Towards a notion of
Cartesian differential storage category

R.A.G. Seely

McGill University & John Abbott College

Joint with Rick Blute & Robin Cockett

http://www.math.mcgill.ca/rags/

Dedication

Durham 1991

Halifax 1995

To our birthday boy, and my long-time collaborator, Robin Cockett as our collaboration enters its maturity (21 years), and he enters his dotage (60 years) . . .

Best wishes for many more productive and enjoyable years!

Preludium

2006 Differential categories—an additive monoidal category of "linear" maps, a (suitable) comonad whose coKleisli maps are "smooth", and a differential combinator. (This gave a "categorical reconstruction" of Ehrhard & Regnier's work)

2007 Talks by JRBC and RAGS on storage, *etc* (*Eg* my FMCS talk at Colgate)

2009 Cartesian differential categories—a left additive Cartesian category with a differential operator, and subcategories of "linear" maps. CDCs are the coalgebras of a "higher order chain rule fibration" comonad Faà [2011].

Preludium

True: The coKliesli category of a (suitable) differential (storage) category is a Cartesian differential category[2009].

Wished: The linear maps of a Cartesian differential category form a differential category

Wished: Any Cartesian differential category may be (ff) embedded into the coKliesli category of a (suitable) differential category.

Wished: Any differential category may be represented as the linear maps of a (suitable) Cartesian differential category.

Preludium

With two notions of differential categories (and their ancillary notions) it'll be convenient (today at least) to put an adjective in front of the tensor notion ("$\otimes$-differential category"), to match that in front of the Cartesian notion. SO:

True: The coKliesli category of a (suitable) $\otimes$-differential (storage) category is a Cartesian differential category[2009].

Wished: The linear maps of a Cartesian differential category form a $\otimes$-differential category

Wished: Any Cartesian differential category may be (ff) embedded into the coKliesli category of a (suitable) $\otimes$-differential category.

Wished: Any $\otimes$-differential category may be represented as the linear maps of a (suitable) Cartesian differential category.

Summary

Cartesian Storage Categories given three equivalent ways:

- in terms of a system of $\mathcal{L}$-linear maps
- abstract coKleisli category
- coKleisli category of a forceful comonad

We can define a Cartesian linear category to be

- the linear maps of a Cartesian storage category
- equivalently, a Cartesian category with an exact forceful comonad

Summary

We define a notion of $\otimes$-representability, similar to the characterization of linear maps in terms of bilinear maps. Then TFAE:

- Cartesian storage category with $\otimes$-representability
- the coKleisli category of a $\otimes$-storage category (AKA a "Seely category")

In this context we define a $\otimes$-linear category as the linear maps of a Cartesian storage category with $\otimes$-representability, which is equivalent to being an exact $\otimes$-storage category.

Summary

Add a **deriving transform** to a CSC to create a **Cartesian Differential Storage category**, defined by:

For a Cartesian storage category $\mathbb{X}$, TFAE:

- $\mathbb{X}$ is a CDC and Diff-linear $= \mathcal{L}$-linear
- $\mathbb{X}$ is a CDC and $\langle 1, 0 \rangle D_\times[\varphi]$ is $\mathcal{L}$-linear
- $\mathbb{X}$ has a deriving transformation

If linear idempotents split, this is also equivalent to being the coKleisli category of a $\otimes$-differential storage category

More precisely: if linear idempotents split, then a Cartesian differential storage category automatically has $\otimes$-representability.

Systems of $\mathcal{L}$-linear maps

Given a Cartesian (*i.e.* having finite products) category $\mathbb{X}$, denote the simple slice fibration by $\mathbb{X}[\ ]$ (so $\mathbb{X}[A]$ has the same objects as $\mathbb{X}$, and morphisms $X \longrightarrow Y$ are $\mathbb{X}$-morphisms $X \times A \longrightarrow Y$).

