Supplement: Generators and Relations for Real Stabilizer Operators

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This supplement to “Generators and Relations for Real Stabilizer Operators” establishes that the reduced relations of Definition 5.6 imply the typed relations of Definition 5.1 (as listed in the 8 figures, from Figure 3 to Figure 10 in Appendix B).

1 Proof of Proposition 5.7

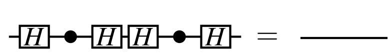
We show that the typed relations, viewed as relations between un-typed circuits, follow from the reduced relations. That is, in this appendix, we *forget* the typing.

Once typing is forgotten, the relations in Figure 6 and Figure 7 coincide so that it is sufficient to derive those in Figure 6. Similarly, of the first 8 rules in Figure 4, the left column is identical to the right column as un-typed rules, so we only prove the left column. Lastly, the next 12 rules of Figure 6 are identical to the 12 rules following them as un-typed rules, so we only derive the first 12 of them.

We start by proving a few useful lemmas before deriving the typed relations. For each one of Figures 3–10, we number the relations left-to-right and row-by-row in order to conveniently refer to them.

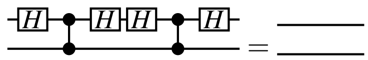
1.1 Lemmas

Lemma 1.1. *The following is a consequence of $R2$ and $R3$.*

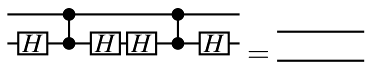


(L1)

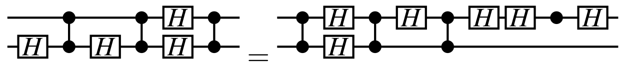
Lemma 1.2. *The following rules are a consequence of $R1 - R18$*



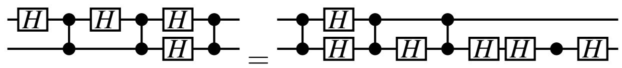
(L2)



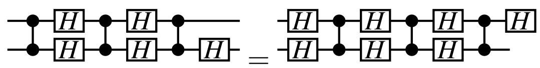
(L3)



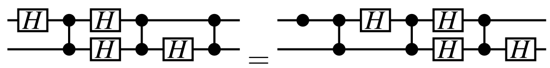
(L4)



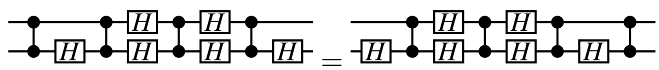
(L5)



(L6)

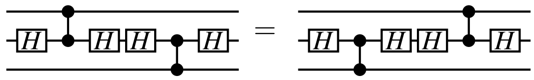


(L7)

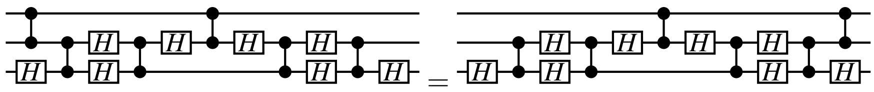


(L8)

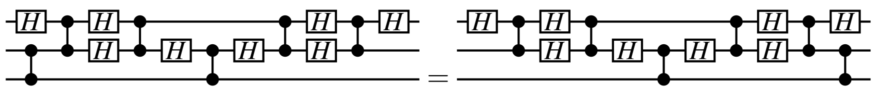
Lemma 1.3. *The following rules are a consequence of $R1-R18$*



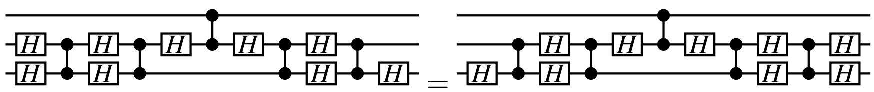
(L9)



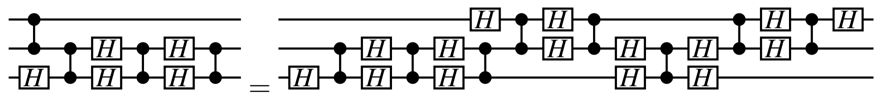
(L10)



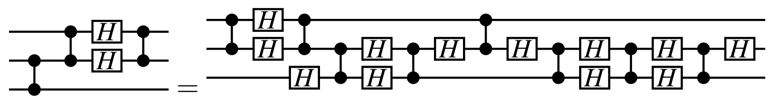
(L11)



(L12)



(L13)



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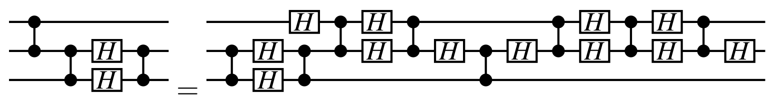

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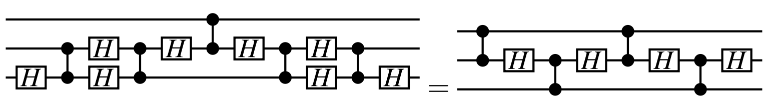

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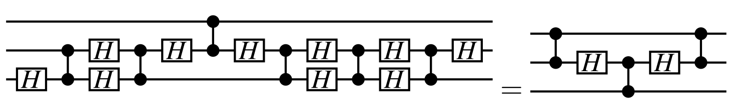
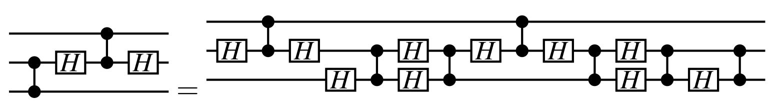


(L17)



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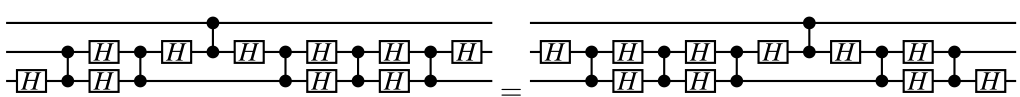

(L19)

1.2 Proofs of Lemmas

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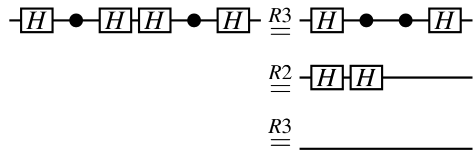

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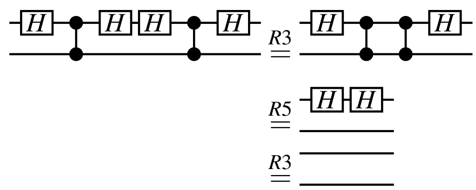

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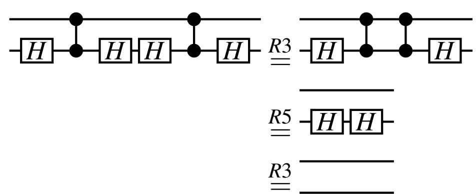

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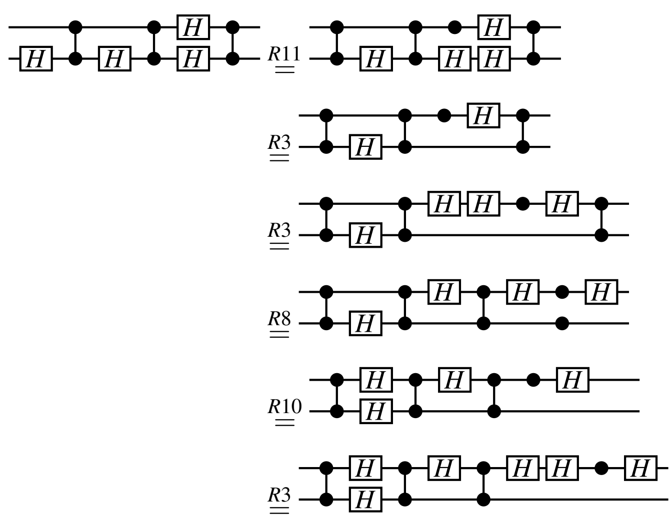

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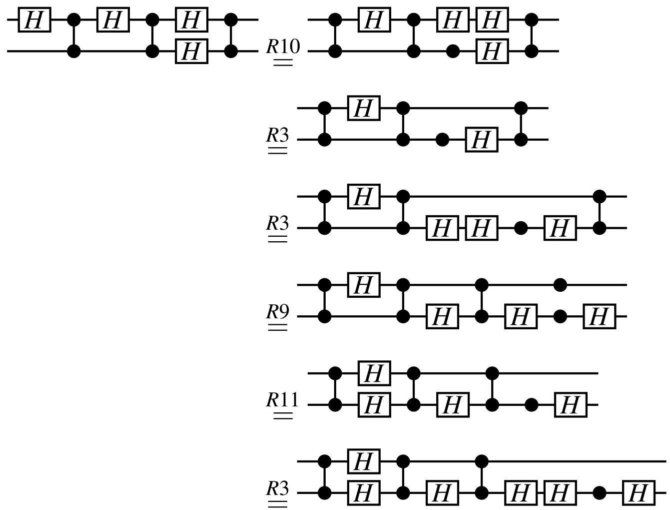

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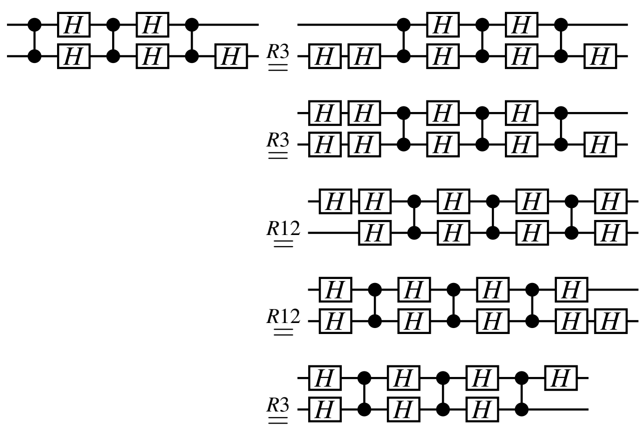

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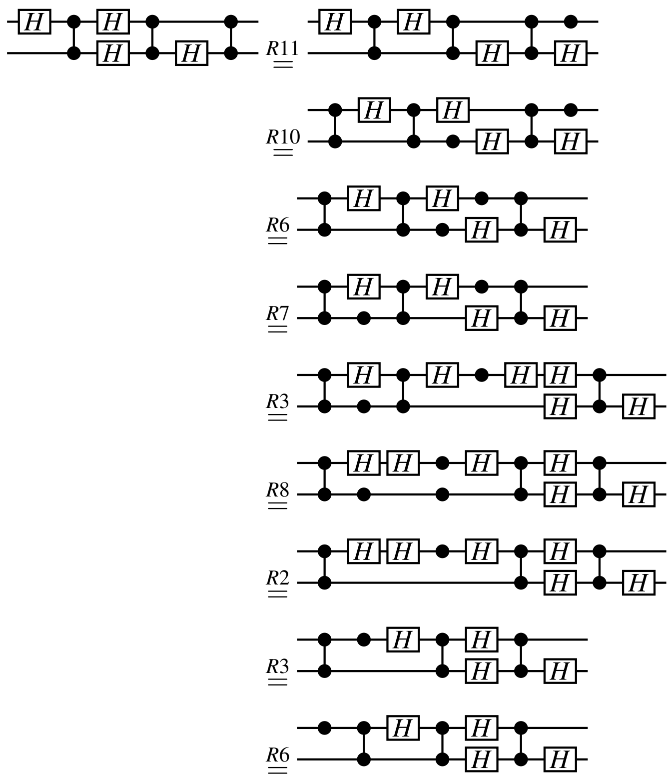

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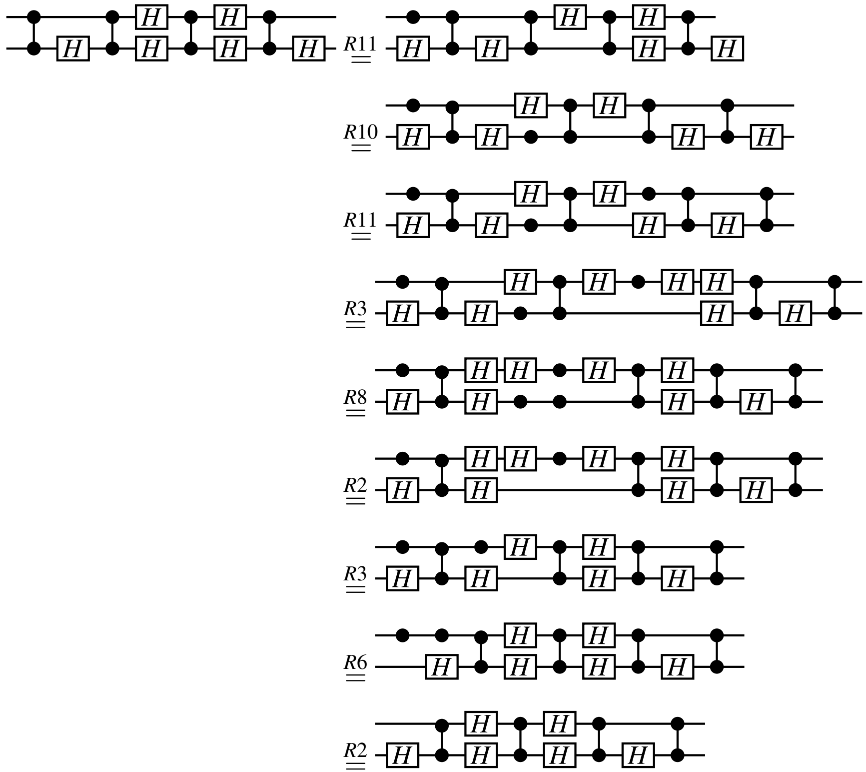

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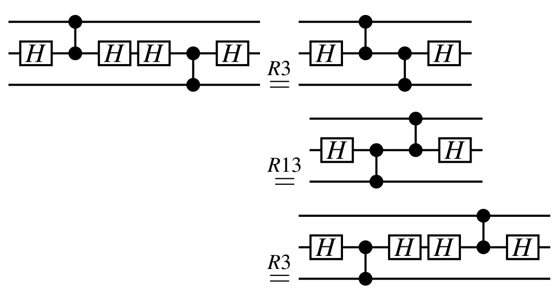

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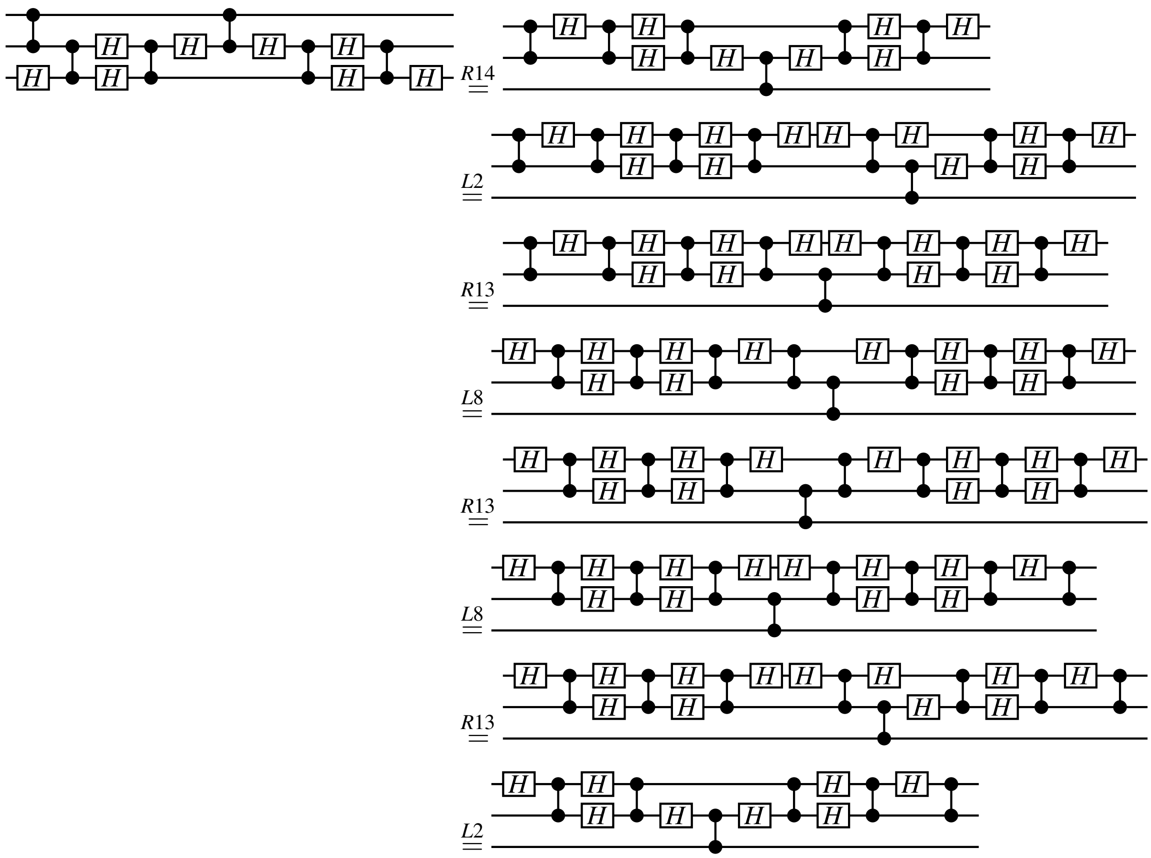

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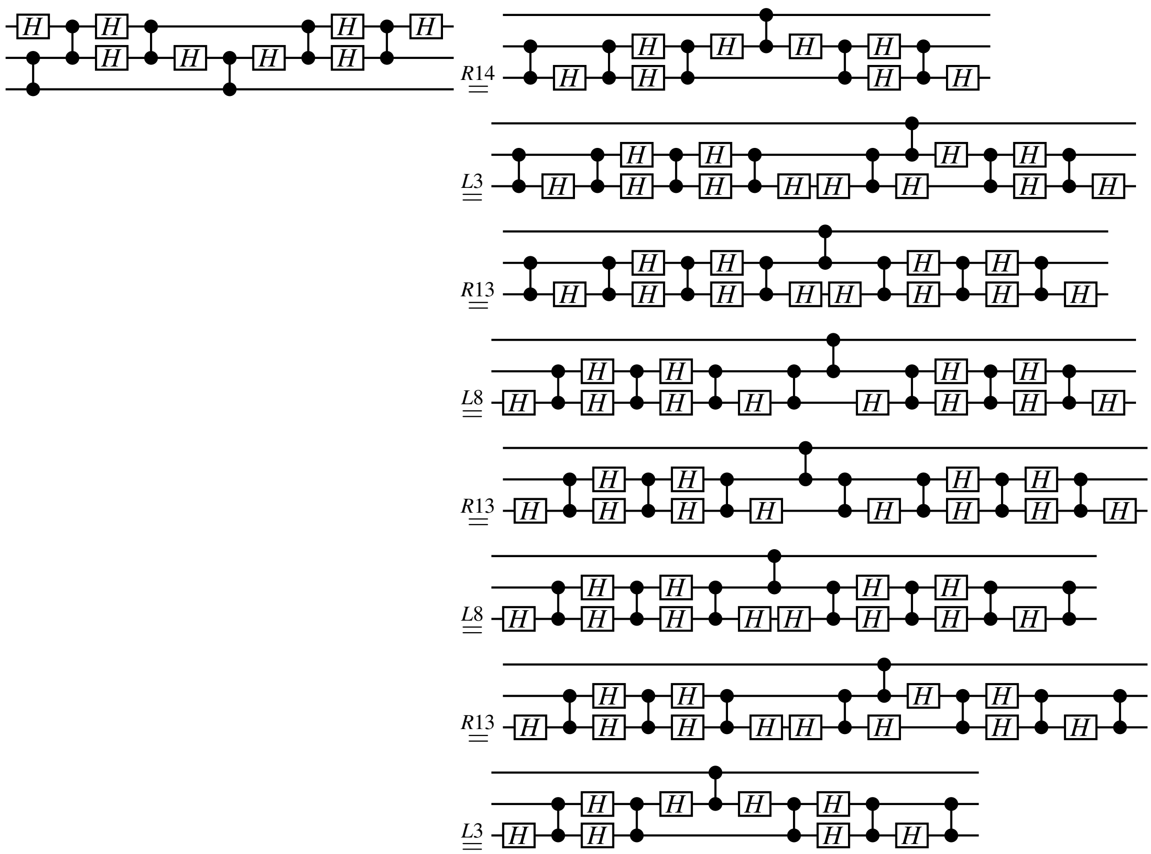

