A finite alternation result for reversible boolean circuits

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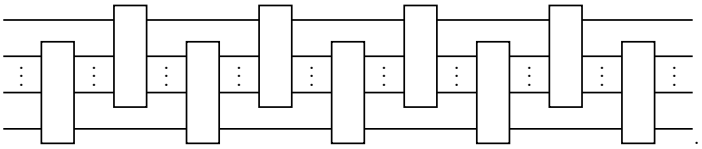
Abstract

We say that a reversible boolean function on n bits has *alternation depth* d if it can be written as the sequential composition of d reversible boolean functions, each of which acts only on the top $n-1$ bits or on the bottom $n-1$ bits. We show that every reversible boolean function of $n \geqslant 4$ bits has alternation depth 9.

1 Introduction

A reversible boolean function on n bits is a permutation of $\{0,1\}^n$. It is well-known that the NOT, controlled NOT, and Toffoli gates form a universal gate set for reversible boolean functions [4, 2, 1]. More precisely, these gates generate (via the operations of composition and cartesian product, and together with the identity functions) all reversible boolean functions on n bits, when $n \leqslant 3$, and all even reversible boolean functions on n bits, when $n \geqslant 4$. A particular representation of a reversible boolean function as a composition of cartesian products of generators and identity functions is called a *reversible circuit*. The problem of finding a (preferably short) circuit to implement a given reversible function is called the *synthesis problem* [3].

When working with reversible boolean functions and circuits, it is not typically possible to reason inductively; we cannot usually reduce a problem about circuits on n bits to a problem about circuits on $n-1$ bits. In this paper, we prove a theorem that may, in some cases, make such inductive reasoning possible: we prove that when $n \geqslant 4$, every even reversible function on n bits can be decomposed into at most 9 reversible functions on $n-1$ bits:



(1)

It is of course not remarkable that n-bit circuits can be decomposed into $(n-1)$-bit circuits: after all, we already know that they can be decomposed into 3-bit

circuits, namely gates. What is perhaps remarkable is that the bound 9 is independent of n.

There are some potential applications of such a result — although admittedly, they may not be very practical. As a first application, one may obtain an alternative proof of universality, by turning any universal gate set on n bits into a universal gate set on $n+1$ bits, provided that $n \geqslant 3$. This also yields a new method for circuit synthesis: given a good procedure for synthesizing n-bit circuits, we obtain a procedure for synthesizing $(n+1)$-bit circuits that is at most 9 times worse. By applying this idea recursively, we obtain circuits of size $O(9^n)$ for any reversible function on n bits. This is worse than what can be obtained by other methods. However, it may be possible to improve this procedure further, for example by noting that the 9 subcircuits need not be completely general; they can be chosen to be of particular forms, which may be easier to synthesize recursively.

Another potential application is the presentation of (even) reversible boolean functions by generators and relations. While the NOT, CNOT, and Toffoli gates are a well-known set of generators, to the author's knowledge, no complete set of relations for these generators is known. For any given n, the group of n-bit reversible functions is a finite group, so finding a complete set of relations for any fixed n is a finite (although very large) problem. However, it is not trivial to find a set of relations that works for all n; at present, it is not even known whether the theory is finitely axiomatizable. If we had a procedure for rewriting every circuit into one of the form (1), then we could obtain a complete set of relations for n-bit circuits by considering (a) a complete set of relations for $(n-1)$-bit circuits, (b) the relations required to do the rewriting, and (c) any relations required to prove equalities between circuits of the form (1). In particular, if it could be shown that a finite set of relations are sufficient for (b) and (c), a finite equational presentation of reversible boolean functions could be derived.

2 Statement of the main result

We write $S(X)$ for the group of permutations of a finite set X. For $f \in S(X)$ and $g \in S(Y)$, let $f \times g \in S(X \times Y)$ be the permutation defined componentwise by $(f \times g)(x, y) = (f(x), g(y))$. We also write $\mathrm{id}_X \in S(X)$ for the identity permutation on X. Recall that a permutation is *even* if it can be written as a product of an even number of 2-cycles.

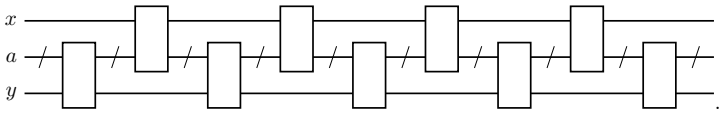
Let $2 = \{0, 1\}$ be the set of booleans, which we identify with the binary digits 0 and 1. By abuse of notation, we also write $2 = \mathrm{id}_2$ for the identity permutation on the set 2.

Definition. Let A be a finite set, and let $\sigma \in S(2 \times A \times 2)$ be a permutation. We say that σ has *alternation depth* d if it can be written as a product of d factors $\sigma = \sigma_1 \sigma_2 \cdots \sigma_d$, where each factor σ_i is either of the form $f \times 2$ for some $f \in S(2 \times A)$ or of the form $2 \times g$ for some $g \in S(A \times 2)$.

The purpose of this paper is to prove the following theorem:

Theorem 2.1. *Let A be a finite set of 3 or more elements. Then every even permutation $\sigma \in S(2 \times A \times 2)$ has alternation depth 9.*

In circuit notation, Theorem 2.1 can be understood as stating that every reversible boolean function on the set $2 \times A \times 2$ can be expressed as a circuit in the following form:



(2)

Here, the lines labelled x and y each represent a bit, and the line labelled a represents an element of the set A. The case of boolean circuits arises as the special case where the cardinality of A is a power of 2.