$\mathbb{X}$ has a system of $\mathcal{L}$-linear maps (or "a system of linear maps", $\mathcal{L}$ being understood) if in each simple slice $\mathbb{X}[A]$ there is a class of maps $\mathcal{L}[\mathcal{A}] \subseteq \mathbb{X}[\mathcal{A}]$, (the **$\mathcal{L}[\mathcal{A}]$-linear** maps), satisfying:

[LS.1] Identity maps and projections are in $\mathcal{L}[A]$, and $\mathcal{L}[\mathcal{A}]$ is closed under ordered pairs;

[LS.2] $\mathcal{L}[\mathcal{A}]$ is closed under composition and whenever $g \in \mathcal{L}[A]$ is a retraction and $gh \in \mathcal{L}[A]$ then $h \in \mathcal{L}[A]$;

[LS.3] all substitution functors $\mathbb{X}[B] \xrightarrow{\mathbb{X}[f]} \mathbb{X}[A]$ (given by $A \xrightarrow{f} B$) preserve linear maps.

Note that $\mathcal{L}[\ ]$ is a Cartesian subfibration of $\mathbb{X}[\ ]$.

Classified[1] $\mathcal{L}$-linear systems

A system of $\mathcal{L}$-linear maps is strongly classified if there is an object function S and maps $X \xrightarrow{\varphi} S(X)$ such that for every $f\colon A \times X \to Y$ there is a unique $f^\sharp\colon A \times S(X) \to Y$ in $\mathcal{L}[\mathcal{A}]$ (*i.e.* $f^\sharp$ is linear in its second argument) making

$$\begin{tikzcd}
A \times X \arrow[rr, "f"] \arrow[d, "1\times\varphi"'] & & Y \\
A \times S(X) \arrow[urr, dotted, "f^\sharp"'] & &
\end{tikzcd}$$

commute. The classification is said to be persistent in case whenever $f\colon A \times B \times X \to Y$ is linear in its second argument B then $f^\sharp\colon A \times B \times S(X) \to Y$ is also linear in its second argument.

[1] Co-classified?

Cartesian storage categories

Definition: A Cartesian storage category is a Cartesian category $\mathbb{X}$ with a persistently classified system of $\mathcal{L}$-linear maps.

Consequences Define
$\epsilon = 1_A^\sharp \colon S(A) \longrightarrow A$ as the linear lifting of the identity on A,
$\theta = \varphi^\sharp \colon A \times S(X) \longrightarrow S(A \times X)$ as the linear lifting of φ,
$\delta = (\varphi\varphi)^\sharp \colon S(A) \longrightarrow S^2(A)$ as the linear lifting of $\varphi\varphi$,
and $\mu = \epsilon_S \colon S^2(A) \longrightarrow S(A)$. Then:

1. S is a strong functor (with strength given by θ).
2. (S, φ, μ) is a commutative monad.
3. (S, ϵ, δ) is a comonad on the category of linear maps.

Strong abstract coKleisli categories

Definition: A strong abstract coKleisli category is a Cartesian category $\mathbb{X}$ equipped with a strong functor S, a strong natural transformation $\varphi\colon X \to S(X)$, and an unnatural transformation $\epsilon\colon S(X) \to X$, satisfying

1. $\epsilon_S\colon S^2(X) \to S(X)$ is a strong natural transformation
2. $\varphi\epsilon = 1$; $S(\varphi)\epsilon = 1$, $\epsilon\epsilon = S(\epsilon)\epsilon$
3. projections are ϵ-natural

In such a category, the ϵ-natural maps form a system of linear maps, classified by (S, φ). In this case, persistence = "S is a commutative monad". But:

Fact: In a strong abstract coKleisli category, the monad S is commutative, and so the classification of linear maps is persistent.

So: If $\mathbb{X}$ is Cartesian, then it is a Cartesian storage category iff it is a strong abstract coKleisli category.

Forceful comonads

We look more carefully at the comonad S on the linear maps: the strength θ is not a strength on the category of linear maps, so S as a comonad is not necessarily strong. We remedy this by assuming the existence of a force on the comonad S, *viz* a natural transformation $\psi\colon S(A \times S(X)) \longrightarrow S(A \times X)$ which generates a strength in the coKleisli category making S a strong monad. (There are 6 axioms on ψ that do this.) A comonad with a force is called forceful.