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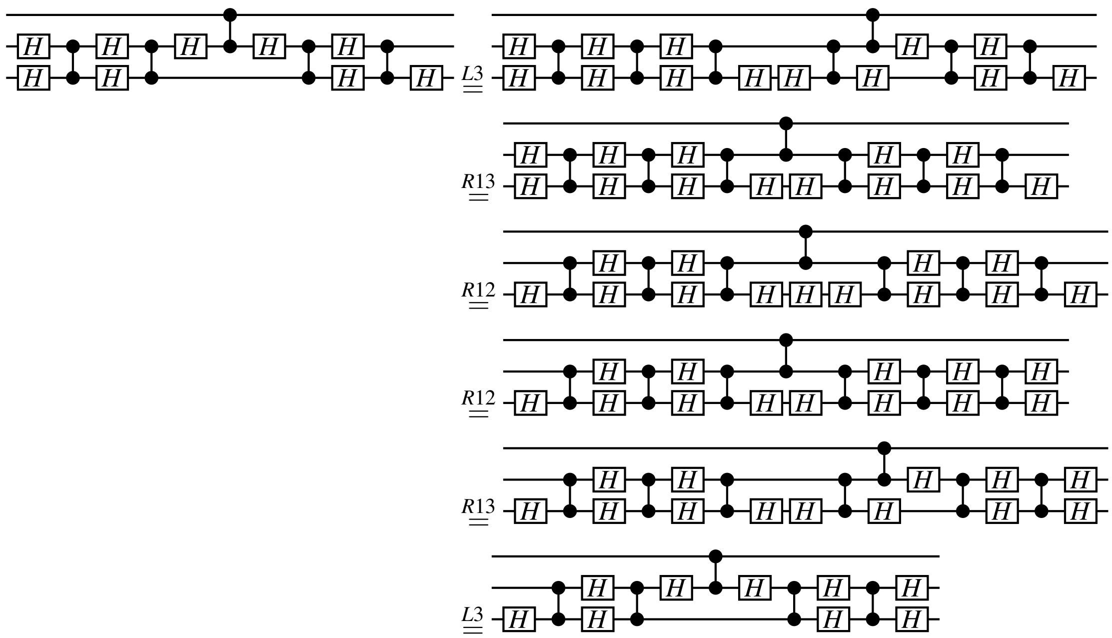

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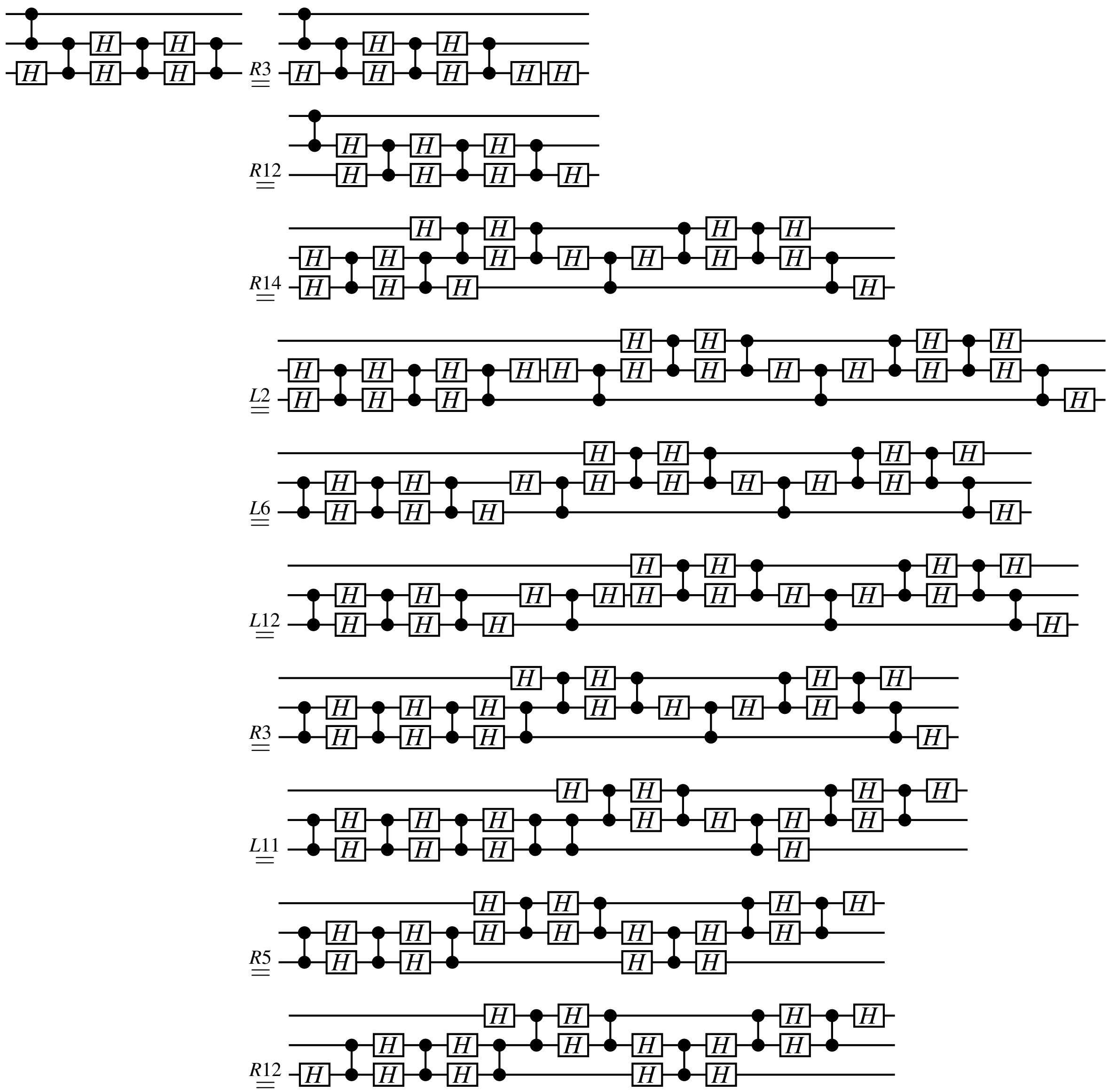

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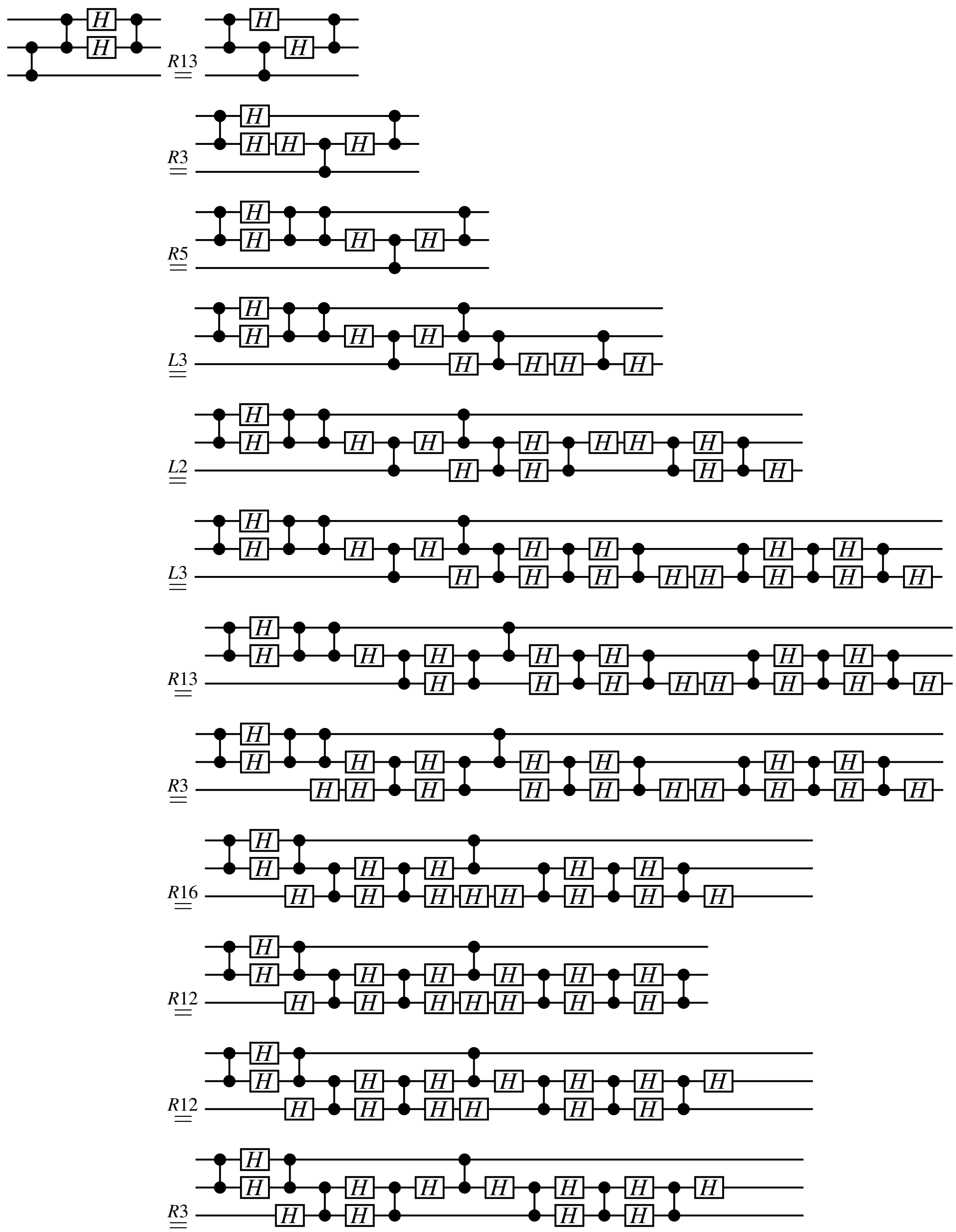

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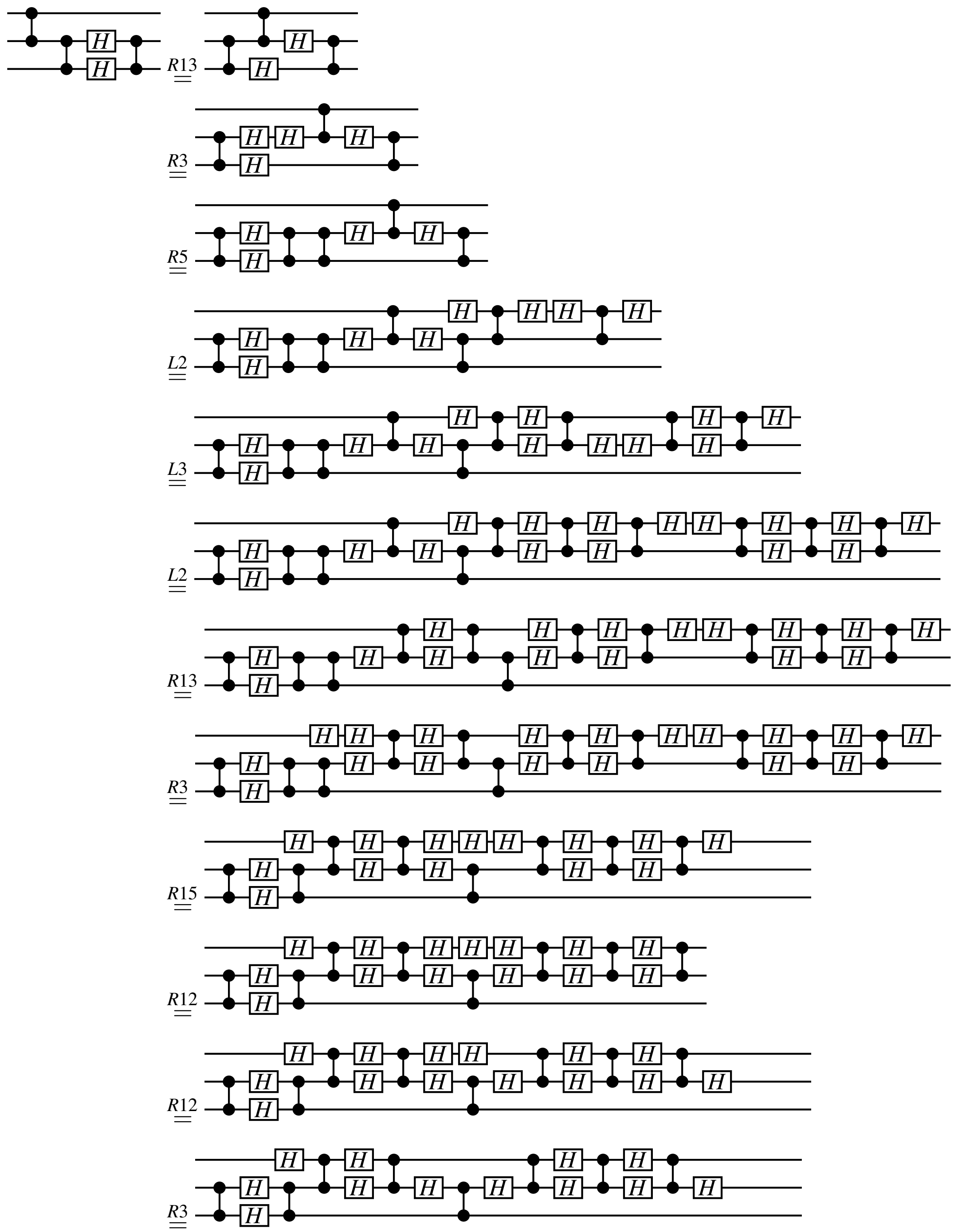

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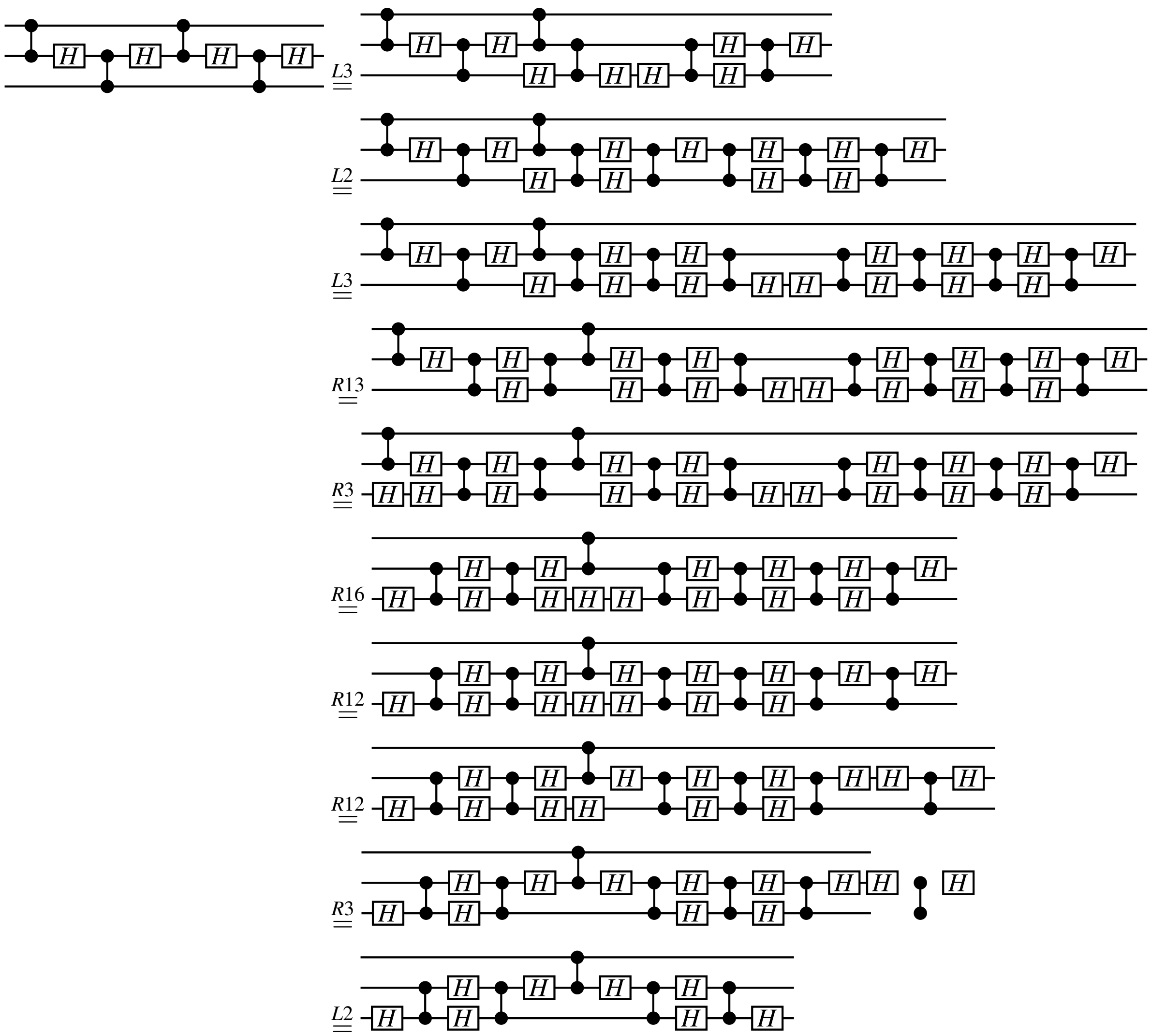

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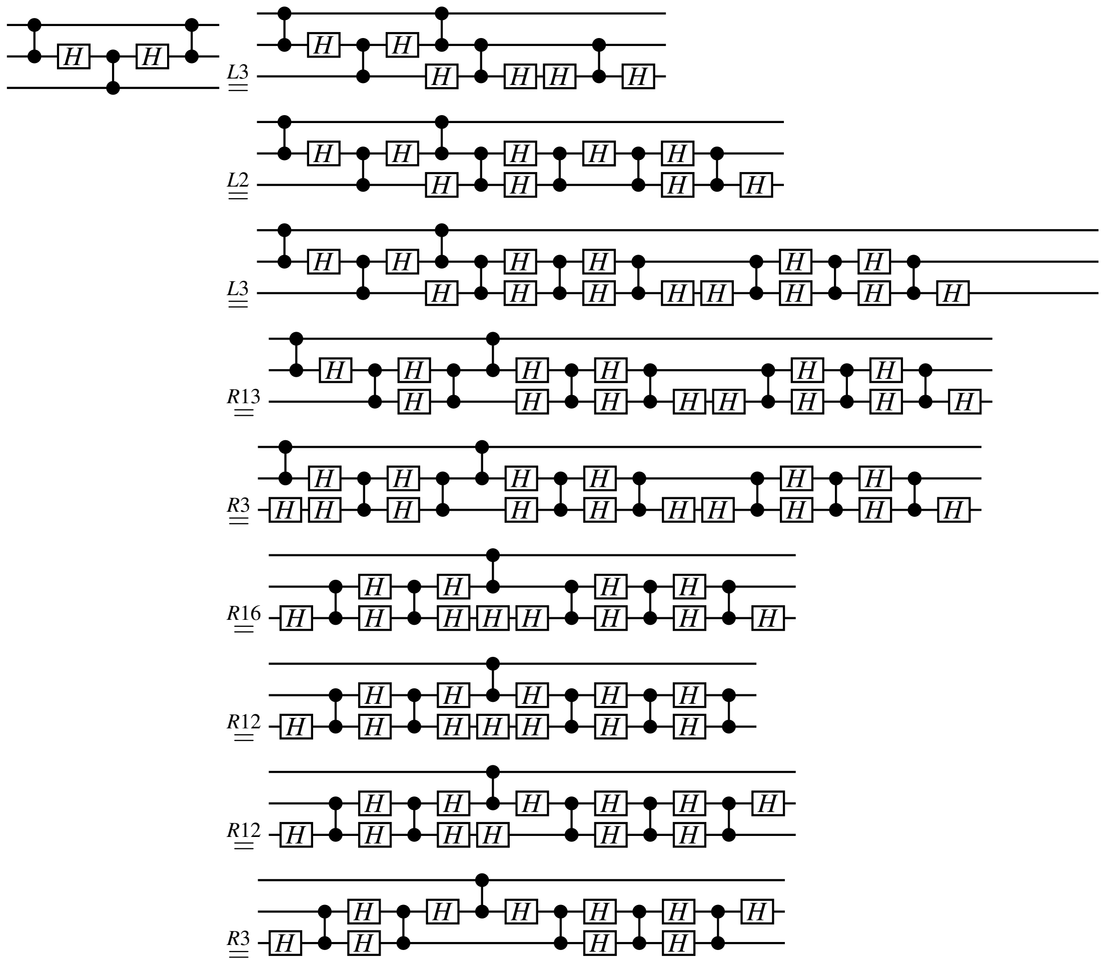