Remark 2.2. The evenness of σ is a necessary condition for Theorem 2.1, because all permutations of the forms $f \times 2$ and $2 \times g$ are even, and therefore only even permutations can have an alternation depth.

Our proof of Theorem 2.1 is in two parts. In Section 3, we will show that every even permutation of a certain form $g + h$ has alternation depth 5. In Section 4, we will show that every even permutation can be decomposed into a permutation of alternation depth 4 and a permutation of the form $g + h$. Together, these results imply Theorem 2.1.

3 First construction: balanced permutations

3.1 Preliminaries

We fix some terminology. The *support* of a permutation $\sigma \in S(X)$ is the set $\operatorname{supp}\sigma = \{x \in X \mid \sigma(x) \neq x\}$. Two permutations $\sigma, \tau \in S(X)$ are *disjoint* if $\operatorname{supp}\sigma \cap \operatorname{supp}\tau = \emptyset$. In this case, σ and τ commute: $\sigma\tau = \tau\sigma$. We also call $\sigma\tau$ a *disjoint product* in this case. Recall the cycle notation for permutations: for $k > 1$, we write $(a_1\ a_2\ \ldots\ a_k)$ for the permutation with support $\{a_1, \ldots, a_k\}$ defined by $a_1 \mapsto a_2$, $a_2 \mapsto a_3$, $\ldots$, $a_{k-1} \mapsto a_k$ and $a_k \mapsto a_1$. Such a permutation is also called a *k-cycle*. Every permutation can be uniquely decomposed (up to the order of the factors) into a product of disjoint cycles. A k-cycle is even if and only if k is odd.

Two permutations $\sigma, \sigma' \in S(X)$ are *similar*, in symbols $\sigma \sim \sigma'$, if there exists τ such that $\sigma' = \tau^{-1}\sigma\tau$. It is easy to see that σ, σ' are similar if and only if their cycle decompositions contain an equal number of k-cycles for every k.

If $g, h \in S(X)$ are permutations on some finite set X, we define their *disjoint sum* $g + h \in S(2 \times X)$ as

$$(g + h)(0, x) = (0, g(x)) \quad \text{and} \quad (g + h)(1, x) = (1, h(x)).$$

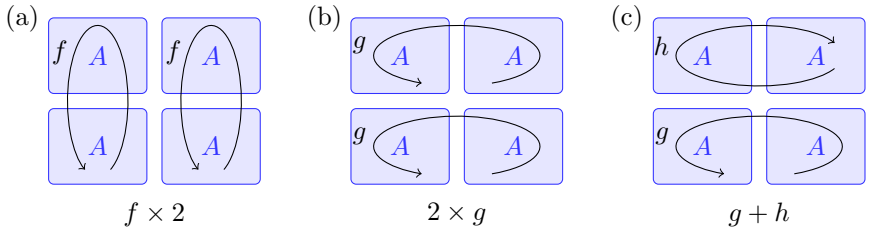


Figure 1: Visualizing permutations of $2 \times A \times 2$

We note the following properties:

$$g + g = 2 \times g, \tag{3}$$
$$(g + h) \times 2 = g \times 2 + h \times 2, \tag{4}$$
$$(g + h)(g' + h') = gg' + hh'. \tag{5}$$

Property (3) also helps explain our choice of writing "2" for the identity permutation in $S(2)$.

Although it will not be strictly necessary for the proofs that follow (which are combinatorial), it may sometimes be helpful to visualize sets of the form $2 \times A \times 2$, and permutations thereon, as follows. We visualize the set $2 \times A$ as two copies of A stacked vertically, with elements of the form $(0, a)$ and $(1, a)$ belonging to the lower and upper copy, respectively. Similarly, we visualize the set $A \times 2$ as two copies of A side by side, with elements of the form $(a, 0)$ and $(a, 1)$ belonging to the left and right copy, respectively. In the same vein, we visualize the set $2 \times A \times 2$ as four copies of A arranged in two rows and columns. The effect of a permutations of the form $f \times 2$ is to apply f separately to the left and right column, as shown in Figure 1(a). Similarly, the effect of $2 \times g$ is to apply g separately to the top and bottom rows, and the effect of $g + h$ is to apply g to the bottom row and h to the top row, as shown in Figure 1(b) and (c).

3.2 Decomposition into balanced permutations

Definition. A permutation σ is *balanced* if the number of k-cycles in its cycle decomposition is even for all $k \geqslant 2$. Moreover, σ is *nearly balanced* if the number of k-cycles in its cycle decomposition is even for all $k \geqslant 3$.

For example, the permutation (1 2)(3 4)(5 6 7)(8 9 10) is balanced, the permutation (1 2)(3 4 5)(8 9 10) is nearly balanced, and (1 2)(3 4)(5 6 7) is neither balanced nor nearly balanced.

Remark 3.1. The disjoint product of any number of (nearly) balanced permutations is (nearly) balanced. Moreover, a nearly balanced permutation is balanced if and only if it is even.

The purpose of this subsection is to prove that every even permutation on a set of 5 or more elements can be decomposed into a product of two balanced permutations. This will be Proposition 3.8 below. The proof requires a sequence of lemmas.

Lemma 3.2. *Let σ be a k-cycle, where $k \geqslant 2$ and $k \neq 3$. Then there exists a balanced permutation ρ and a nearly balanced permutation τ such that $\sigma = \tau\rho$. Moreover,* $\operatorname{supp}\tau \cup \operatorname{supp}\rho \subseteq \operatorname{supp}\sigma$.