Proposition: In any Cartesian storage category, the comonad S on the linear maps has a force given by $\psi = S(\theta)\epsilon$ (the canonical coKleisli image of θ).

Proposition: Given a Cartesian category with a forceful comonad, its coKleisli category is a Cartesian storage category.

Theorem: Cartesian Storage Categories

A category is a Cartesian storage category
iff it is the coKleisli category of a forceful comonad
iff it is a strong abstract coKleisli category.

Moreover: a category is the linear maps of a Cartesian storage category iff it is a Cartesian category with an exact forceful comonad (it's tempting to call such categories "linear" ...)

where "exact" means that

$$S(S(X)) \underset{\epsilon}{\overset{S(\epsilon)}{\rightrightarrows}} S(X) \xrightarrow{\epsilon} X$$

is a coequalizer. (A category with an exact comonad is always the subcategory of ϵ-natural maps of its coKleisli category.)

Representing tensors

A Cartesian storage category is ⊗-representable[2] if, in each slice $\mathbb{X}[A]$, for each X and Y there is an object $X \otimes Y$ and a bilinear map $\varphi_\otimes : X \times Y \longrightarrow X \otimes Y$ such that for every bilinear map $g : X \times Y \longrightarrow Z$ in $\mathbb{X}[A]$ there is a unique linear map (in $\mathbb{X}[A]$) making the following diagram commute:

$$\begin{tikzcd} X \times Y \arrow[r, "g"] \arrow[d, "\varphi_\otimes"'] & Z \\ X \otimes Y \arrow[ur, dotted, "g^\otimes"'] & \end{tikzcd}$$

Note that this means in $\mathbb{X}$ we have

$$\begin{tikzcd} A \times X \times Y \arrow[r, "g"] \arrow[d, "1\times\varphi_\otimes"'] & Z \\ A \times (X \otimes Y) \arrow[ur, dotted, "g^\otimes"'] & \end{tikzcd}$$

[2]We are sorely tempted to call these Bilinear Cartesian Storage Categories!

Persistence

There is a corresponding notion of unit representable, which we shall always assume when assuming $\otimes$-representability; also we say $\otimes$-representability is persistent if linearity in other parameters (in A) is preserved.

It turns out that persistence is automatic in coKleisli categories, and so in Cartesian storage categories.

Proposition: If $\mathbb{X}$ has a system of linear maps with persistent $\otimes$-representation, then $\otimes$ is a symmetric tensor product with unit on the subcategory of linear maps. Furthermore S is a monoidal functor.

(The proof uses the universal lifting property given by $\otimes$-representability, in "evident ways".)

$\otimes$-Storage categories

A $\otimes$-storage category (AKA "Seely" category) is a symmetric monoidal category with:

- products and
- a comonad (S, ϵ, δ)
- which has a storage natural isomorphism $s\colon S \longrightarrow S$, *i.e.* a comonoidal transformation from $\mathbb{X}$ as a smc wrt $\times$ to $\mathbb{X}$ as a smc wrt $\otimes$.

This gives natural isos

$$s_1\colon S(1) \longrightarrow \top \text{ and } s_2\colon S(X \times Y) \longrightarrow S(X) \otimes S(Y)$$

satisfying "obvious" coherence conditions.

(These categories are precisely what is needed to model MELL without $\multimap$ but with products.)

$\otimes$-linear categories

Proposition: The linear maps of a Cartesian storage category with $\otimes$-representability form a $\otimes$-storage category; conversely, the coKleisli category of a $\otimes$-storage category is a Cartesian storage category with $\otimes$-representability.

The second claim follows from showing that the comonad is canonically forceful; the force is given by

$$S(A \times S(X)) \xrightarrow{s_2} S(A) \otimes S^2(X) \xrightarrow{1 \otimes \epsilon} S(A) \otimes S(X) \xrightarrow{s_2^{-1}} S(A \times X)$$

We will say $\mathbb{X}$ is a $\otimes$-linear category if it is an exact $\otimes$-storage category, or equivalently, the linear maps of a $\otimes$-representable Cartesian storage category.