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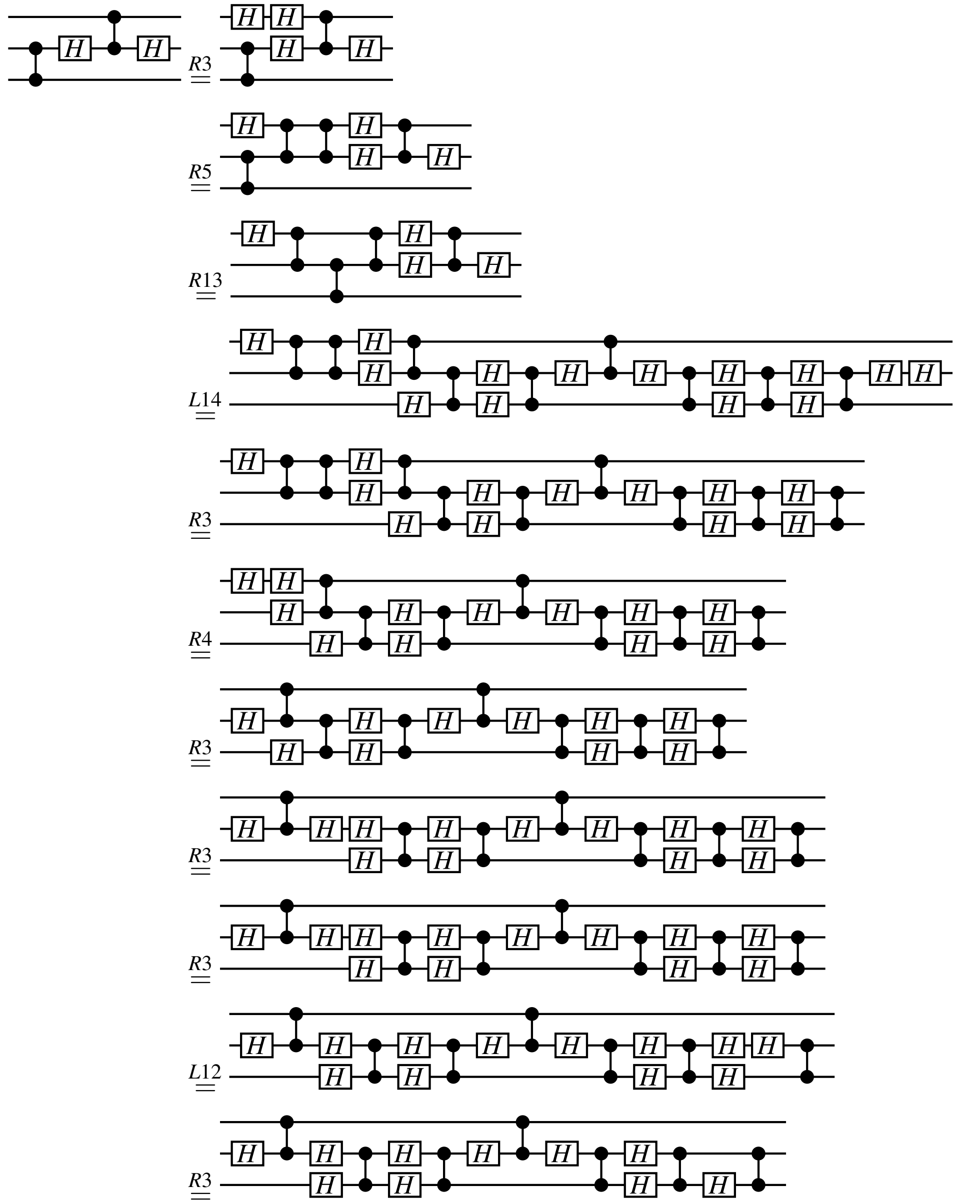

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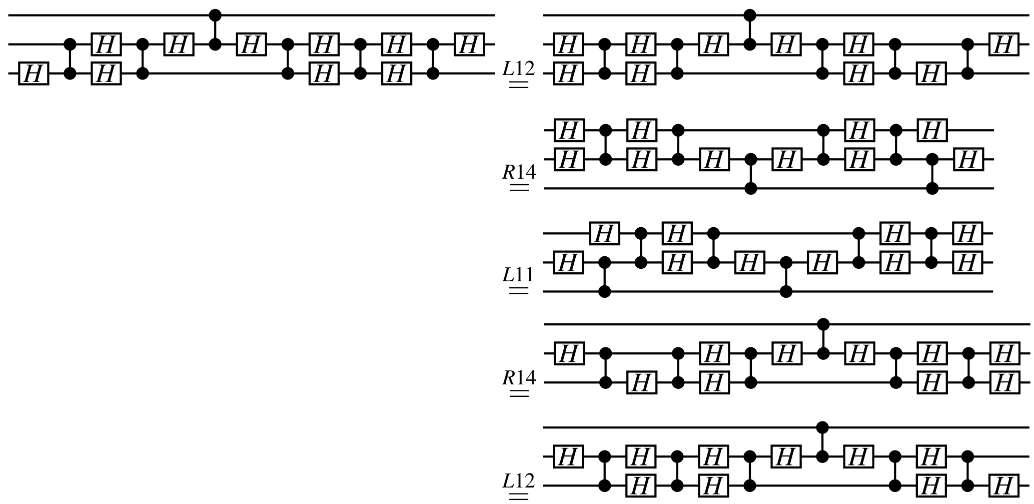
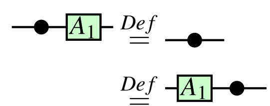


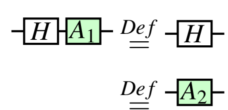
1.3 Proofs of Rewrite Rules

Figure 3:

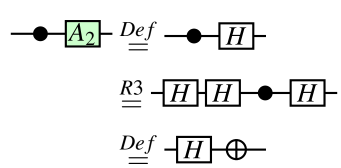
3.1



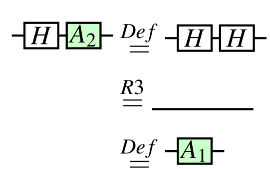
3.2



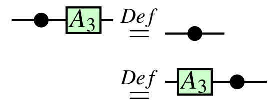
3.3



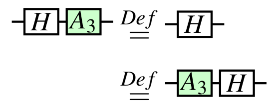
3.4



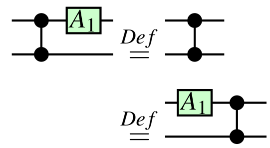
3.5


A_3
$\overset{Def}{=}$
$\overset{Def}{=}$
A_3

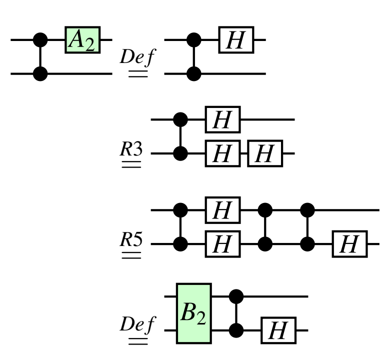
3.6


H
A_3
$\overset{Def}{=}$
H
$\overset{Def}{=}$
A_3
H

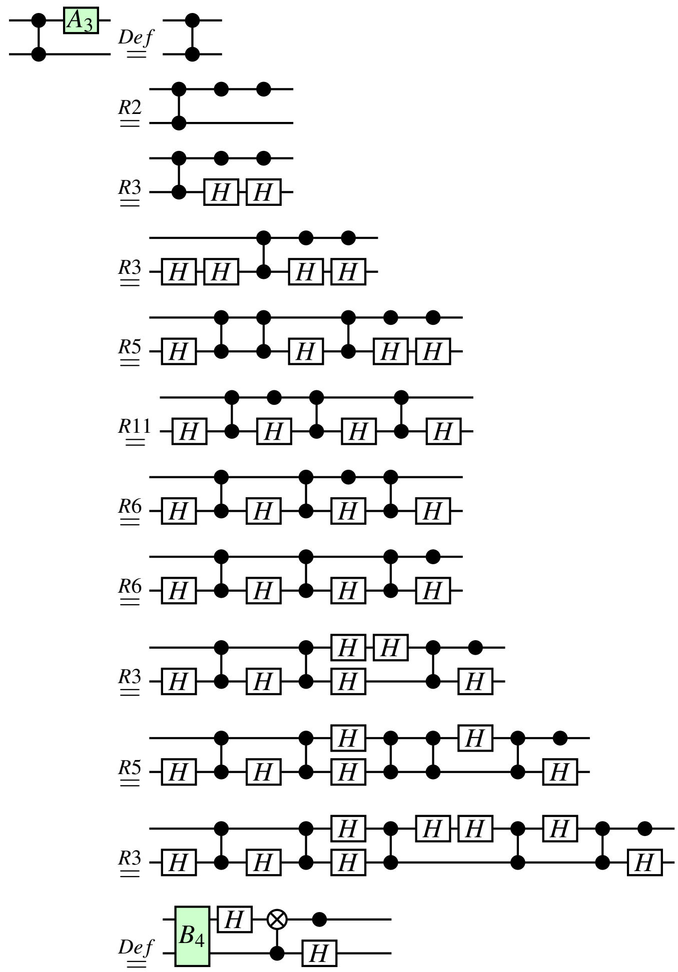
3.7


A_1
$\overset{Def}{=}$
$\overset{Def}{=}$
A_1

3.8

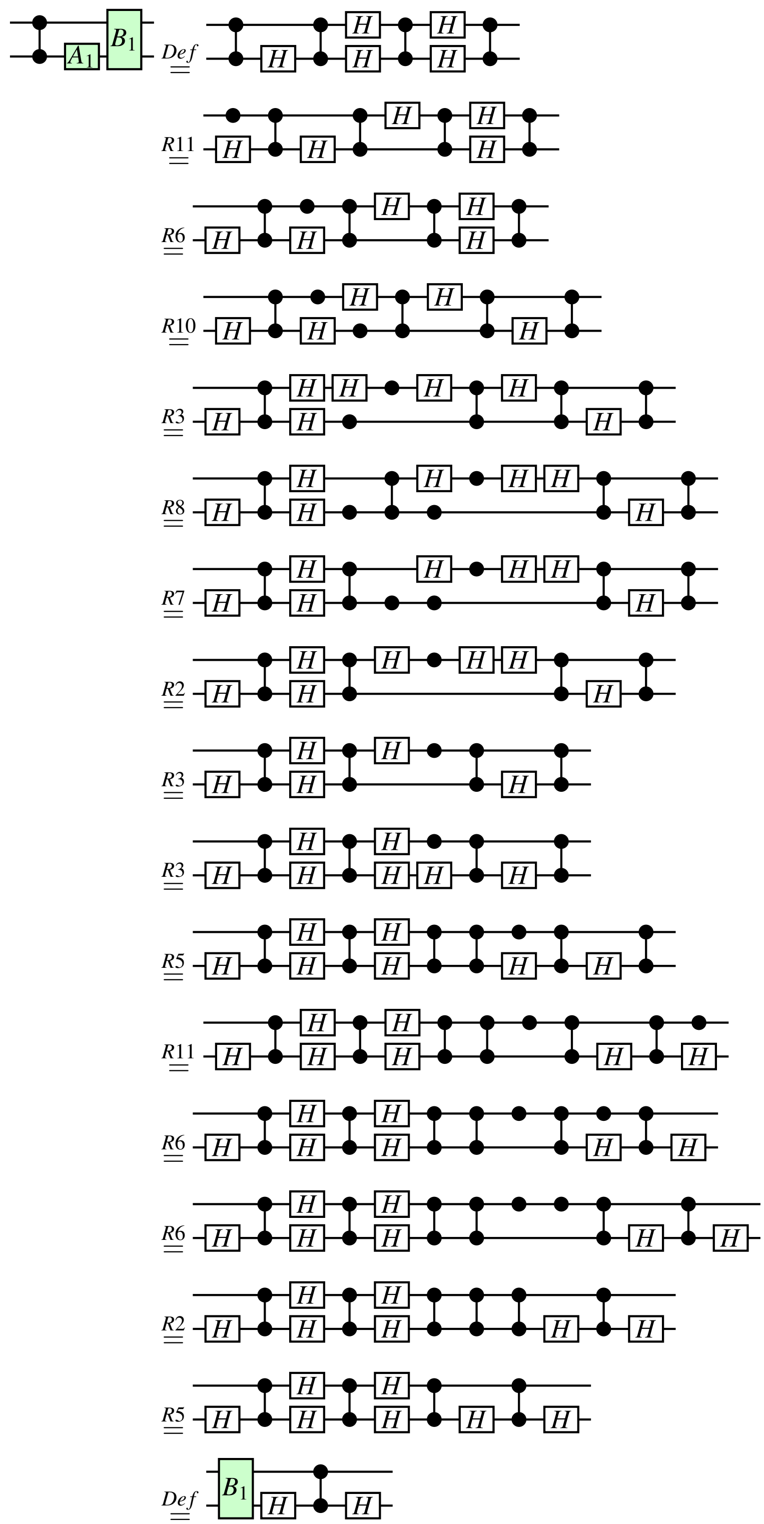

A_2
$\overset{Def}{=}$
H
$\overset{R3}{=}$
H
H
H
$\overset{R5}{=}$
H
H
H
$\overset{Def}{=}$
B_2
H

3.9

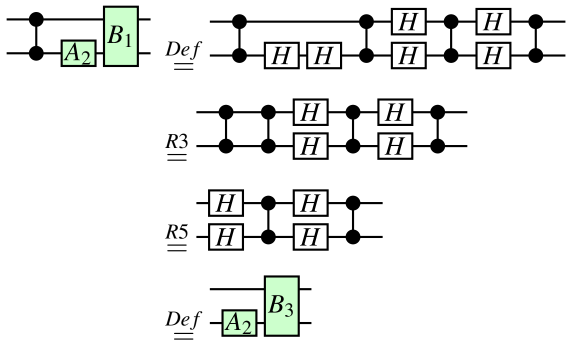


A_3
$\overset{Def}{\equiv}$
$\overset{R2}{\equiv}$
$\overset{R3}{\equiv}$
$\overset{R3}{\equiv}$
$\overset{R5}{\equiv}$
$\overset{R11}{\equiv}$
$\overset{R6}{\equiv}$
$\overset{R6}{\equiv}$
$\overset{R3}{\equiv}$
$\overset{R5}{\equiv}$
$\overset{R3}{\equiv}$
$\overset{Def}{\equiv}$
B_4

3.10



3.11



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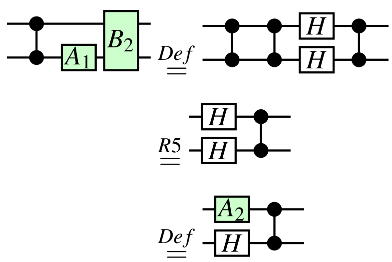

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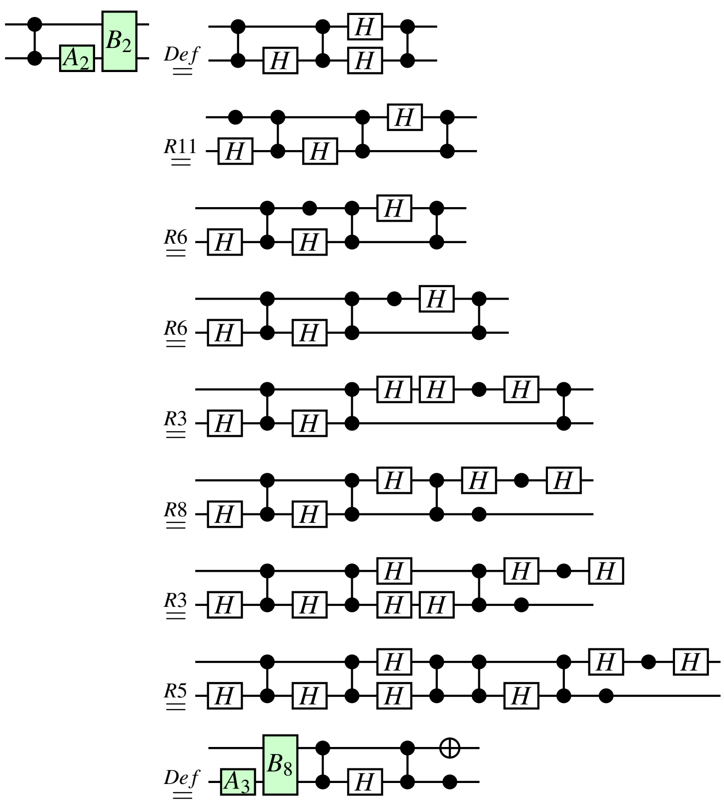

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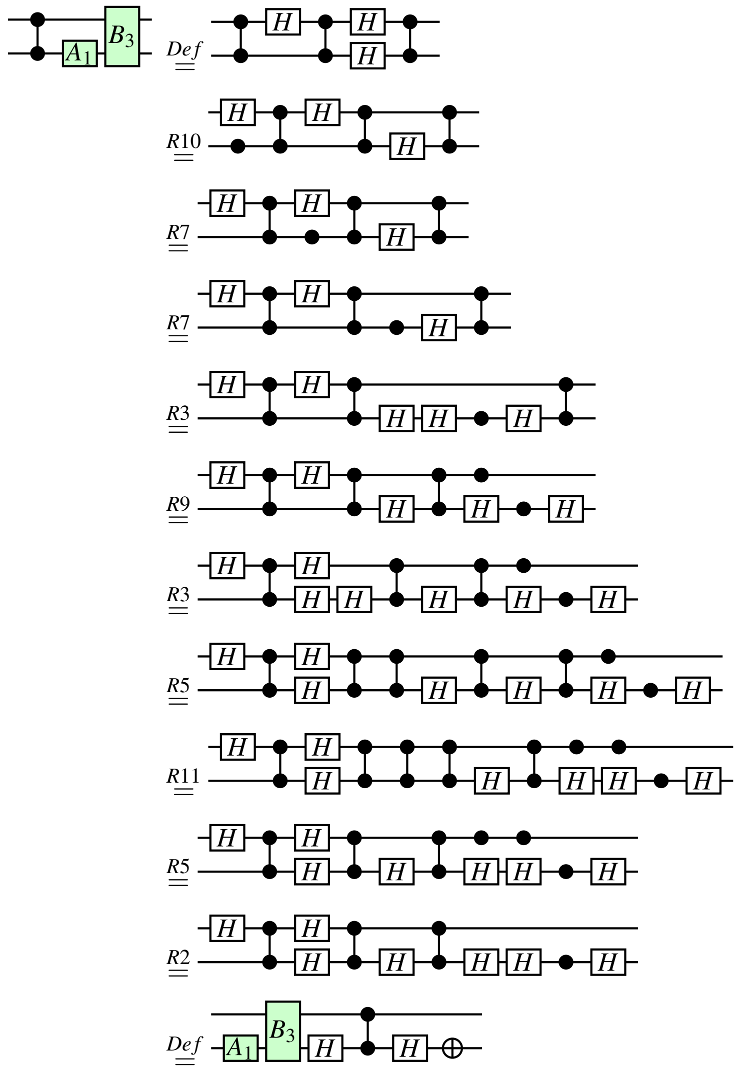

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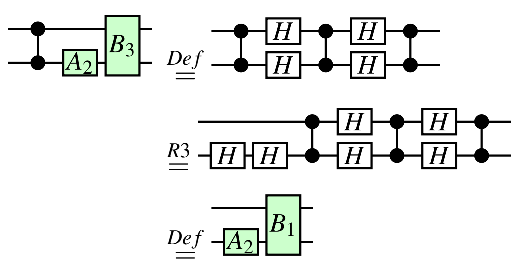

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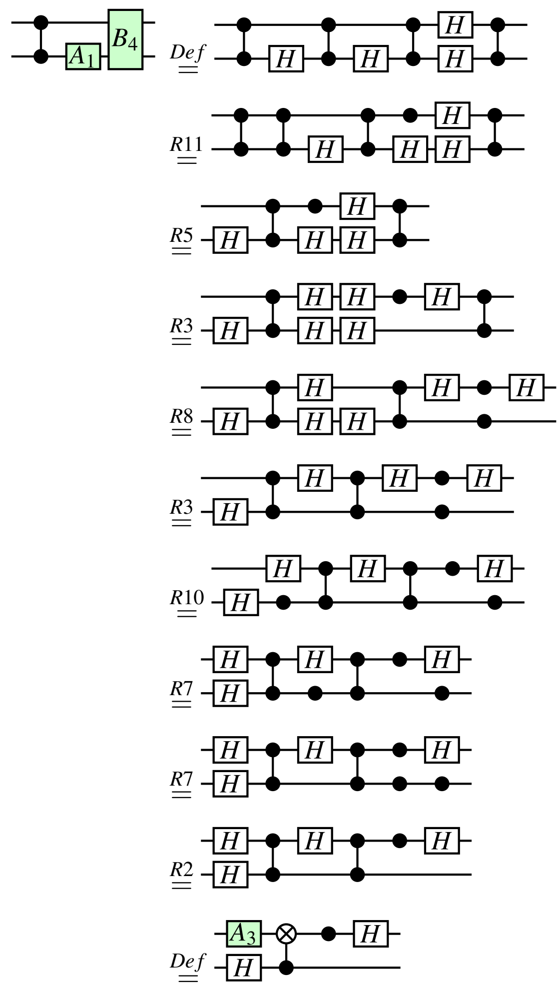