Proof. Let $\sigma = (a_1\ a_2\ \ldots\ a_k)$. If $k = 2t$ is even, let

$$\begin{aligned}\rho &= (a_1\ a_2\ \ldots\ a_t)(a_{t+1}\ a_{t+2}\ \ldots\ a_{2t}),\\ \tau &= (a_1\ a_{t+1}).\end{aligned}$$

If $k = 2t+1$ is odd (and therefore, by assumption, $t \geqslant 2$), let

$$\begin{aligned}\rho &= (a_1\ a_2\ \ldots\ a_t)(a_{t+1}\ a_{t+3}\ a_{t+4}\ \ldots\ a_{2t+1}),\\ \tau &= (a_1\ a_{t+1})(a_{t+2}\ a_{t+3}).\end{aligned}$$

In both cases, the conclusion of the lemma is satisfied. □

Lemma 3.3. *Let σ be the disjoint product of a 3-cycle and a k-cycle, where $k \geqslant 2$. Then there exists a balanced permutation ρ and a nearly balanced permutation τ such that $\sigma = \tau\rho$. Moreover,* $\operatorname{supp}\tau \cup \operatorname{supp}\rho \subseteq \operatorname{supp}\sigma$.

Proof. Let $\sigma = (b_1\ b_2\ b_3)(a_1\ a_2\ \ldots\ a_k)$. If $k = 2$, let

$$\begin{aligned}\rho &= (b_1\ b_2)(a_1\ a_2),\\ \tau &= (b_1\ b_3).\end{aligned}$$

If $k = 3$, let

$$\begin{aligned}\rho &= (b_1\ b_2)(a_1\ a_2),\\ \tau &= (b_1\ b_3)(a_1\ a_3),\end{aligned}$$

If $k = 4$, let

$$\begin{aligned}\rho &= (b_1\ b_2\ b_3)(a_1\ a_2\ a_3),\\ \tau &= (a_1\ a_4).\end{aligned}$$

If $k = 2t+1$ is odd, with $t \geqslant 2$, let

$$\begin{aligned}\rho &= (b_1\ b_2\ b_3)(a_t\ a_{t+1}\ a_{2t+1}),\\ \tau &= (a_1\ a_2\ \ldots\ a_t)(a_{t+2}\ a_{t+3}\ \ldots\ a_{2t+1}).\end{aligned}$$

If $k = 2t$ is even, with $t \geqslant 3$, let

$$\begin{aligned}\rho &= (a_3\ a_4\ \ldots\ a_t\ a_{2t})(a_{t+1}\ a_{t+2}\ \ldots\ a_{2t-1}),\\ \tau &= (b_1\ b_2\ b_3)(a_1\ a_2\ a_3)(a_{t+1}\ a_{2t}).\end{aligned}$$

In all cases, the conclusion of the lemma is satisfied. □

Lemma 3.4. *Let σ be a disjoint product of two or more 3-cycles. Then there exist balanced permutations ρ, τ such that $\sigma = \tau\rho$. Moreover,* $\operatorname{supp}\tau \cup \operatorname{supp}\rho \subseteq \operatorname{supp}\sigma$.

Proof. By assumption, σ factors as $\sigma = \gamma_1\gamma_2\cdots\gamma_\ell$, where $\gamma_1, \ldots, \gamma_\ell$ are pairwise disjoint 3-cycles and $\ell \geqslant 2$. Note that γ_i^2 is also a 3-cycle, and $\gamma_i^4 = \gamma_i$, for all i.

If ℓ is even, let $\rho = \tau = \gamma_1^2\gamma_2^2\cdots\gamma_\ell^2$. If ℓ is odd, let $\rho = \gamma_1\gamma_2^2\gamma_3^2\cdots\gamma_{\ell-1}^2$ and $\tau = \gamma_2^2\gamma_3^2\cdots\gamma_{\ell-1}^2\gamma_\ell$. In both cases, the conclusion of the lemma is satisfied. □

Lemma 3.5. *Let σ be an even permutation, other than a 3-cycle. Then σ can be written as $\sigma = \tau\rho$, where ρ, τ are balanced.*

Proof. By considering the cycle decomposition of σ, it is easy to see that σ can be factored into disjoint factors such that each factor satisfies the premise of one of Lemmas 3.2, 3.3, or 3.4. Let $\sigma = \sigma_1\cdots\sigma_\ell$ be such a factorization. Using the lemmas, each σ_i can be written as $\sigma_i = \tau_i\rho_i$, where ρ_i is balanced and τ_i is nearly balanced. Moreover, since the support of each ρ_i and τ_i is contained in that of σ_i, the ρ_i are pairwise disjoint, the τ_i are pairwise disjoint, and $\rho_i\tau_j = \tau_j\rho_i$ whenever $i \neq j$. Let $\rho = \rho_1\cdots\rho_\ell$ and $\tau = \tau_1\cdots\tau_\ell$. Then we have $\sigma = \tau\rho$. Moreover, by Remark 3.1, ρ is balanced and τ is nearly balanced. Finally, since σ and ρ are even permutations, so is τ, and it follows, again by Remark 3.1, that τ is balanced. □

Lemma 3.6. *Let σ be a 3-cycle in $S(X)$, where $|X| \geqslant 5$. Then there exist balanced permutations ρ, τ such that $\sigma = \tau\rho$.*

Proof. Let $\sigma = (a_1\ a_2\ a_3)$. Since $|X| \geqslant 5$, there exist elements a_4, a_5 of X that are different from each other and from $a_1, \ldots, a_3$. Let

$$\rho = (a_1\ a_2)(a_4\ a_5),$$
$$\tau = (a_1\ a_3)(a_4\ a_5).$$

Then the conclusion of the lemma is satisfied. □

Remark 3.7. Unlike the situation in Lemmas 3.2–3.5, it is not possible to choose ρ and τ in Lemma 3.6 so that their support is contained in that of σ. An easy case distinction shows that Lemma 3.6 is false when $|X| \leqslant 4$.

