquad
\begin{tikzcd}
x_0 \arrow[d, dash] \\
x_1 \arrow[d, dash] \\
x_2 \arrow[d, dash, dashed] \\
x_\infty \arrow[d, dash] \\
0
\end{tikzcd}
$$

$$
\mathcal{R} \qquad
\begin{tikzcd}
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \mathbb{T} \arrow[d, dash] \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I} \arrow[d, dash] \\
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{\perp}
\end{tikzcd}
$$

Thank you for your attention.