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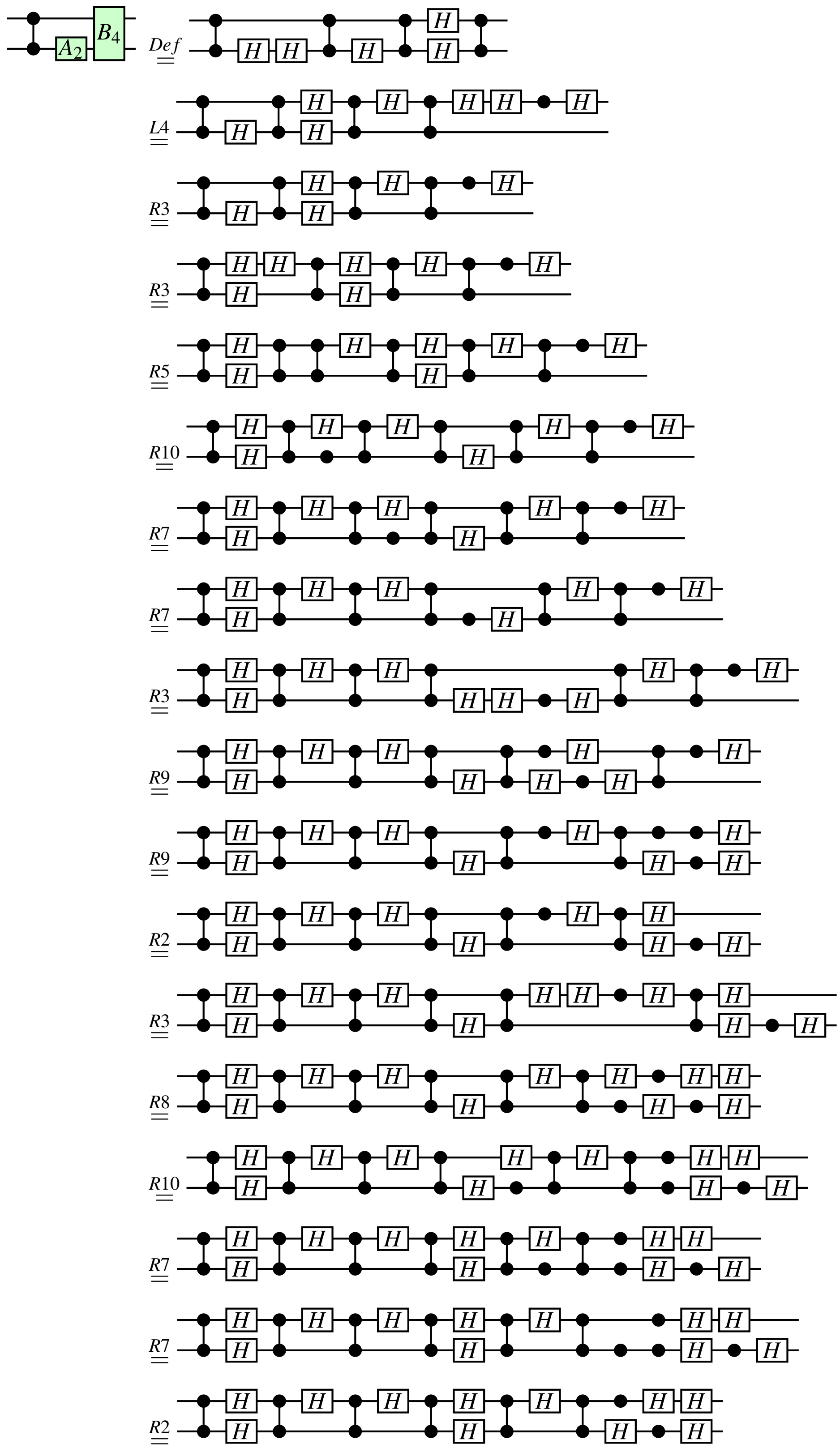
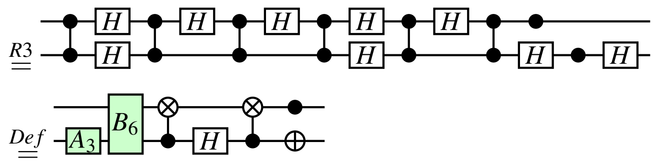
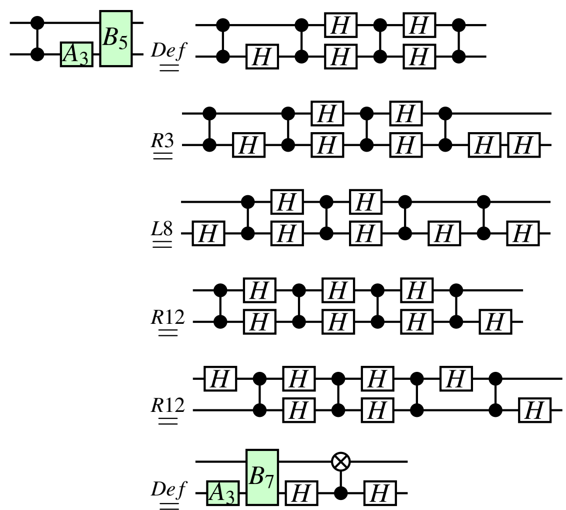


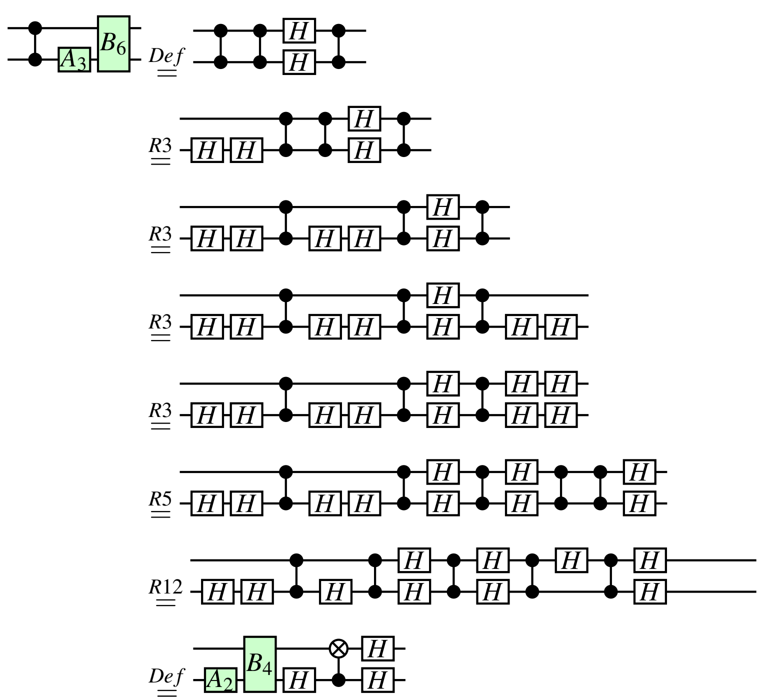
A_2
B_4
$\stackrel{Def}{=}$
$\stackrel{L4}{=}$
$\stackrel{R3}{=}$
$\stackrel{R3}{=}$
$\stackrel{R5}{=}$
$\stackrel{R10}{=}$
$\stackrel{R7}{=}$
$\stackrel{R7}{=}$
$\stackrel{R3}{=}$
$\stackrel{R9}{=}$
$\stackrel{R9}{=}$
$\stackrel{R2}{=}$
$\stackrel{R3}{=}$
$\stackrel{R8}{=}$
$\stackrel{R10}{=}$
$\stackrel{R7}{=}$
$\stackrel{R7}{=}$
$\stackrel{R2}{=}$
H


R3
Def
A3
B6

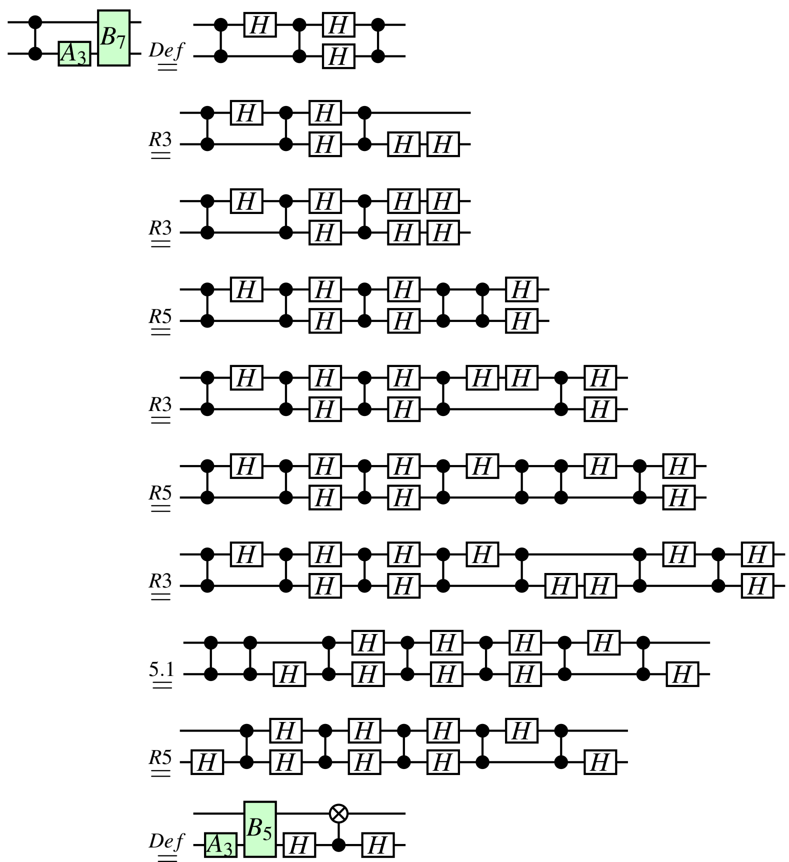
3.18


A3
B5
Def
R3
L8
R12
R12
Def
A3
B7

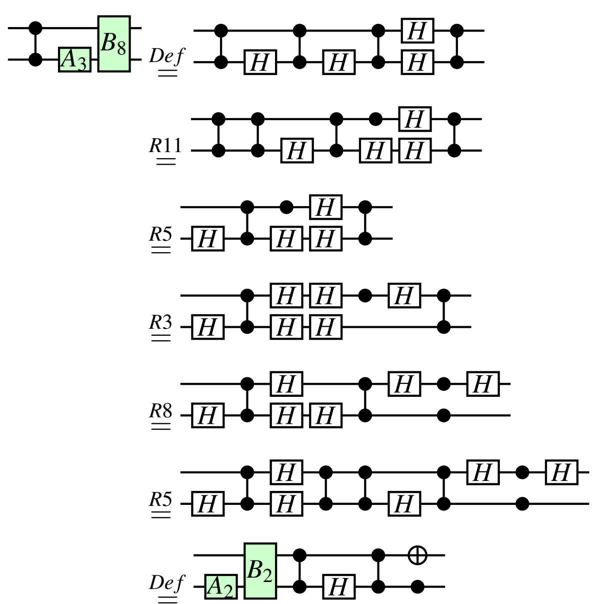
3.19


A3
B6
Def
R3
R3
R3
R3
R5
R12
Def
A2
B4

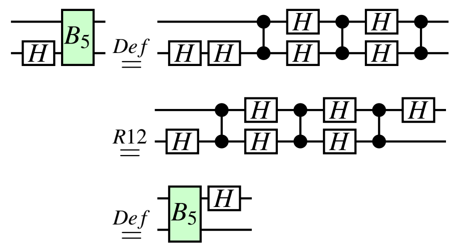
3.20


A3
B7
Def
R3
R3
R5
R3
R5
R3
5.1
R5
Def
A3
B5

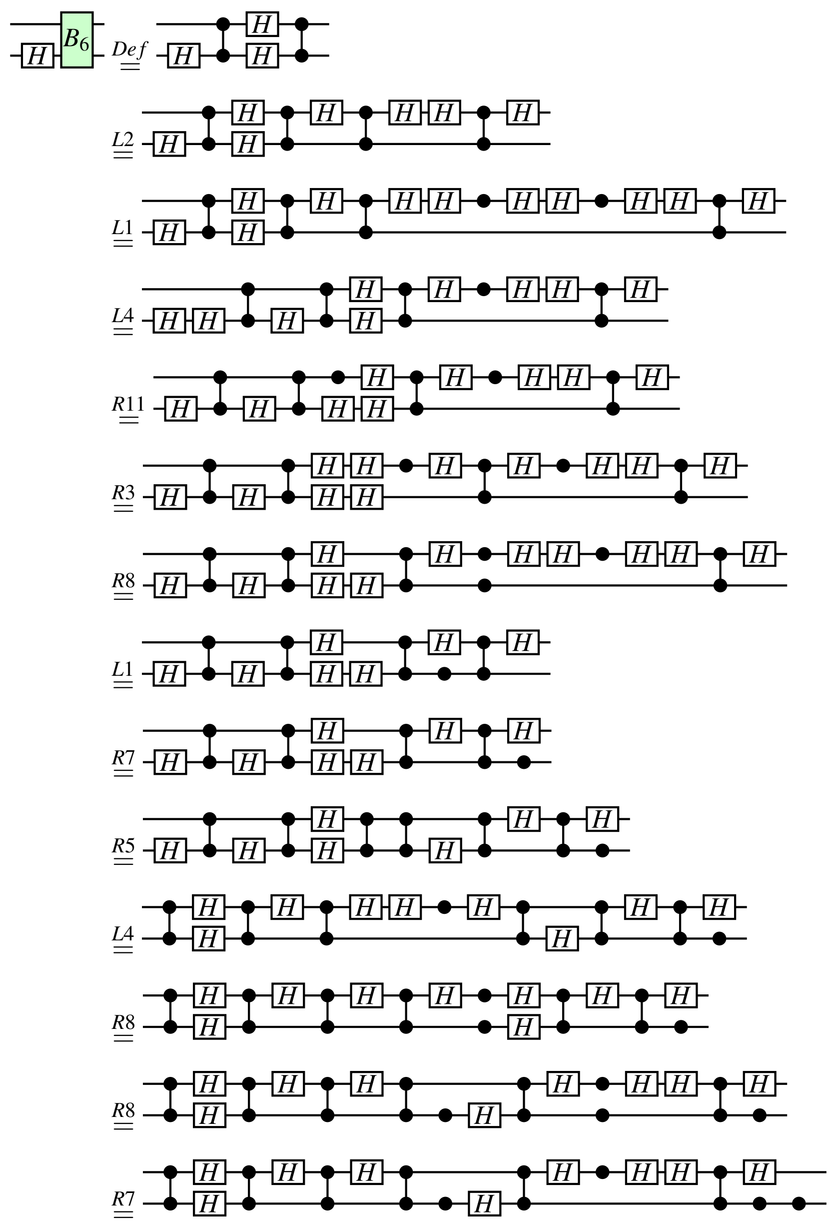
3.21

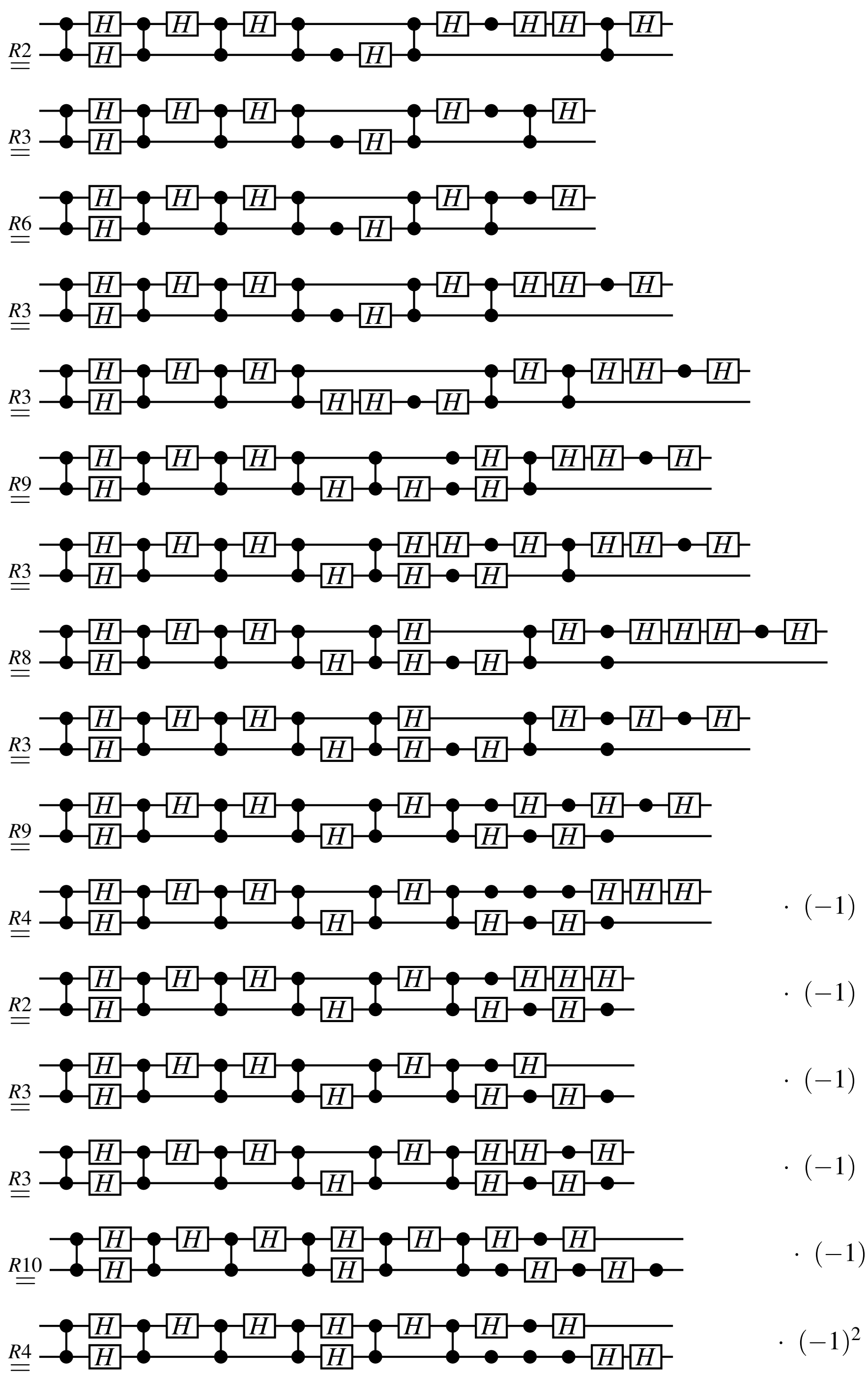

A3
B8
Def
R11
R5
R3
R8
R5
Def
A2
B2

3.22

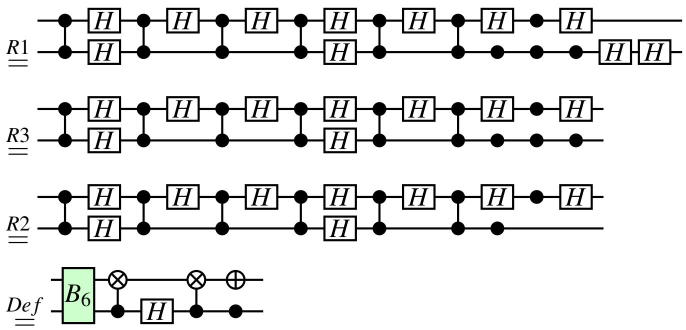


3.23

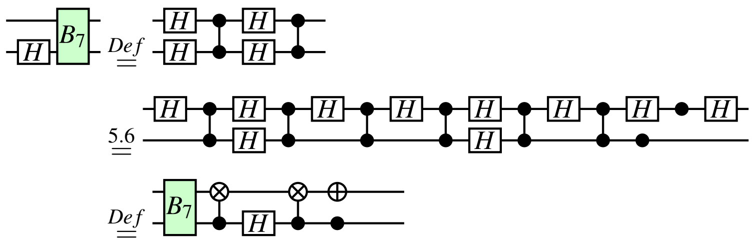




R2
R3
R6
R3
R3
R9
R3
R8
R3
R9
R4 · (−1)
R2 · (−1)
R3 · (−1)
R3 · (−1)
R10 · (−1)
R4 · (−1)²



3.24



3.25

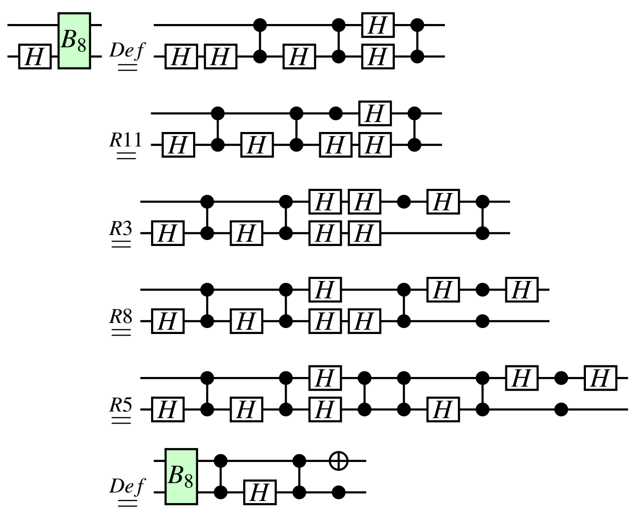
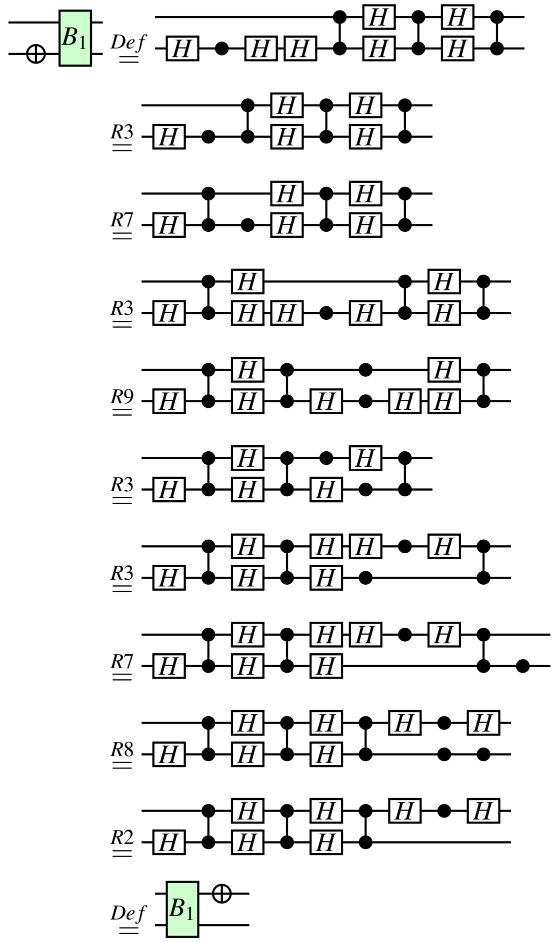
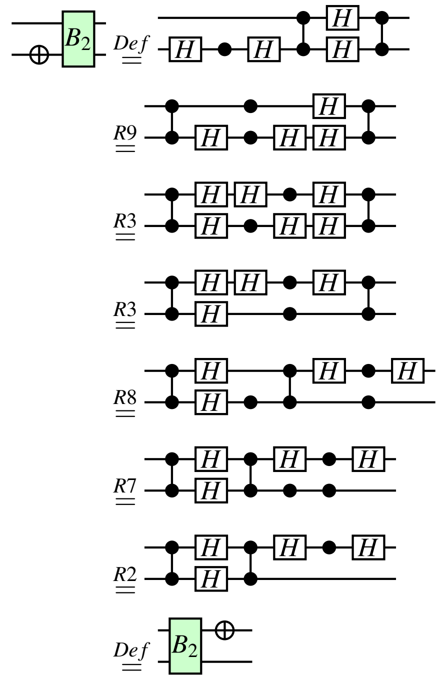