Proposition 3.8. *Let σ be an even permutation in $S(X)$, where $|X| \geqslant 5$. Then there exist balanced permutations ρ, τ such that $\sigma = \tau\rho$.*

Proof. By Lemma 3.6 if σ is a 3-cycle, and by Lemma 3.5 otherwise. □

3.3 Alternation depth of permutations of the form $g + h$

We now come to the main result of Section 3, which is that every even permutation of the form $g + h$ has alternation depth 5.

Proposition 3.9. *Let A be a finite set of 3 or more elements, and let $g, h \in S(A \times 2)$ be permutations such that $\sigma = g + h$ is even. Then σ has alternation depth 5.*

The proof requires two lemmas.

Lemma 3.10. *Let $\tau \in S(A \times 2)$ be a balanced permutation. Then there exist permutations $g \in S(A \times 2)$ and $h \in S(A)$ such that*

$$\tau = g^{-1}(h \times 2)g.$$

Proof. For all $k \geqslant 2$, let y_k be the number of k-cycles in the cycle decomposition of τ. Since the cycles are disjoint, we have $\sum_k k y_k \leqslant 2|A|$. Since τ is balanced, all y_k are even. We can therefore find a permutation $h \in S(A)$ whose number of k-cycles is exactly $y_k/2$, for all k. Since $h \times 2$ and τ have, by construction, the same number of k-cycles for all k, we have $h \times 2 \sim \tau$. By definition of similarity, it follows that there exists some g with $\tau = g^{-1}(h \times 2)g$, as claimed. □

Lemma 3.11. *Let $\tau \in S(A \times 2)$ be a balanced permutation, and let $\sigma = \mathrm{id}_{A\times 2} + \tau \in S(2 \times A \times 2)$. Then there exist permutations $g \in S(A \times 2)$ and $f \in S(2 \times A)$ such that*

$$\sigma = (2 \times g^{-1})(f \times 2)(2 \times g).$$

Proof. By Lemma 3.10, we can find $g \in S(A \times 2)$ and $h \in S(A)$ such that $\tau = g^{-1}(h \times 2)g$. Let $f = \mathrm{id}_A + h \in S(2 \times A)$. Then

$$\begin{aligned}
&(2 \times g^{-1})(f \times 2)(2 \times g) \\
&\quad = (2 \times g^{-1})((\mathrm{id}_A + h) \times 2)(2 \times g) \\
&\quad = (g^{-1} + g^{-1})(\mathrm{id}_A \times 2 + h \times 2)(g + g) \\
&\quad = (g^{-1}\,\mathrm{id}_{A\times 2}\,g) + (g^{-1}(h \times 2)g) \\
&\quad = \mathrm{id}_{A\times 2} + \tau \\
&\quad = \sigma.
\end{aligned}$$

Here, in addition to the defining properties of f, g, h, and σ, we have also used (3) and (4) in the second step and (5) in the third step. □

Proof of Proposition 3.9. Let $\tau = hg^{-1} \in S(A \times 2)$, and note that τ is even. By Proposition 3.8, there exist balanced permutations $\tau_1, \tau_2 \in S(A \times 2)$ such that $\tau = \tau_2\tau_1$. By Lemma 3.11, there exist $g_1, g_2 \in S(A \times 2)$ and $f_1, f_2 \in S(2 \times A)$ such that $\mathrm{id}_{A\times 2} + \tau_i = (2 \times g_i^{-1})(f_i \times 2)(2 \times g_i)$, for $i = 1, 2$. Then we have:

$$\begin{aligned}
\sigma &= g + h \\
&= \mathrm{id}_{A\times 2} g + \tau g \\
&= (\mathrm{id}_{A\times 2} + \tau)(g + g) \\
&= (\mathrm{id}_{A\times 2} + \tau_2\tau_1)(2 \times g) \\
&= (\mathrm{id}_{A\times 2} + \tau_2)(\mathrm{id}_{A\times 2} + \tau_1)(2 \times g) \\
&= (2 \times g_2^{-1})(f_2 \times 2)(2 \times g_2)(2 \times g_1^{-1})(f_1 \times 2)(2 \times g_1)(2 \times g) \\
&= (2 \times g_2^{-1})(f_2 \times 2)(2 \times g_2 g_1^{-1})(f_1 \times 2)(2 \times g_1 g),
\end{aligned}$$

which is of alternation depth 5 as desired. □

4 Second construction: colorings

4.1 Colorings

As before, let $2 = \{0, 1\}$. If X is any finite set, a *coloring* of X is a map $c : X \to 2$. Here, we think of the binary digits 0 and 1 as *colors*, i.e., $x \in X$

(a)

p	q	r	p	q	r
0	1	1	1	1	1
0	0	0	1	0	0

(b)

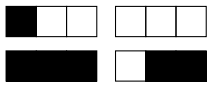


Figure 2: Visualizing colorings of $2 \times A \times 2$

has color $c(x)$. We say that the coloring c is *fair* if there is an equal number of elements of each color, i.e., $|c^{-1}\{0\}| = |c^{-1}\{1\}|$.

The group $S(X)$ acts in a natural way on the colorings of X as follows: we define $\sigma \bullet c = c'$, where $c'(x) = c(\sigma^{-1}(x))$. Note that $(\sigma\tau) \bullet c = \sigma \bullet (\tau \bullet c)$. Also, $\sigma \bullet c$ is fair if and only if c is fair.