$\otimes$-representable Cartesian storage categories

To repeat:

Proposition: A $\otimes$-representable Cartesian storage category is precisely the coKleisli category of a $\otimes$-storage category.

Proposition: Any $\otimes$-linear category may be represented as the subcategory of linear maps of a $\otimes$-representable Cartesian storage category.

It now remains to add a differential operator to this setup . . .

Cartesian differential storage categories

Proposition: Suppose $\mathbb{X}$ is a Cartesian storage category. TFAE:

1. $\mathbb{X}$ is a Cartesian differential category for which Diff-linear $=$ $\mathcal{L}$-linear
2. $\mathbb{X}$ is a Cartesian differential category for which $\eta = \langle 1, 0 \rangle D_\times[\varphi]$ is $\mathcal{L}$-linear
3. $\mathbb{X}$ has a deriving transformation

where in a CDC, f being Diff-linear means $D_\times[f] = \pi_0 f$, and where a deriving transformation is an (unnatural) transformation $d_\times : A \times A \longrightarrow S(A)$ satisfying a number (10!) of axioms.

A Cartesian differential storage category is a CSC satisfying any of the above equivalent conditions.

Deriving transformation axioms

[cd.1] $d_\times S(0)\epsilon = 0$ and $d_\times S(f+g)\epsilon = d_\times(S(f)+S(g))\epsilon$

[cd.2] $\langle h+k, f\rangle d_\times = \langle h, f\rangle d_\times + \langle k, f\rangle d_\times$ and $\langle 0, f\rangle d_\times = 0$

[cd.3] $d_\times \epsilon = \pi_0$

[cd.4] $d_\times S(\langle f, g\rangle)\epsilon = d_\times \langle S(f)\epsilon, S(g)\epsilon\rangle$

[cd.5] $d_\times S(fg)\epsilon = \langle d_\times S(f)\epsilon, \pi_1 f\rangle d_\times S(g)\epsilon$

[cd.6] $\langle\langle g, 0\rangle, \langle h, k\rangle\rangle d_\times S(d_\times S(f)\epsilon)\epsilon = \langle g, k\rangle d_\times S(f)\epsilon$

[cd.7] $\langle\langle 0, h\rangle, \langle g, k\rangle\rangle d_\times S(d_\times S(f)\epsilon)\epsilon =$
$\langle\langle 0, g\rangle, \langle h, k\rangle\rangle d_\times S(d_\times S(f)\epsilon)\epsilon$

[cd.8] $\langle 1, 0\rangle d_\times$ is ϵ-natural

[cd.9] $(\epsilon \otimes 1)s_2^{-1}S(d_\times)\epsilon = (S(\epsilon)\otimes 1)s_2^{-1}S(d_\times)\epsilon$

[cd.10] $(\eta \otimes 1)\nabla = (\eta \otimes 1)s_2^{-1}S(d_\times)\epsilon$

where ∇ is the codiagonal in the canonical bialgebra structure on a $\otimes$-storage category (see our [2006] paper for details).

Split linear idempotents

Any Cartesian storage category in which linear idempotents split in every slice and which has "codereliction" $\eta \colon A \longrightarrow S(A)$ (a natural transformation which is linear and splits ϵ, so $\eta_A \epsilon_A = 1_A$) automatically is $\otimes$-representable.

Hence if linear idempotents split, being a Cartesian differential storage category is equivalent to being the coKleisli category of a $\otimes$-differential $\otimes$-storage category.

And the linear maps of such a Cartesian differential storage category form a $\otimes$-differential $\otimes$-storage category.

Conclusion

So, in the “abstract” world we have the correspondance we hoped for. What remains is the embedding theorems which put “concrete” differential categories (Cartesian and $\otimes$) into that abstract world.

To be continued ...