Figure 4:

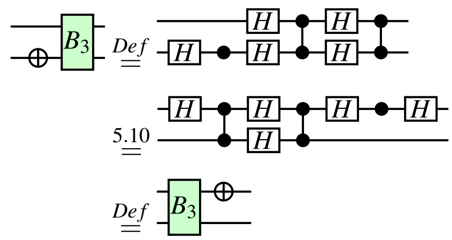
4.1



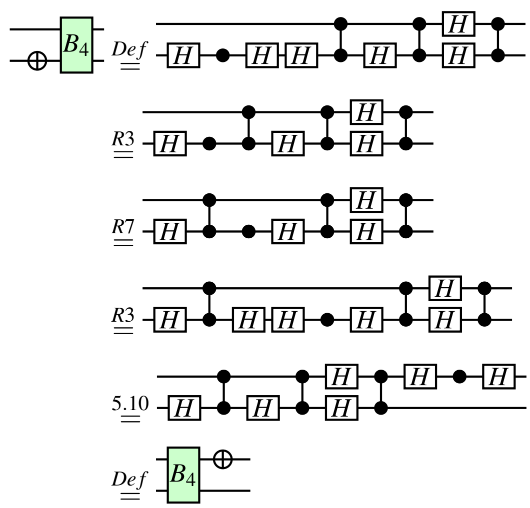
4.2


B_2
Def
R9
R3
R3
R8
R7
R2
Def
B_2

4.3

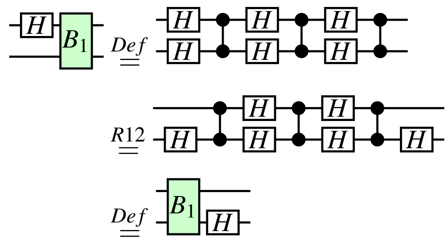

B_3
Def
5.10
Def
B_3

4.4

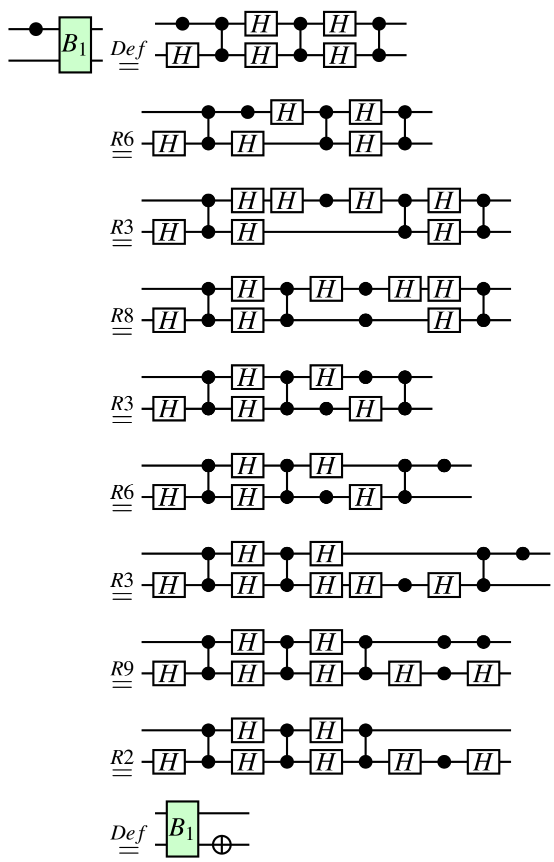

B_4
Def
R3
R7
R3
5.10
Def
B_4

The proofs of 4.5–4.8 are the same as those of 4.1–4.4.

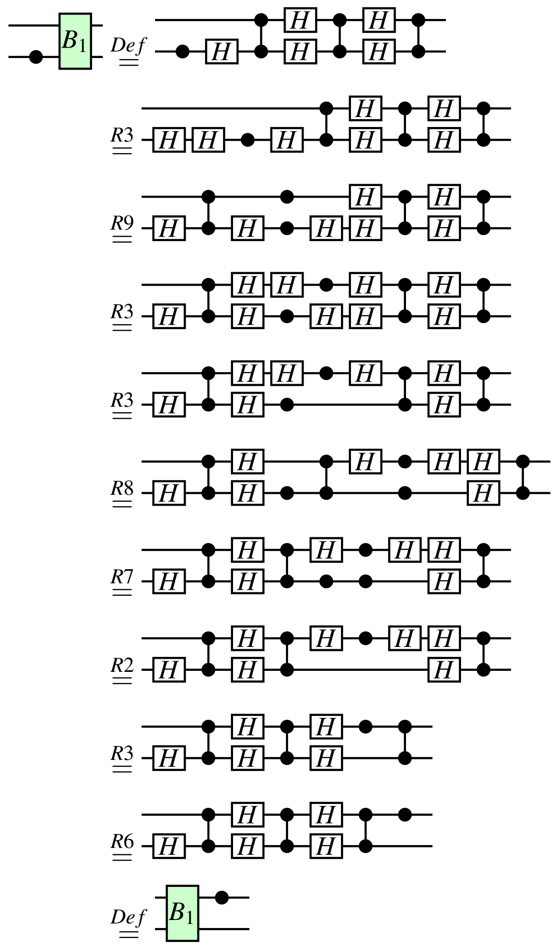
4.9



4.10

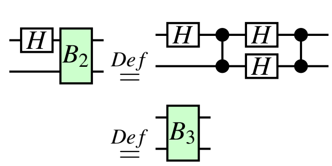


4.11



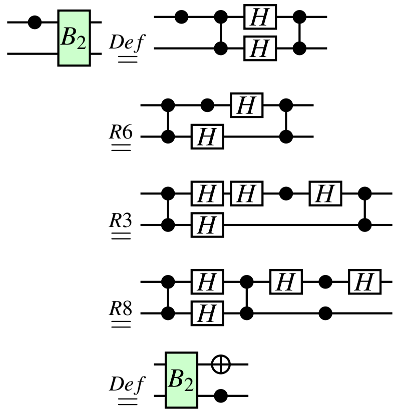
B_1
$\stackrel{Def}{=}$
H
$\stackrel{R3}{=}$
$\stackrel{R9}{=}$
$\stackrel{R3}{=}$
$\stackrel{R3}{=}$
$\stackrel{R8}{=}$
$\stackrel{R7}{=}$
$\stackrel{R2}{=}$
$\stackrel{R3}{=}$
$\stackrel{R6}{=}$
$\stackrel{Def}{=}$
B_1

4.12

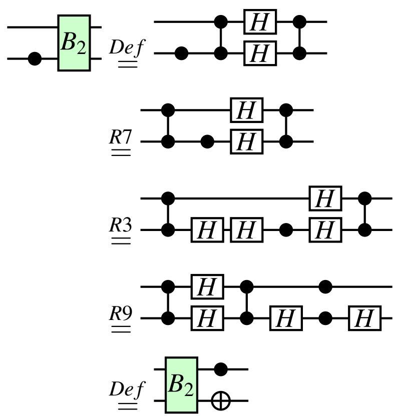


H
B_2
$\stackrel{Def}{=}$
H
$\stackrel{Def}{=}$
B_3

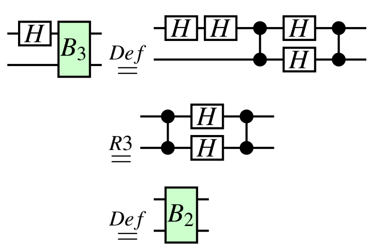
4.13


B2
Def
H
H
R6
H
H
R3
H
H
H
H
R8
H
H
H
H
Def
B2

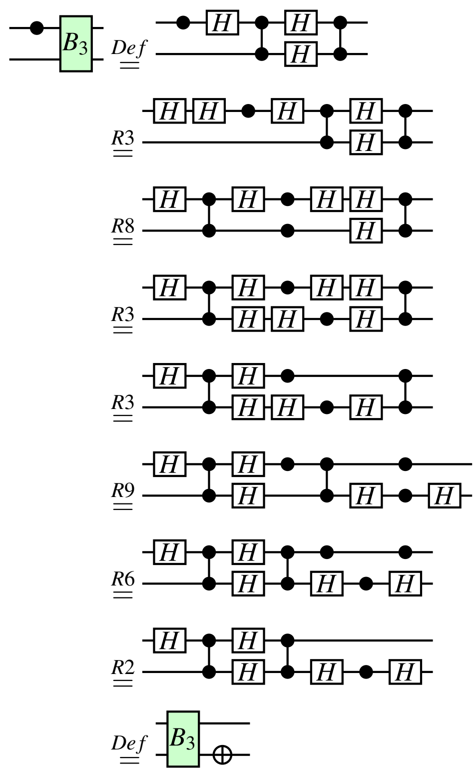
4.14


B2
Def
H
H
R7
H
H
R3
H
H
H
H
R9
H
H
H
H
Def
B2

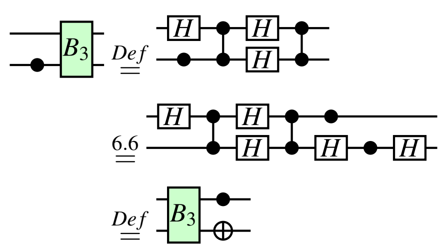
4.15


H
B3
Def
H
H
H
H
R3
H
H
Def
B2

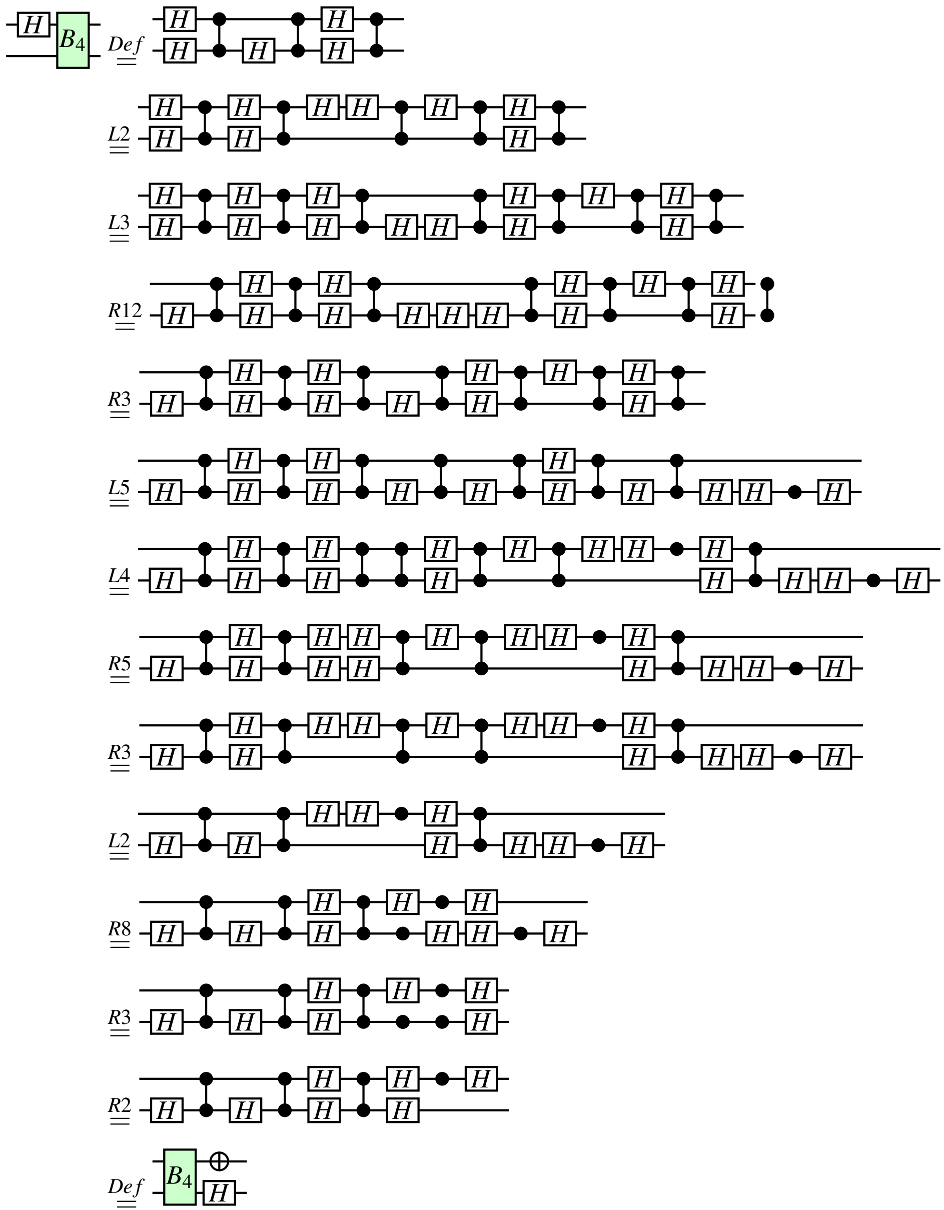
4.16



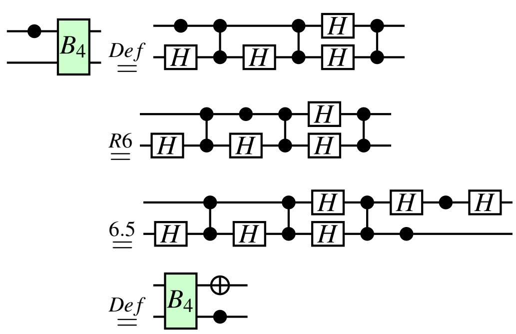
4.17



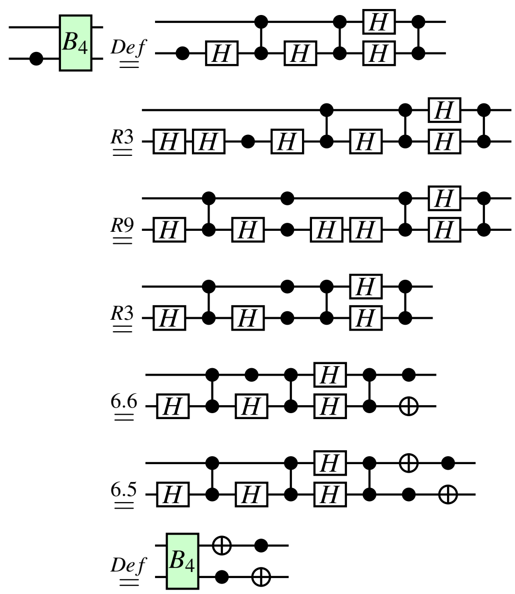
4.18



4.19

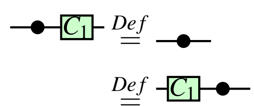


4.20

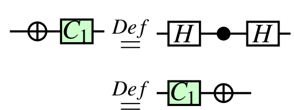


The proofs of 4.21–4.32 are the same as those of 4.9–4.20.