On a set of the form $2 \times X$, the *standard coloring* is the one given by $c_{\mathrm{st}}(0, x) = 0$ and $c_{\mathrm{st}}(1, x) = 1$, for all x.

Remark 4.1. The standard coloring is fair. Conversely, if c is a fair coloring of $2 \times X$, there exists a permutation $f \in S(2 \times X)$ such that $f \bullet c = c_{\mathrm{st}}$.

The following lemma relates colorings to permutations of the form $g + h$ considered in the previous section.

Lemma 4.2. *A permutation $\sigma \in S(2 \times X)$ is of the form $\sigma = g + h$, for some $g, h \in S(X)$, if and only if $\sigma \bullet c_{\mathrm{st}} = c_{\mathrm{st}}$.*

Proof. This is elementary. We have $\sigma \bullet c_{\mathrm{st}} = c_{\mathrm{st}}$ if and only if for all x, $\sigma(0, x)$ is of the form $(0, y)$, and $\sigma(1, x)$ is of the form $(1, z)$. By setting $g(x) = y$ and $h(x) = z$, this is equivalent to σ being of the form $g + h$. □

We are now ready to state the main result of Section 4, which is that every fair coloring of $2 \times A \times 2$ can be converted to the standard coloring by the action of a permutation of alternation depth 4.

Proposition 4.3. *Let A be a finite set of 3 or more elements, and let c be a fair coloring of $2 \times A \times 2$. Then there exists a permutation $\sigma \in S(2 \times A \times 2)$ such that $\sigma \bullet c = c_{\mathrm{st}}$ and σ has alternation depth 4.*

The proof of Proposition 4.3 will take up the remainder of Section 4.

4.2 Visualizing colorings

Colorings on $2 \times A \times 2$ can be visualized in the same row-and-column format we used in Figure 1. An example of a coloring, where $A = \{p, q, r\}$, is shown in Figure 2(a). The figure indicates, for example, that $c(1, p, 0) = 0$, $c(1, q, 0) = 1$, and so on. When the names of the elements of A are not important, we omit them. Additionally, we sometimes represent the colors 0 and 1 by black and white squares, respectively, as in Figure 2(b).

4.3 Color pairs

We begin by characterizing when two colorings c, c' of $2 \times X$ are related by the action of a permutation of the form $2 \times g$ for $g \in S(X)$. This is the case if and only if c and c' have the same *color pair distribution*.

Definition. Let X be a set, and consider a coloring c of $2 \times X$. We define a function $c^* : X \to 2 \times 2$ by $c^*(x) = (c(0, x), c(1, x))$. We call $c^*(x)$ the *color pair* of x.

Informally, a color pair corresponds to a single column of digits in Figure 2(a). We note that the action of permutations $g \in S(X)$ respects color pairs in the following sense: let $c' = (2 \times g) \bullet c$. Then

$$c'^*(g(x)) = (c'(0, g(x)), c'(1, g(x))) = (c(0, x), c(1, x)) = c^*(x). \tag{6}$$

In particular, the action of $2 \times g$ on colorings does not change the *number* of elements of X with each color pair. Conversely, whenever two colorings c, c' have this property, then they are related by the action of $2 \times g$, for some g. The following definition helps us state this more precisely.

Definition. Let X be a set, and c a coloring of $2 \times X$. For any $i, j \in 2$, define $N_c(i, j) \subseteq X$ to be the set of elements with color pair (i, j), i.e.,

$$N_c(i, j) = \{x \in X \mid c^*(x) = (i, j)\}.$$

Note that X is the disjoint union of the $N_c(i, j)$, for $i, j \in 2$. Let $n_c(i, j) = |N_c(i, j)|$ be the number of elements with color pair (i, j). Then the *color pair distribution* of c is the 4-tuple

$$(n_c(0, 0), n_c(0, 1), n_c(1, 0), n_c(1, 1)).$$

For example, the coloring from Figure 2 has color pair distribution $(1, 4, 0, 1)$, because the color pair $(0, 0)$ occurs once, the color pair $(0, 1)$ occurs four times, and so on. The following lemma is then obvious.

Lemma 4.4. *Let c, c' be colorings of $2 \times X$. Then c, c' have the same color pair distribution if and only if there exists a permutation $g \in S(X)$ such that $c' = (2 \times g) \bullet c$.* □

4.4 Color standardization

Definition. Let A be a set, and let c be a coloring of $2 \times A \times 2$. We say that c is

- *standard* if $c = c_{\mathrm{st}}$, i.e., if $c^*(a, 0) = c^*(a, 1) = (0, 1)$ for all $a \in A$;
- *symmetric* if $c^*(a, 0) = c^*(a, 1)$ for all $a \in A$;
- *regular* if each color pair occurs an even number of times, i.e., if $n_c(0, 0)$, $n_c(0, 1)$, $n_c(1, 0)$, and $n_c(1, 1)$ are even;