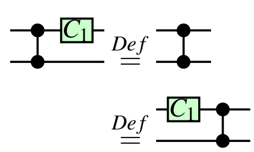
4.33



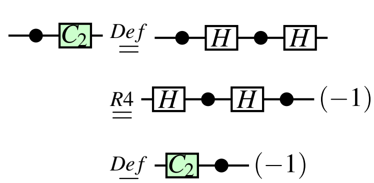
4.34



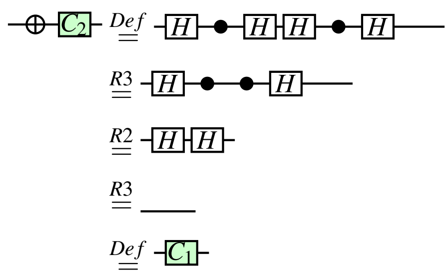
4.35



4.36



4.37



4.38

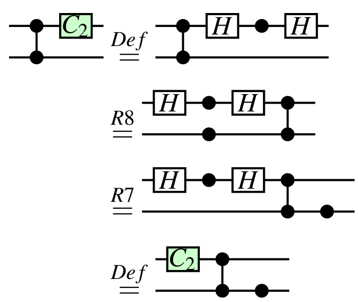
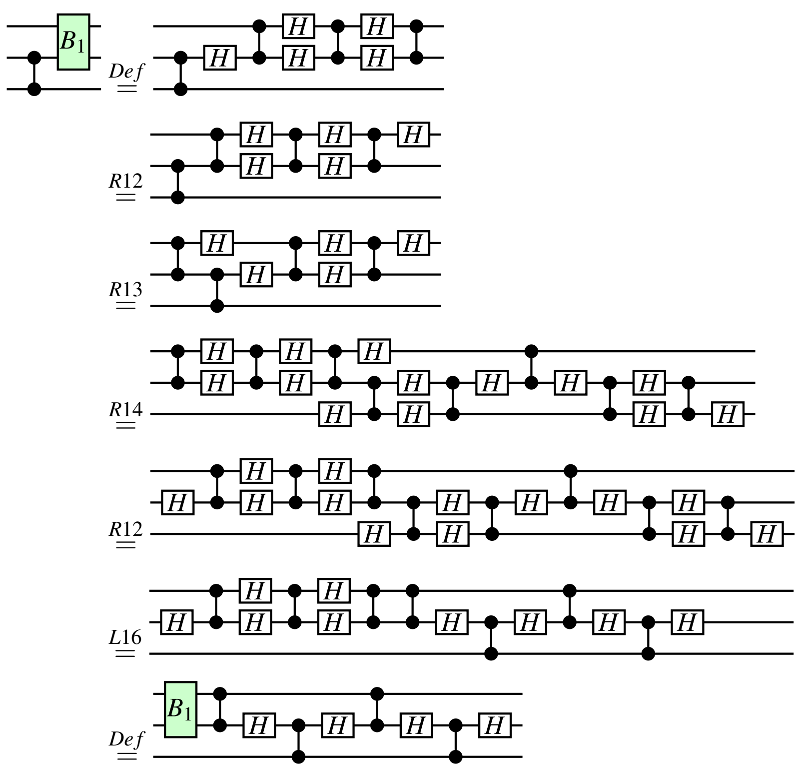
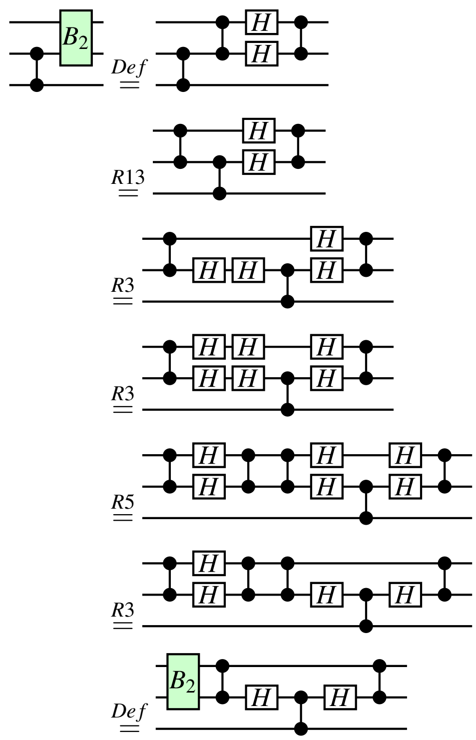


Figure 5:

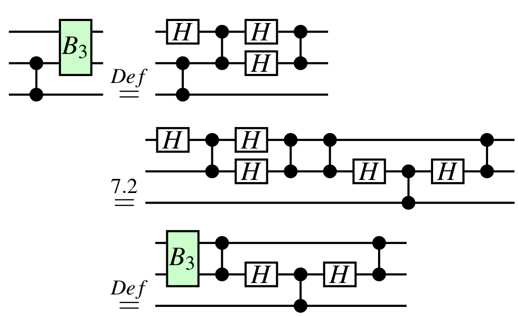
5.1



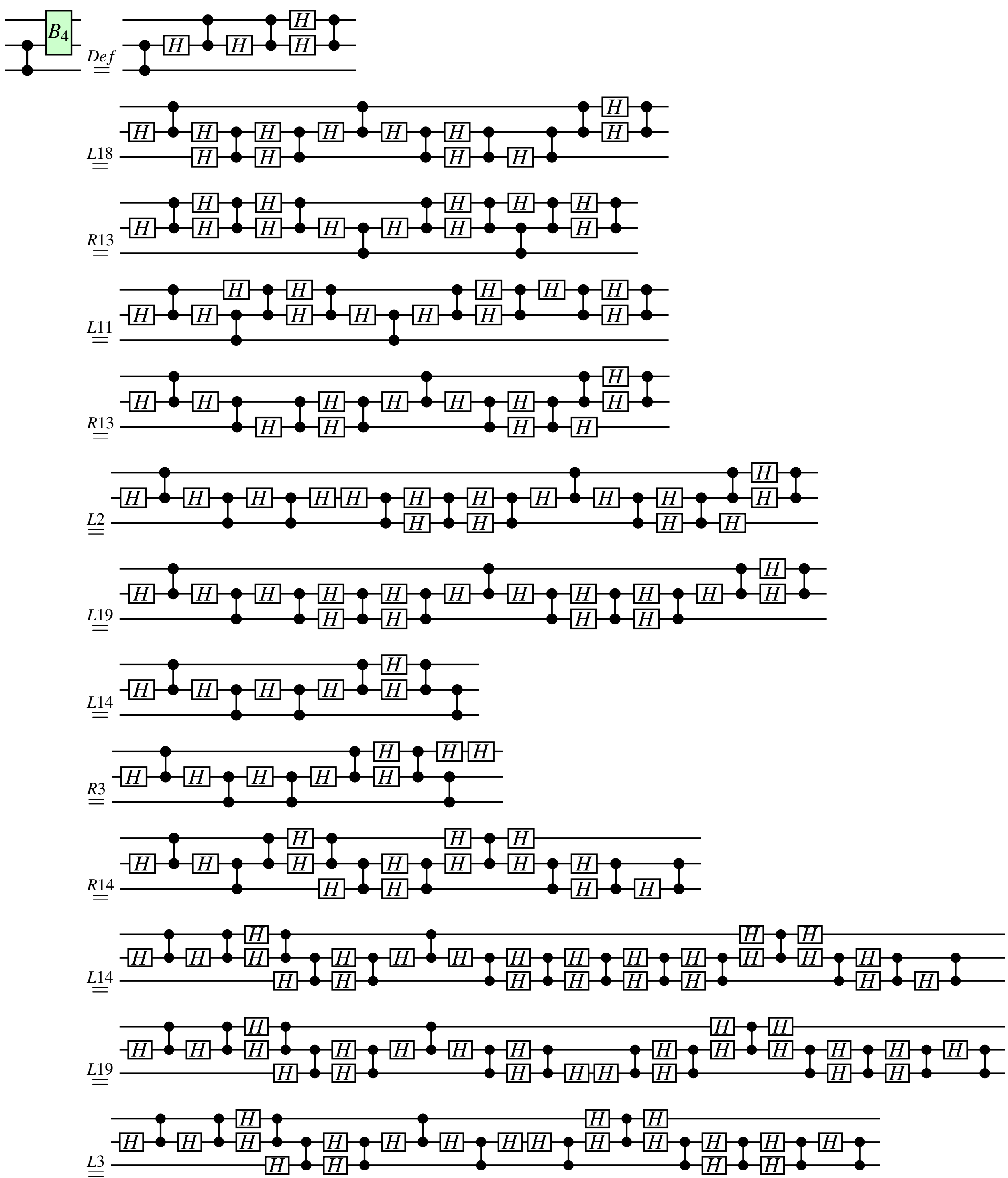
5.2

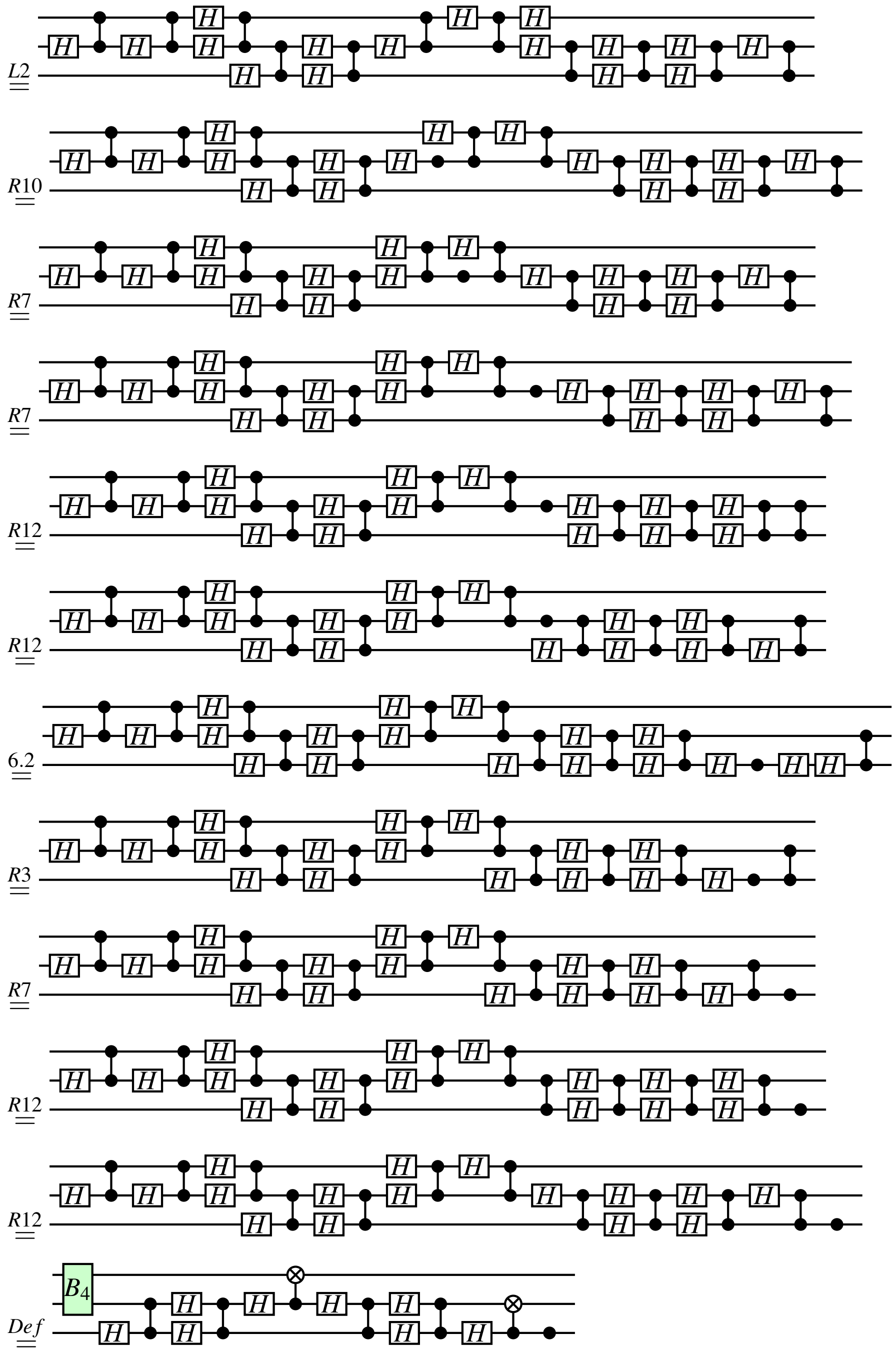


5.3

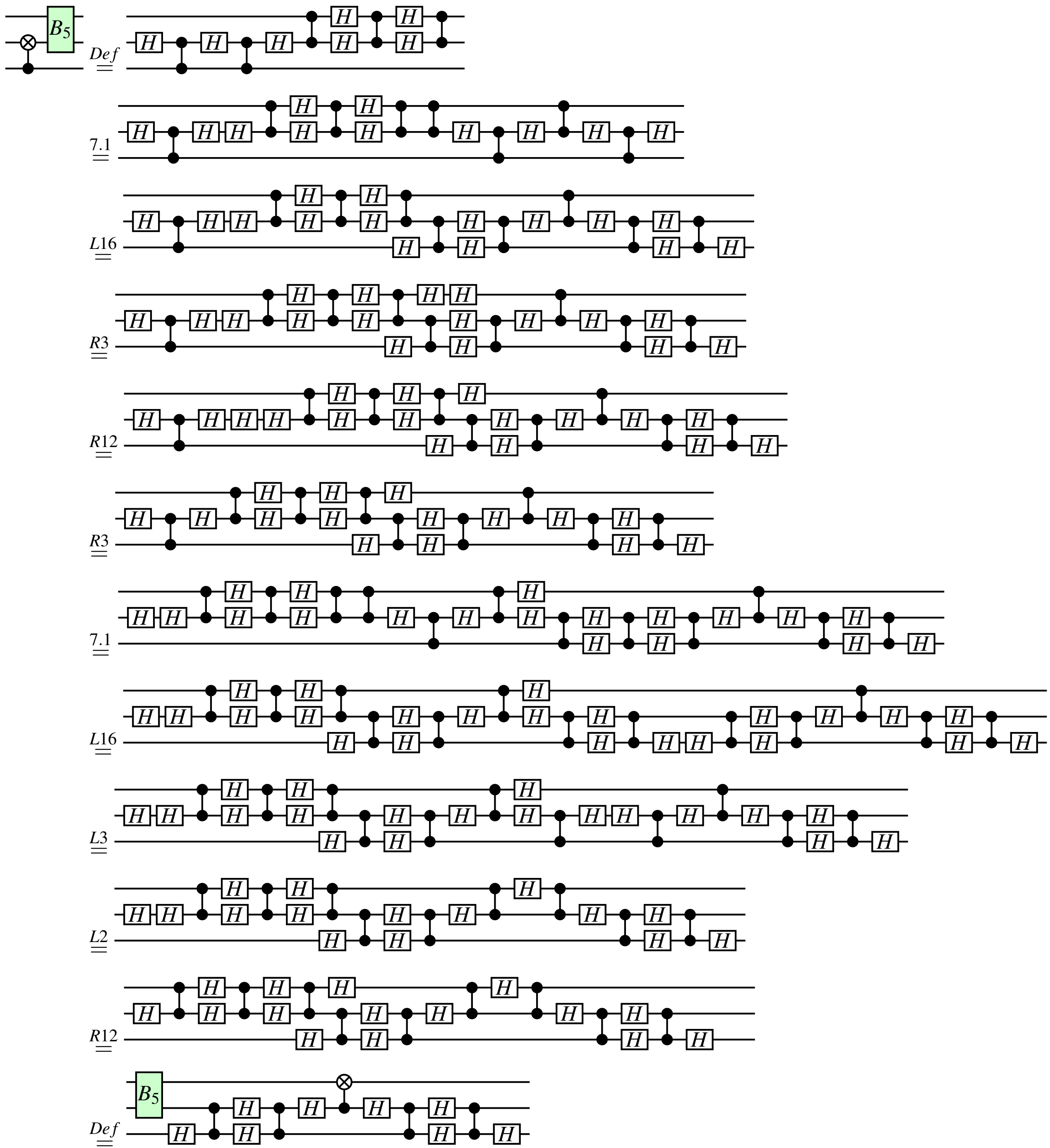


5.4

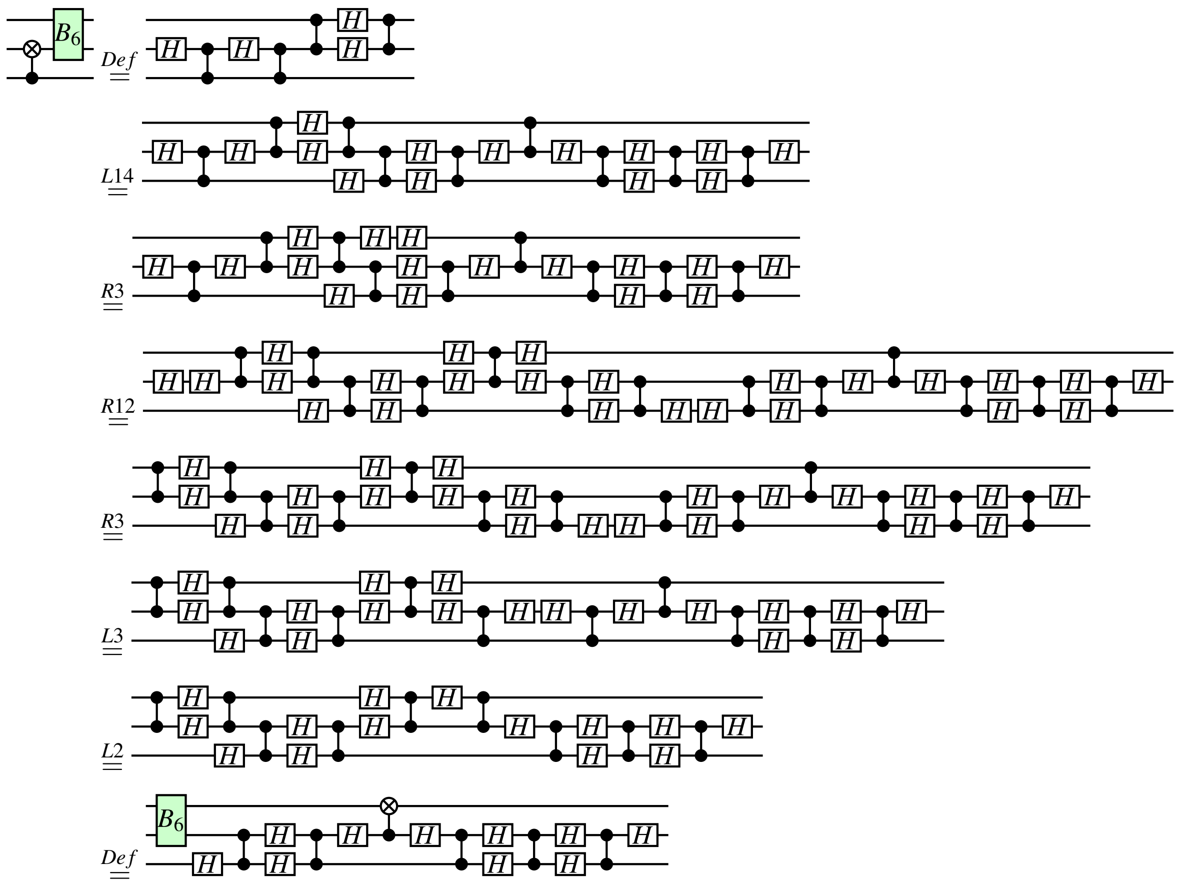



L2
R10
R7
R7
R12
R12
6.2
R3
R7
R12
R12
Def
B_4

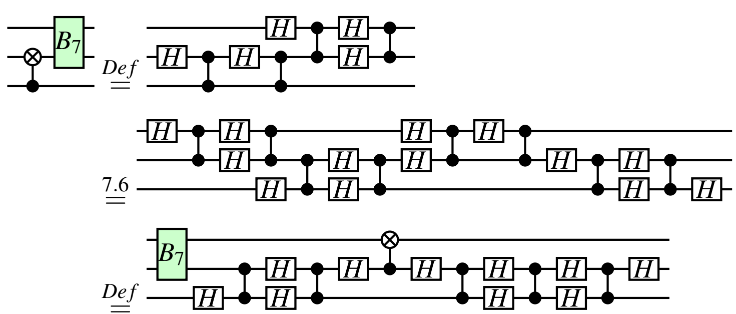
5.5


B_5
$\stackrel{Def}{=}$
$\stackrel{7.1}{=}$
$\stackrel{L16}{=}$
$\stackrel{R3}{=}$
$\stackrel{R12}{=}$
$\stackrel{R3}{=}$
$\stackrel{7.1}{=}$
$\stackrel{L16}{=}$
$\stackrel{L3}{=}$
$\stackrel{L2}{=}$
$\stackrel{R12}{=}$
$\stackrel{Def}{=}$
B_5

5.6



5.7



5.8

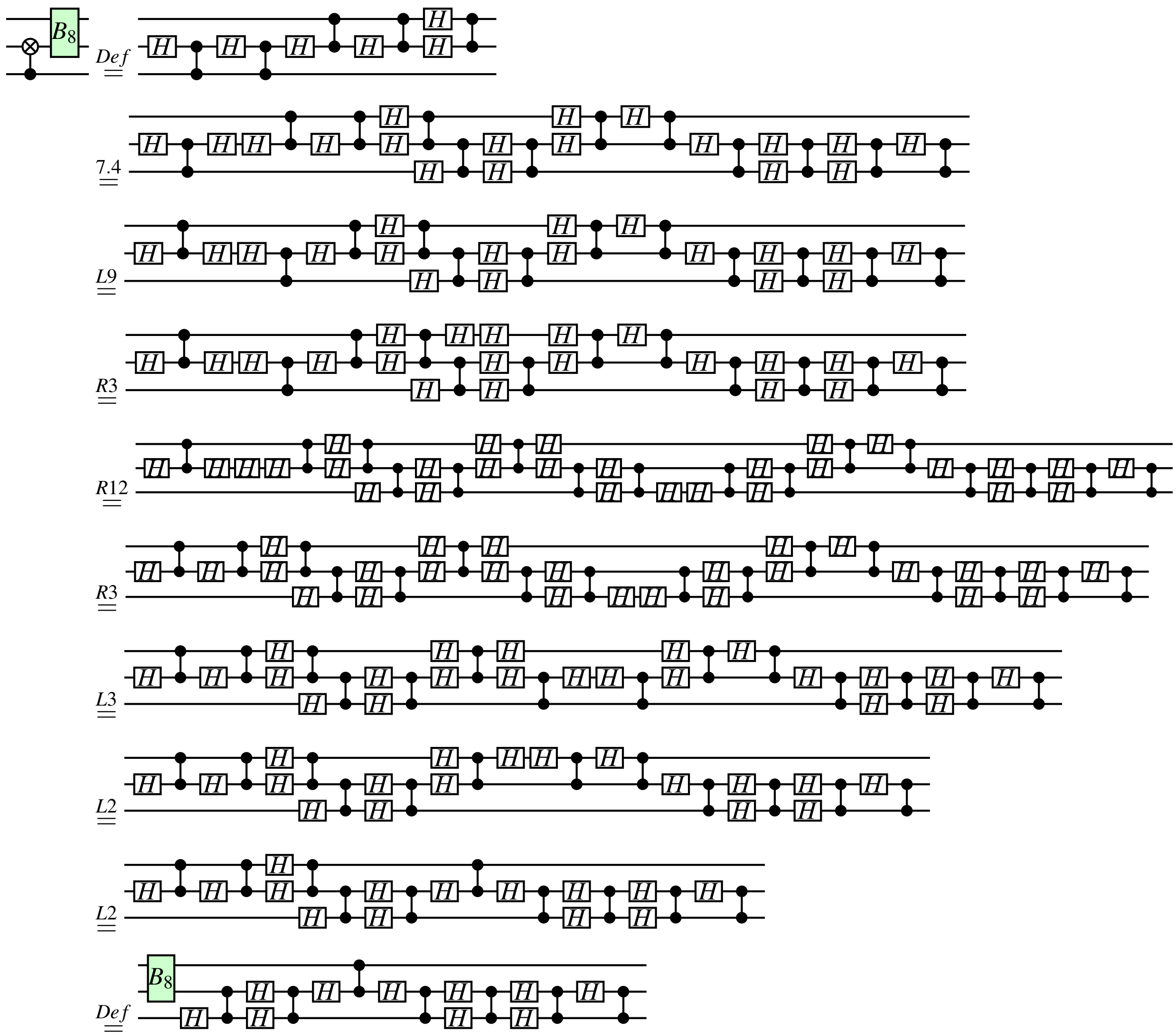
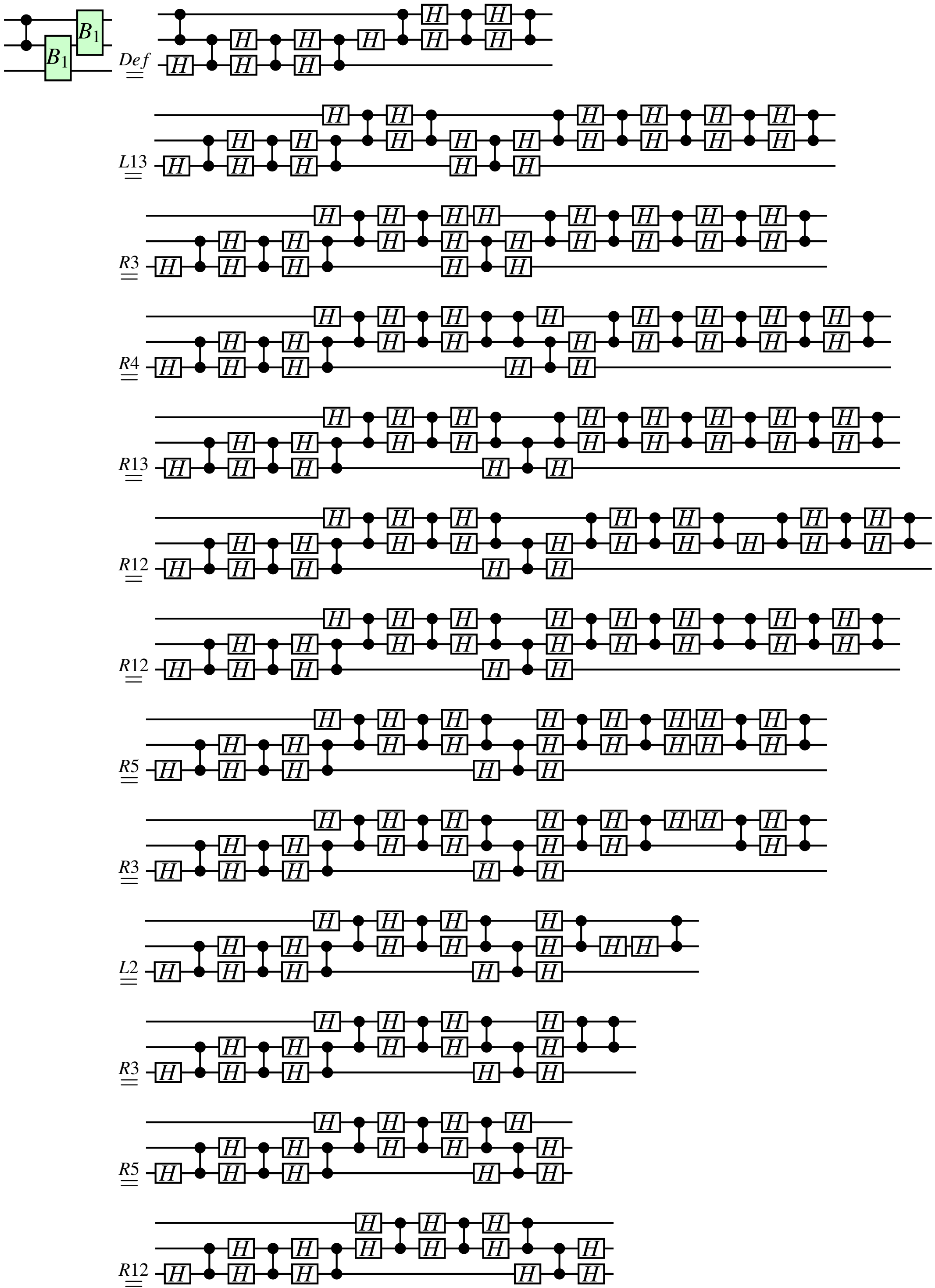
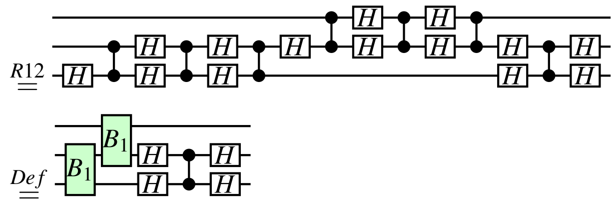


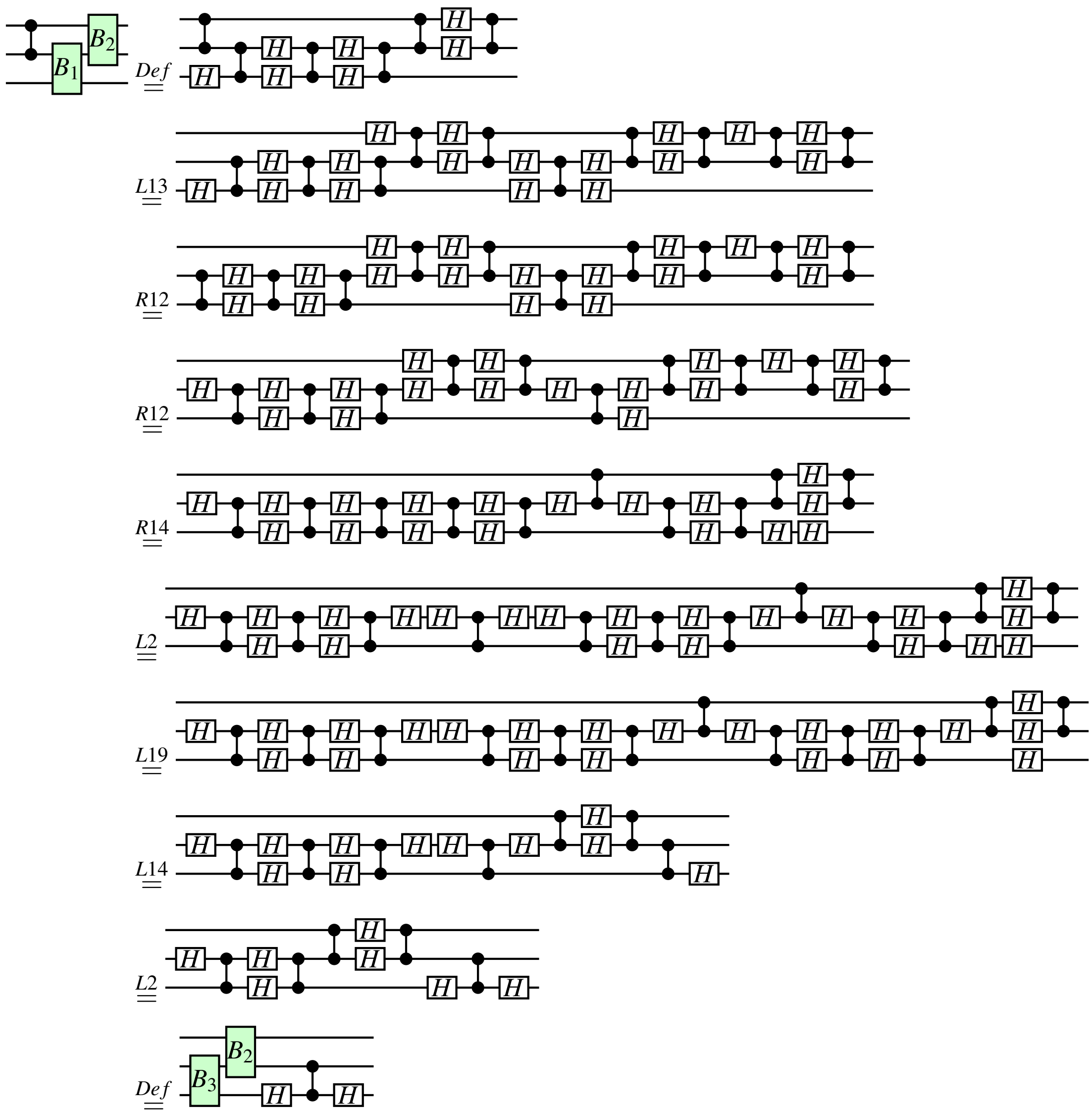
Figure 6:

6.1

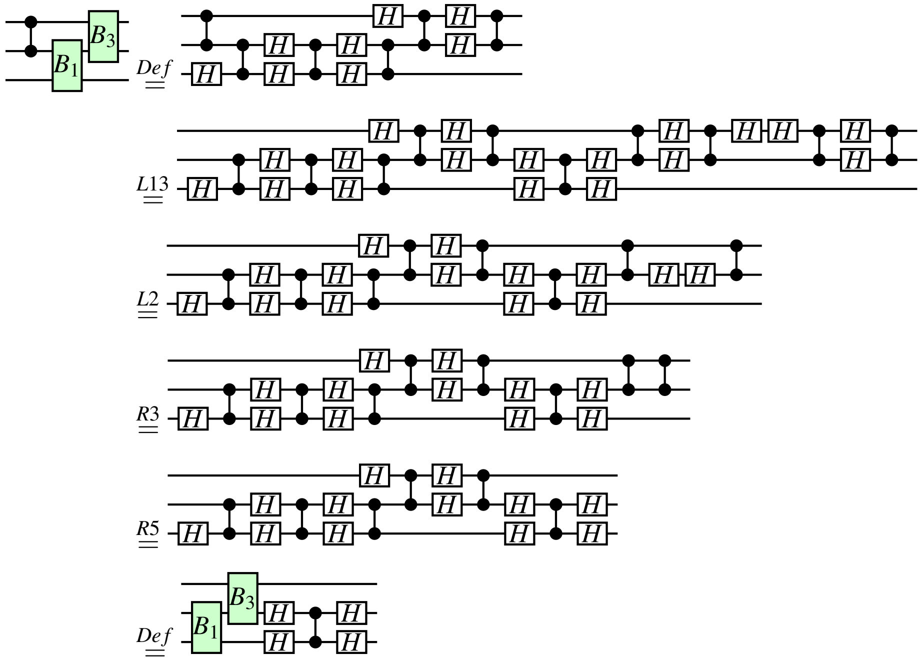




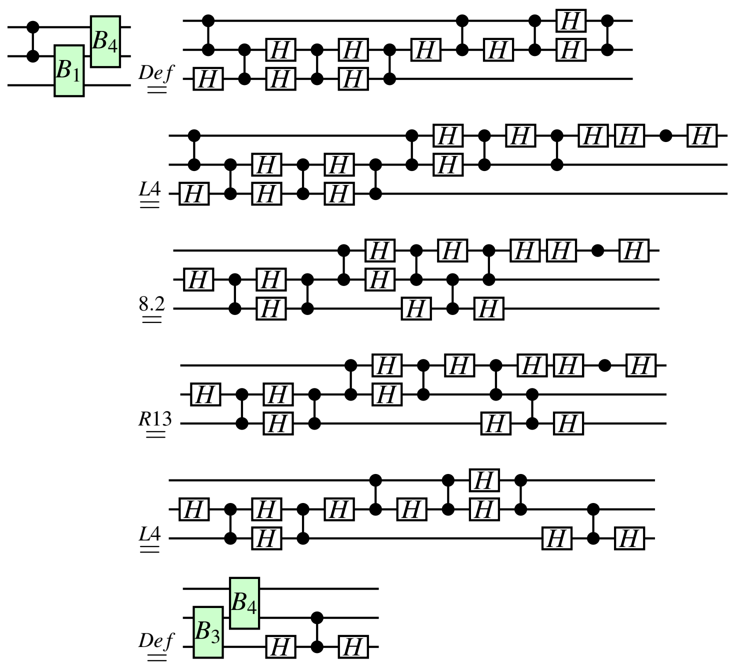
6.2



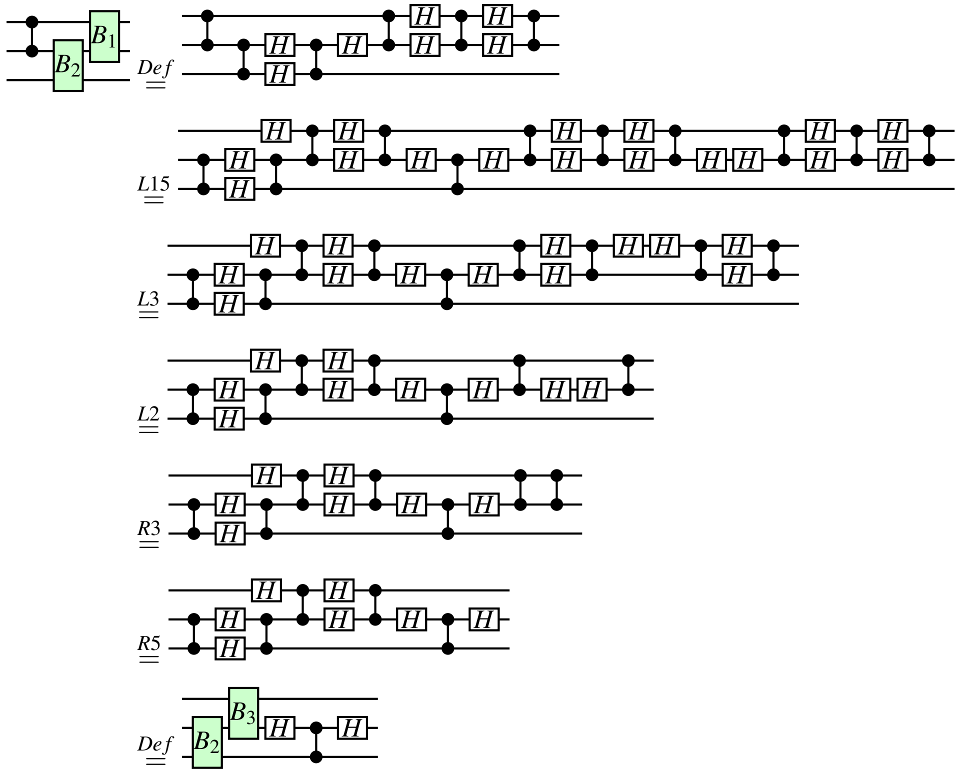
6.3



6.4

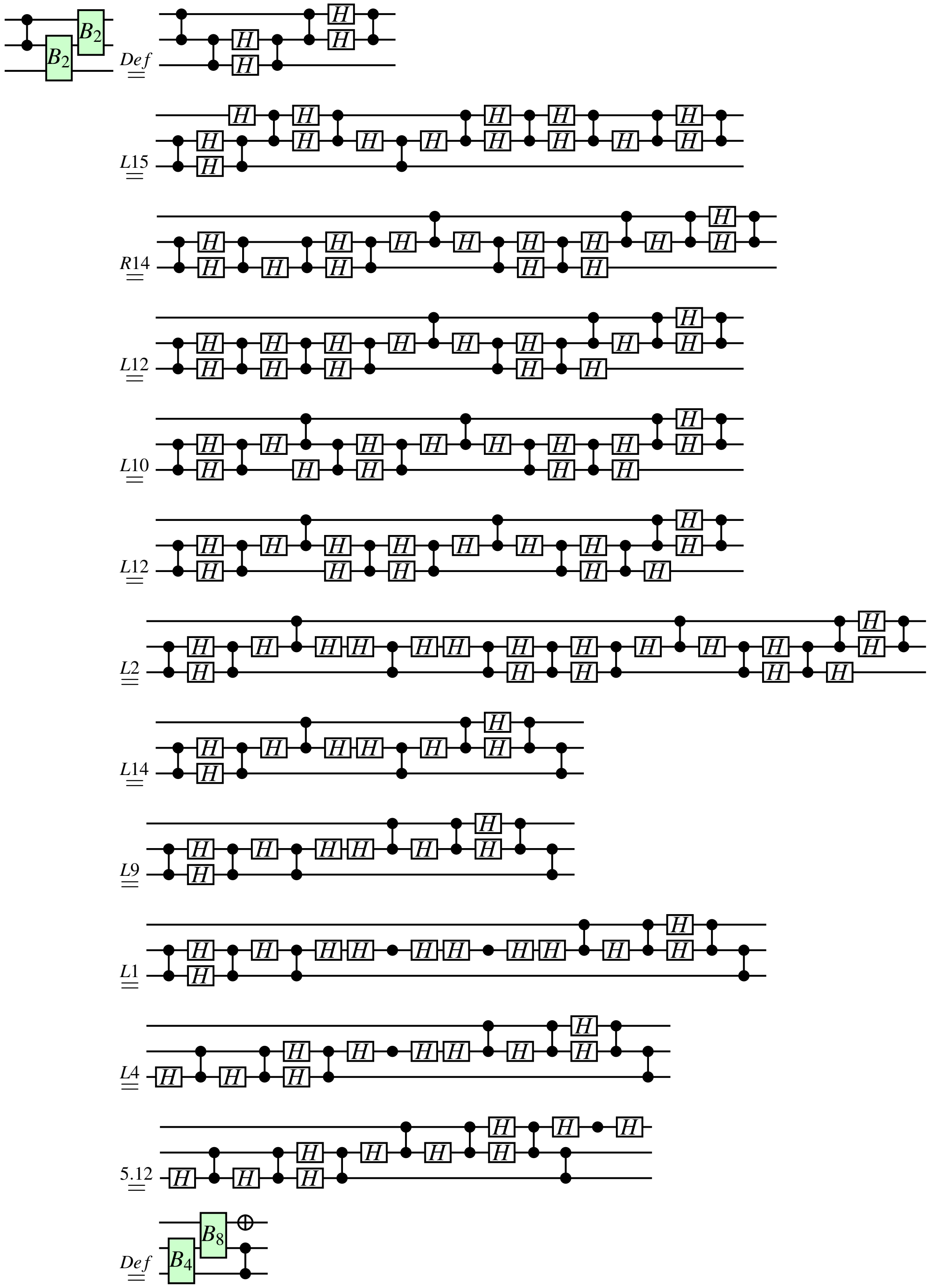


6.5

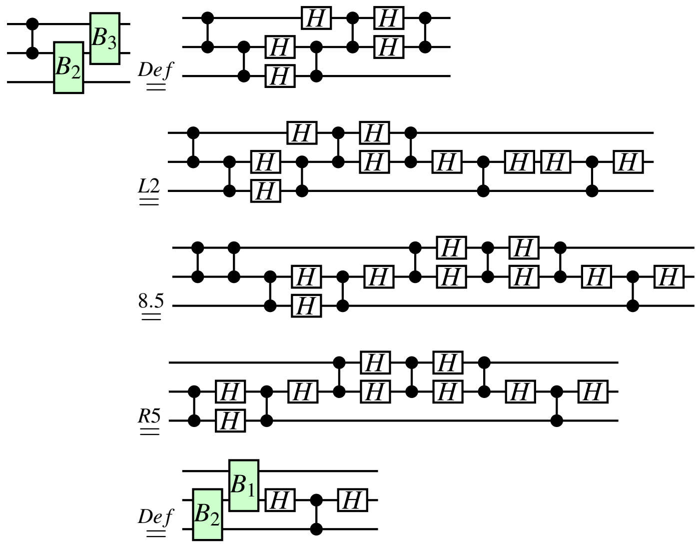


B_1
B_2
$\overset{Def}{=}$
$\overset{L15}{=}$
$\overset{L3}{=}$
$\overset{L2}{=}$
$\overset{R3}{=}$
$\overset{R5}{=}$
$\overset{Def}{=}$
B_3
B_2
H