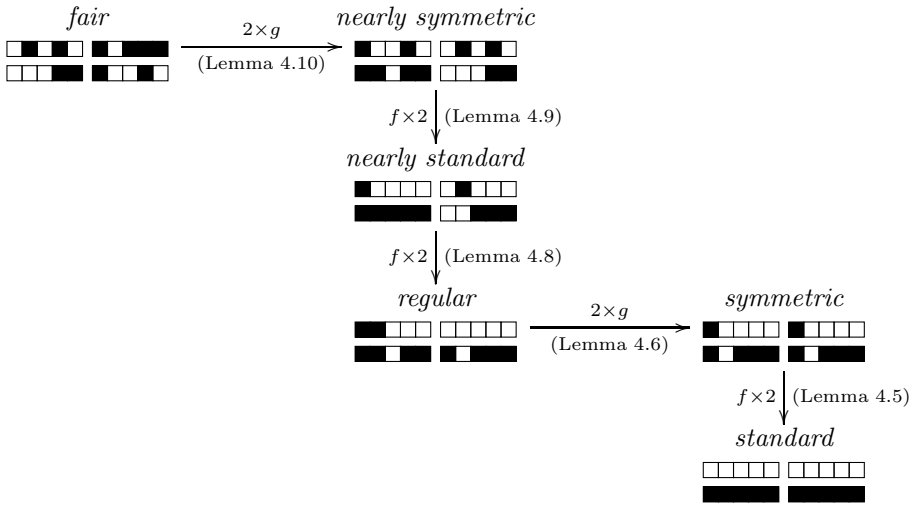


Figure 3: Standardizing a fair permutation

- *nearly standard* if $c^*(a,0) = c^*(a,1) = (0,1)$ for almost all $a \in A$, except that there is at most one $a_1 \in A$ such that $c^*(a_1,0) = (0,0)$ and $c^*(a_1,1) = (1,1)$, and at most one $a_2 \in A$ such that $c^*(a_2,0) = (0,1)$ and $c^*(a_2,1) = (1,0)$;
- *nearly symmetric* if $c^*(a,0) = c^*(a,1)$ for almost all $a \in A$, except that there is at most one $a_1 \in A$ such that $c^*(a_1,0) = (0,0)$ and $c^*(a_1,1) = (1,1)$, and at most one $a_2 \in A$ such that $c^*(a_2,0) = (0,1)$ and $c^*(a_2,1) = (1,0)$.

An example of each of these properties is shown in Figure 3. Our strategy for proving Proposition 4.3 is to use the action of permutations of the forms $2 \times g$ and $f \times 2$ to successively improve the properties of a coloring until it is standard. This procedure is also outlined in Figure 3, along with the number of the lemma that will be used in each step. The remainder of this section is devoted to the statements and proofs of these lemmas, culminating in the proof of Proposition 4.3 in Section 4.5.

Lemma 4.5. *Let c be a symmetric fair coloring of $2 \times A \times 2$. Then there exists $f \in S(2 \times A)$ such that $(f \times 2) \bullet c$ is standard.*

Proof. Since c is symmetric, we have $c(i,a,0) = c(i,a,1)$ for all $(i,a) \in 2 \times A$; write $p(i,a) = c(i,a,0)$. Since c is fair, $p : 2 \times A \to 2$ is also fair. By Remark 4.1, there exists a permutation $f \in S(2 \times A)$ such that $f \bullet p$ is the standard coloring of $2 \times A$. It follows that $(f \times 2) \bullet c$ is the standard coloring of $2 \times A \times 2$. □

Lemma 4.6. *Let c be a regular coloring of $2 \times A \times 2$. Then there exists $g \in S(A \times 2)$ such that $(2 \times g) \bullet c$ is symmetric.*

Proof. Since c is regular, we can find integers p, q, r, s such that $n_c(0,0) = 2p$, $n_c(1,1) = 2q$, $n_c(0,1) = 2r$, and $n_c(1,0) = 2s$. Note that $n_c(0,0) + n_c(0,1) + n_c(1,0) + n_c(1,1) = 2|A|$, and therefore $p+q+r+s = |A|$. Write A as a disjoint union of sets $P \cup Q \cup R \cup S$, where $|P| = p$, $|Q| = q$, $|R| = r$, and $|S| = s$. Define a coloring c' by

$$\begin{aligned} c'^*(a,0) = c'^*(a,1) &= (0,0), \quad \text{if } a \in P, \\ c'^*(a,0) = c'^*(a,1) &= (1,1), \quad \text{if } a \in Q, \\ c'^*(a,0) = c'^*(a,1) &= (0,1), \quad \text{if } a \in R, \\ c'^*(a,0) = c'^*(a,1) &= (1,0), \quad \text{if } a \in S. \end{aligned}$$

Then by construction, c' is symmetric and has the same color pair distribution as c. Hence by Lemma 4.4, there exists $g \in S(A \times 2)$ such that $c' = (2 \times g) \bullet c$, which was to be shown. □

Lemma 4.7. *Suppose $|A| = 3$ and c is a nearly standard coloring of $2 \times A \times 2$. Then there exists $f \in S(2 \times A)$ such that $(f \times 2) \bullet c$ is regular.*

Proof. Write A as the disjoint union $A_1 \cup A_2 \cup A_3$, where $c^*(a_1, 0) = (0,0)$ and $c^*(a_1, 1) = (1,1)$ for all $a_1 \in A_1$, $c^*(a_2, 0) = (0,1)$ and $c^*(a_2, 1) = (1,0)$ for all $a_2 \in A_2$, and $c^*(a,0) = c^*(a,1) = (0,1)$ for all $a \in A_3$. By the definition of nearly standard, we know that A_1 and A_2 have at most one element each. Since we assumed $|A| = 3$, this leaves us with four cases.

- Case 1. Assume $|A_1| = |A_2| = 0$. Say $A_3 = \{p, q, r\}$. Since $c^*(a,0) = c^*(a,1) = (0,1)$ for all $a \in A$, c is the following coloring (using the notation of Section 4.2):

p	q	r	p	q	r
1	1	1	1	1	1
0	0	0	0	0	0.

  Since c is already regular (in fact, standard), we can take f to be the identity permutation.

- Case 2. Assume $|A_1| = 1$ and $|A_2| = 0$. Say $A_1 = \{a_1\}$ and $A_3 = \{p, q\}$. Then c is the coloring

a_1	p	q	a_1	p	q
0	1	1	1	1	1
0	0	0	1	0	0.

  Define $f : 2 \times A \to 2 \times A$ by $f(0, a_1) = (1, p)$, $f(1, p) = (0, q)$, $f(0, q) = (0, a_1)$, and the identity elsewhere. Then $(f \times 2) \bullet c$ is the coloring

a_1	p	q	a_1	p	q
0	0	1	1	1	1
0	0	1	0	0	1.

- Case 3. Assume $|A_1| = 0$ and $|A_2| = 1$. Say $A_2 = \{a_2\}$ and $A_3 = \{p, q\}$. Then c is the coloring

a_2	p	q	a_2	p	q
1	1	1	0	1	1
0	0	0	1	0	0.

  Define $f : 2 \times A \to 2 \times A$ by $f(0, a_2) = (1, p)$, $f(1, p) = (0, q)$, $f(0, q) = (0, a_2)$, and the identity elsewhere. Then $(f \times 2) \bullet c$ is the coloring

a_2	p	q	a_2	p	q
1	0	1	0	1	1
0	0	1	0	0	1.

- Case 4. Assume $|A_1| = |A_2| = 1$. Say $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, and $A_3 = \{p\}$. Then c is the coloring

a_1	a_2	p	a_1	a_2	p
0	1	1	1	0	1
0	0	0	1	1	0.

  Define $f : 2 \times A \to 2 \times A$ by $f(0, a_1) = (1, a_2)$, $f(1, a_2) = (0, p)$, $f(0, p) = (0, a_1)$, and the identity elsewhere. Then $(f \times 2) \bullet c$ is the coloring

a_1	a_2	p	a_1	a_2	p
0	0	1	1	1	1
0	0	1	0	1	0.

In all four cases, $(f \times 2) \bullet c$ is regular, as desired. □

Lemma 4.8. *Suppose $|A| \geqslant 3$ and c is a nearly standard coloring of $2 \times A \times 2$. Then there exists $f \in S(2 \times A)$ such that $(f \times 2) \bullet c$ is regular.*

Proof. The only difference to Lemma 4.7 is that A may have more than 3 elements. However, by the definition of nearly standard, c is already standard (hence regular) on the excess elements. Therefore, we can ignore all but 3 elements of A and proceed as in Lemma 4.7. □

Lemma 4.9. *Let c be a nearly symmetric fair coloring of $2 \times A \times 2$. Then there exists $f \in S(2 \times A)$ such that $(f \times 2) \bullet c$ is nearly standard.*

Proof. Let $A' = \{a \in A \mid c^*(a, 0) = c^*(a, 1)\}$, and let c' be the coloring of $2 \times A' \times 2$ obtained by restricting c to the domain $2 \times A' \times 2$. Then c' is symmetric. By definition of "nearly symmetric", there exists at most two elements $a_1, a_2 \in A \setminus A'$; moreover, the element a_1, if any, satisfies $c^*(a_1, 0) = (0, 0)$ and $c^*(a_1, 1) = (1, 1)$ and the element a_2, if any, satisfies $c^*(a_2, 0) = (0, 1)$ and $c^*(a_2, 1) = (1, 0)$. By assumption, c is fair. Since c restricted to $2 \times (A \setminus A') \times 2$ is evidently fair as well, it follows that c' is also fair. We will choose the permutation f so that its support does not touch the elements a_1 and a_2; it therefore suffices to find some permutation $f' \in S(2 \times A')$ such that $(f' \times 2) \bullet c'$ is standard. But such an f' exists by Lemma 4.5. □

Lemma 4.10. *Let c be a fair coloring of $2\times A\times 2$. Then there exists $g \in S(A\times 2)$ such that $(2 \times g) \bullet c$ is nearly symmetric.*

Proof. The proof is very similar to that of Lemma 4.6. Consider the color pair distribution $(n_c(0,0), n_c(0,1), n_c(1,0), n_c(1,1))$ of the given coloring c, and note that

$$n_c(0,0) + n_c(0,1) + n_c(1,0) + n_c(1,1) = 2|A|. \tag{7}$$

Because c is fair, we must have $n_c(0,0) = n_c(1,1)$, and in particular, $n_c(0,0)$ and $n_c(1,1)$ have the same parity (even or odd). From (7), it follows that $n_c(0,1)$ and $n_c(1,0)$ have the same parity. Therefore, there exist natural numbers p, q, r, s, t, u, with $t, u \in \{0,1\}$, such that

$$n_c(0,0) = 2p + t, \quad n_c(1,1) = 2q + t, \quad n_c(0,1) = 2r + u, \quad n_c(1,0) = 2s + u.$$

(As a matter of fact, $p = q$, but we will not make further use of this fact). From (7), we have that $p + q + r + s + t + u = |A|$. Write A as a disjoint union $P \cup Q \cup R \cup S \cup T \cup U$, where $|P| = p$, $|Q| = q$, $|R| = r$, $|S| = s$, $|T| = t$, and $|U| = u$. Define a coloring c' by

$$\begin{array}{rcll}
c'^*(a,0) = c'^*(a,1) = (0,0), & & & \text{if } a \in P,\\
c'^*(a,0) = c'^*(a,1) = (1,1), & & & \text{if } a \in Q,\\
c'^*(a,0) = c'^*(a,1) = (0,1), & & & \text{if } a \in R,\\
c'^*(a,0) = c'^*(a,1) = (1,0), & & & \text{if } a \in S,\\
c'^*(a,0) = (0,0) & \text{and} & c'^*(a,1) = (1,1), & \text{if } a \in T,\\
c'^*(a,0) = (0,1) & \text{and} & c'^*(a,1) = (1,0), & \text{if } a \in U.
\end{array}$$

By construction, c' has the same color pair distribution as c. Hence by Lemma 4.4, there exists $g \in S(A \times 2)$ such that $c' = (2 \times g) \bullet c$. On the other hand, by construction, c' is nearly symmetric (with a_1 being the unique element of T, if any, and a_2 being the unique element of U, if any). □

4.5 Proof of Proposition 4.3

Proposition 4.3 is now an easy consequence of Lemmas 4.5–4.10. Figure 3 contains a proof "without words". For readers who prefer a proof "with words", we give it here:

Assume $|A| \geqslant 3$ and let c be a fair coloring of $2 \times A \times 2$. By Lemma 4.10, there exists $g_1 \in S(A \times 2)$ such that $c_1 = (2 \times g_1) \bullet c$ is nearly symmetric. By Lemma 4.9, there exists $f_2 \in S(2 \times A)$ such that $c_2 = (f_2 \times 2) \bullet c_1$ is nearly standard. By Lemma 4.8, there exists $g_3 \in S(A\times 2)$ such that $c_3 = (2\times g_3)\bullet c_2$ is regular. By Lemma 4.6, there exists $g_4 \in S(A \times 2)$ such that $c_4 = (2 \times g_4) \bullet c_3$ is symmetric. By Lemma 4.5, there exists $f_5 \in S(2\times A)$ such that $c_5 = (f_5 \times 2) \bullet c_4$ is standard. Let

$$\begin{array}{rcl}
\sigma & = & (f_5 \times 2)(2 \times g_4)(2 \times g_3)(f_2 \times 2)(2 \times g_1)\\
& = & (f_5 \times 2)(2 \times g_4 g_3)(f_2 \times 2)(2 \times g_1).
\end{array}$$

Then $\sigma \bullet c = c_{\mathrm{st}}$, and σ has alternation depth 4, as claimed. □

5 Proof of the main theorem

Our main result, Theorem 2.1, follows from Propositions 3.9 and 4.3. Specifically, let $\sigma \in S(2 \times A \times 2)$ be an even permutation, and let $c = \sigma^{-1} \bullet c_{\mathrm{st}}$. By Proposition 4.3, we can find $\tau \in S(2 \times A \times 2)$ of alternation depth 4 such that $\tau \bullet c = c_{\mathrm{st}}$. Note that τ is even by Remark 2.2. Let $\rho = \sigma\tau^{-1}$. Then ρ is also even, and $\rho \bullet c_{\mathrm{st}} = \sigma \bullet (\tau^{-1} \bullet c_{\mathrm{st}}) = \sigma \bullet c = c_{\mathrm{st}}$. Therefore, by Lemma 4.2, ρ is of the form $g + h$, for $g, h \in S(A \times 2)$. By Proposition 3.9, ρ has alternation depth 5, and it follows that $\sigma = \rho\tau$ has alternation depth 9, as claimed. □

6 Conclusion and further work

We showed that every even permutation of $2 \times A \times 2$ has alternation depth 9. The bound 9 is probably not tight. The constructions of Sections 3 and 4 have many degrees of freedom, making it plausible that a tighter bound on alternation depth can be found.

The best lower bound for alternation depth known to the author is 5. An exhaustive search shows that for $A = \{a, b, c\}$, a 3-cycle with support $\{0\} \times A \times \{0\}$ cannot be written with alternation depth 4. Of course, this particular permutation can be realized with alternation depth 5 by Proposition 3.9.

It is reasonable to conjecture that there is nothing special about the number 2 in Theorem 2.1. Specifically, if N and M are finite sets, I conjecture that there exists a finite bound on the alternation depth of all permutations $\sigma \in S(N \times A \times M)$ (or all even permutations, when N and M are even), for large enough A, independently of the size of A.

The reader may have noticed that in our definition of alternation depth, in the factors of the form $f \times 2$ and $2 \times g$, we did not require the permutations f and g to be even. However, if the construction is to be used recursively (as required, for example, by some potential applications mentioned in the introduction), each f and g must be even. We say that $\sigma \in S(2 \times A \times 2)$ has *even alternation depth* d if it can be written as a product of d factors of the forms $f \times 2$ or $2 \times g$, where each such $f \in S(2 \times A)$ and $g \in S(A \times 2)$ is an even permutation. Then an analogue of Theorem 2.1 holds for even alternation depth. A very inefficient proof is the following: first, it is easy to find some odd permutation $g \in S(A \times 2)$ and even permutations $f_1, f_3, f_5 \in S(2 \times A)$ and $g_2, g_4 \in S(A \times 2)$ such that $(2 \times g) = (f_1 \times 2)(2 \times g_2)(f_3 \times 2)(2 \times g_4)(f_5 \times 2)$. Using this, every permutation of alternation depth d can be rewritten as a permutation of even alternation depth at most $5d + 1$. Naturally, this naive proof yields a bound on even alternation depth that is not very tight, namely, $d = 5 \cdot 9 + 1 = 46$. With a more careful argument, it can be shown that the even alternation depth is in fact bounded by 13, and I expect that it is bounded by 9 or less. But the details of this are left for future work.

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