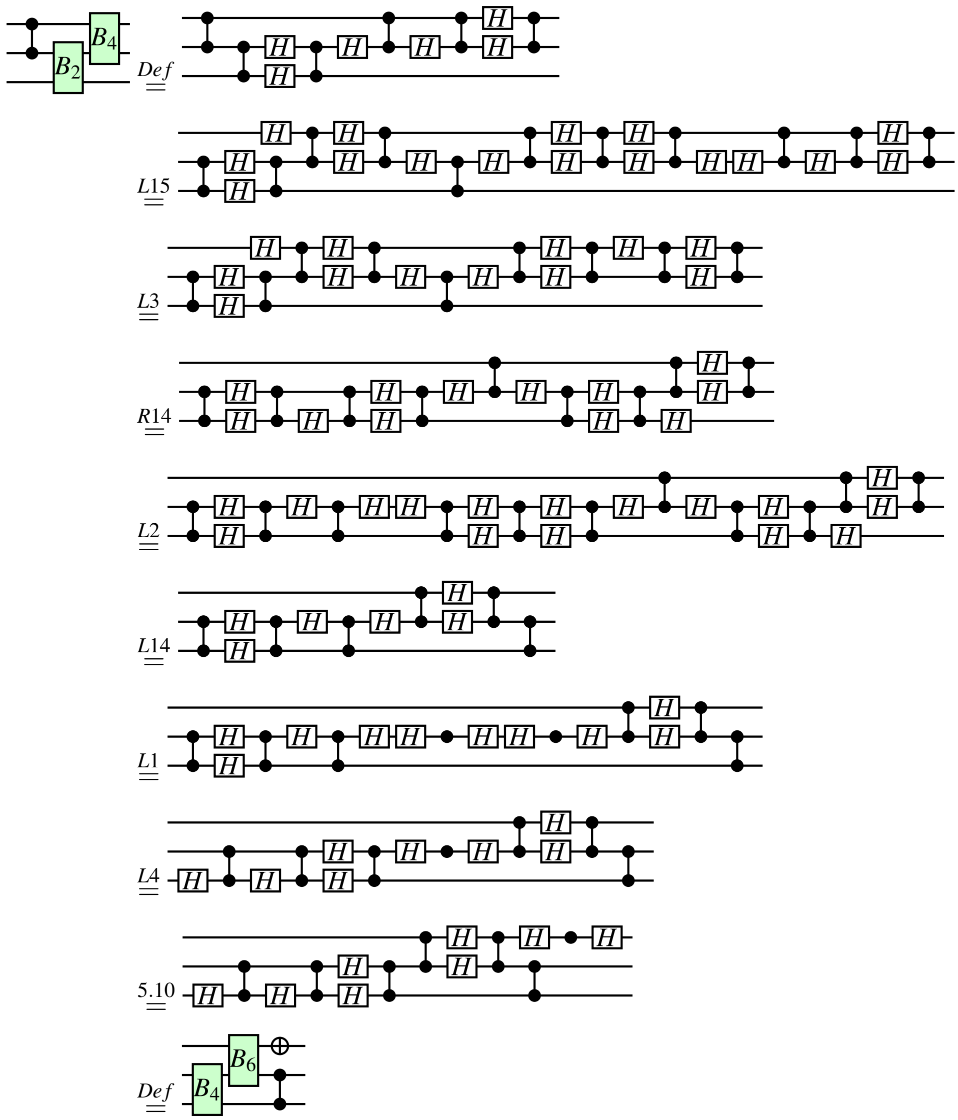
6.6


B_2
B_2
$\stackrel{Def}{=}$
$\stackrel{L15}{=}$
$\stackrel{R14}{=}$
$\stackrel{L12}{=}$
$\stackrel{L10}{=}$
$\stackrel{L12}{=}$
$\stackrel{L2}{=}$
$\stackrel{L14}{=}$
$\stackrel{L9}{=}$
$\stackrel{L1}{=}$
$\stackrel{L4}{=}$
$\stackrel{5.12}{=}$
$\stackrel{Def}{=}$
B_4
B_8

6.7

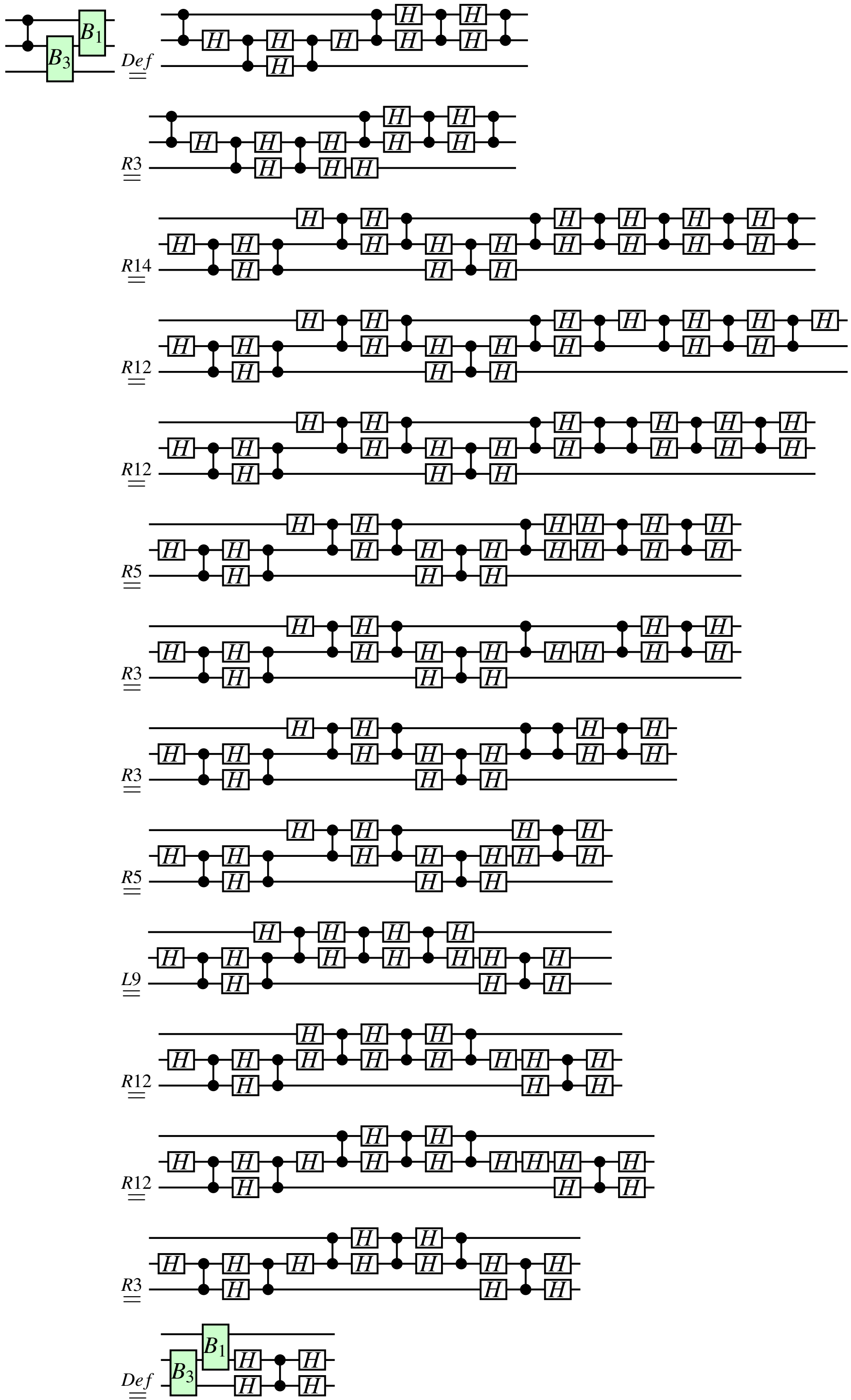


6.8

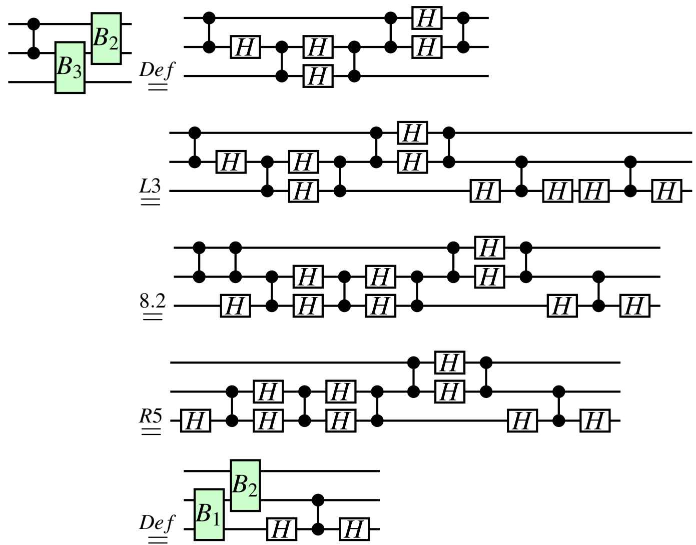


B_2
B_4
$\overset{Def}{=}$
$\overset{L15}{=}$
$\overset{L3}{=}$
$\overset{R14}{=}$
$\overset{L2}{=}$
$\overset{L14}{=}$
$\overset{L1}{=}$
$\overset{L4}{=}$
$\overset{5.10}{=}$
$\overset{Def}{=}$
B_4
B_6

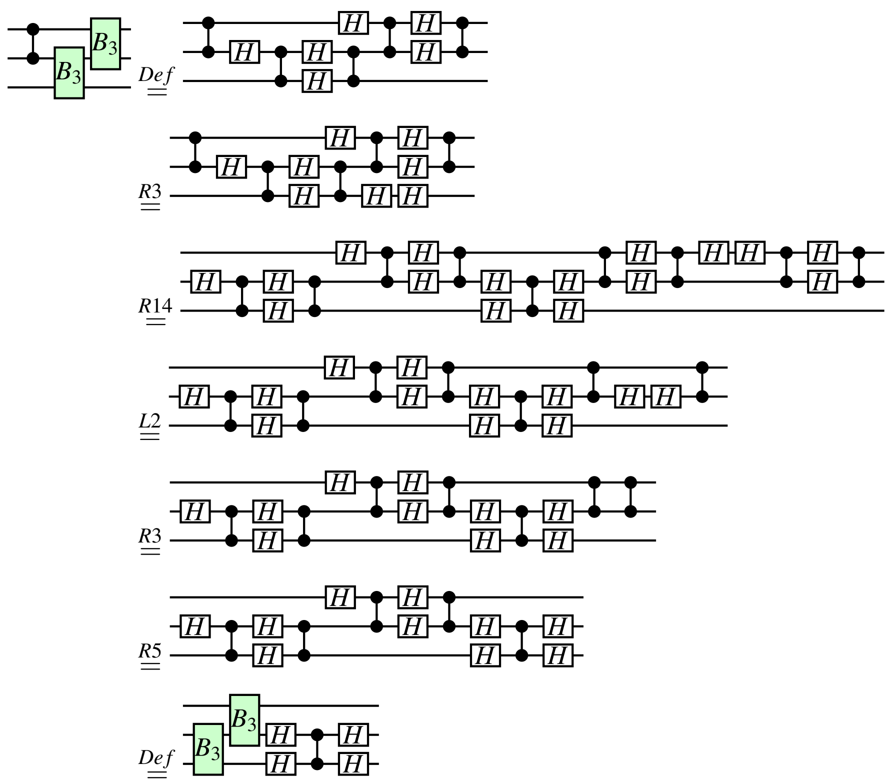
6.9



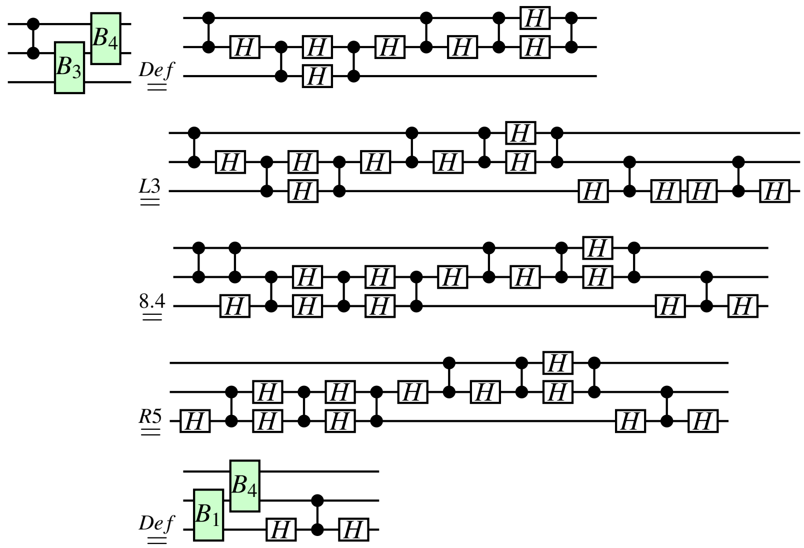
6.10



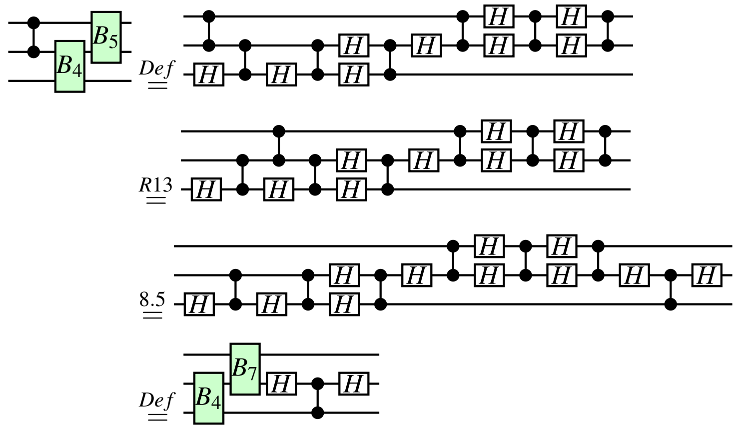
6.11



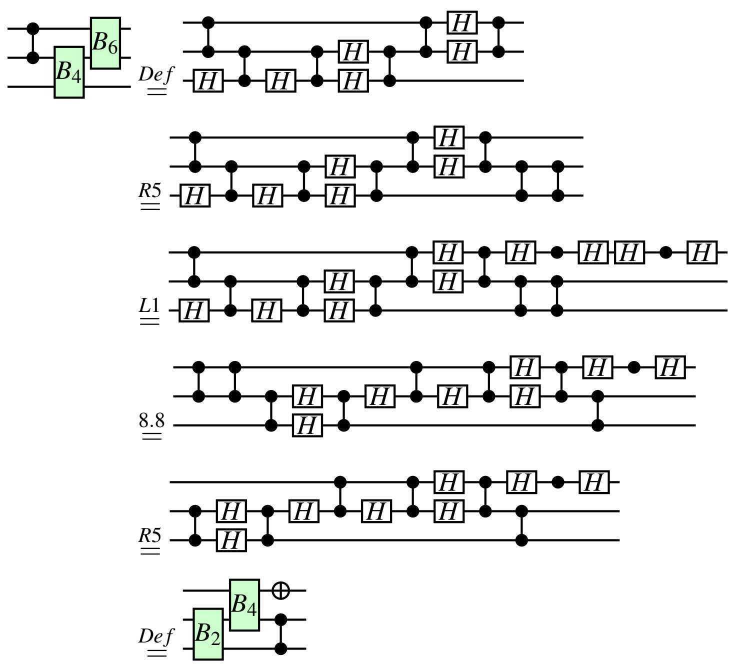
6.12



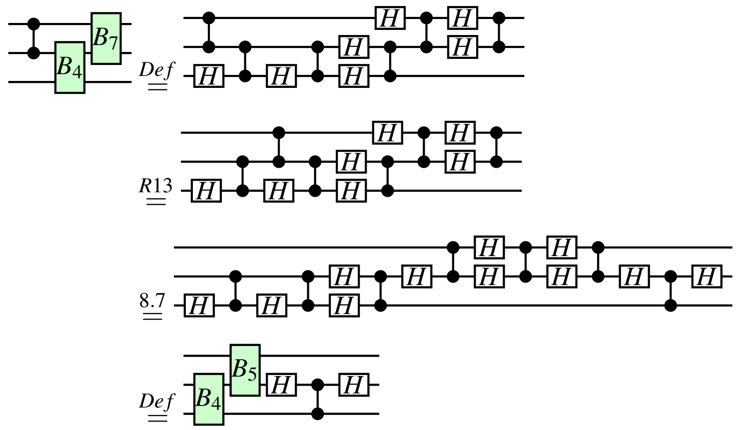
6.13



6.14



6.15



6.16

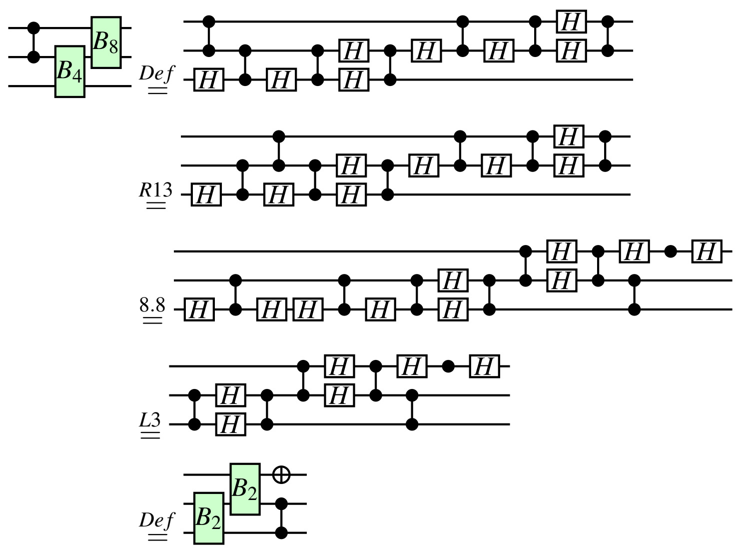
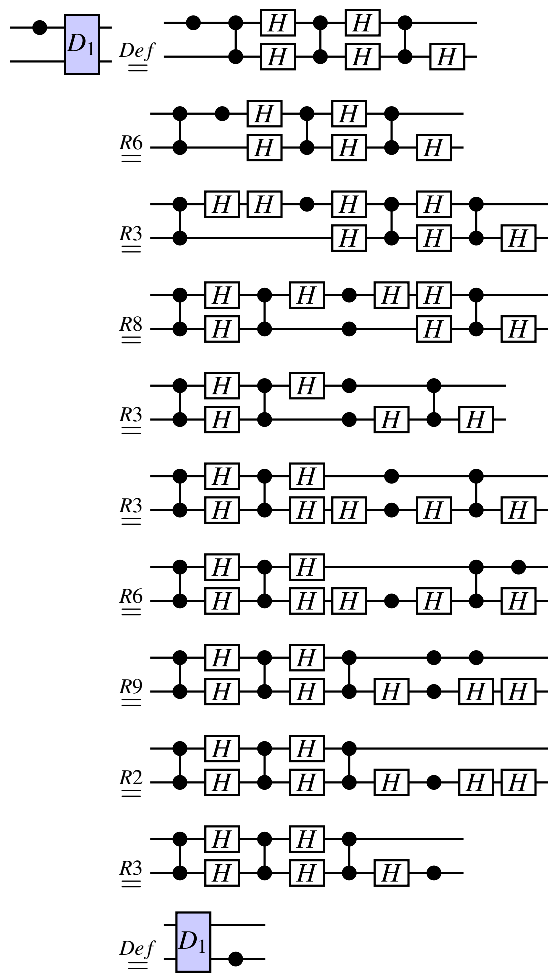


Figure 7:

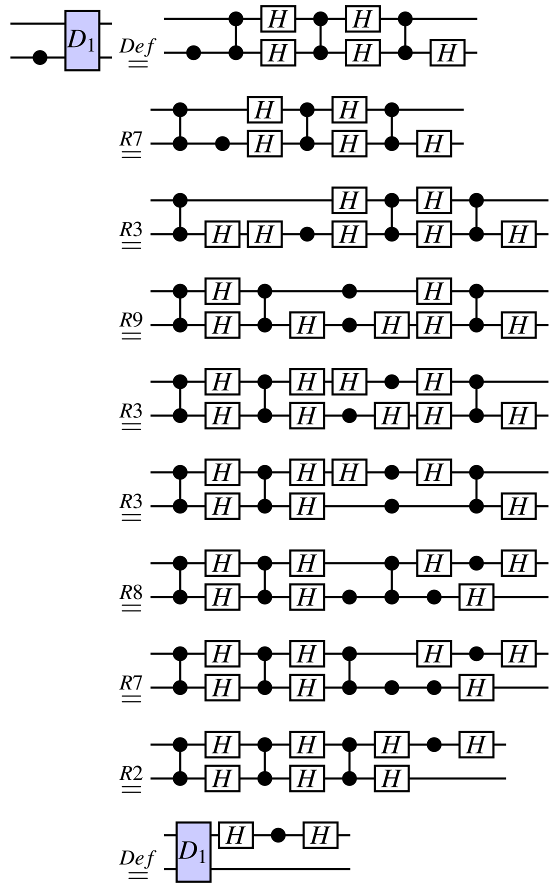
The proofs of 7.1–7.16 are the same as those of 6.1–6.16.

Figure 8:

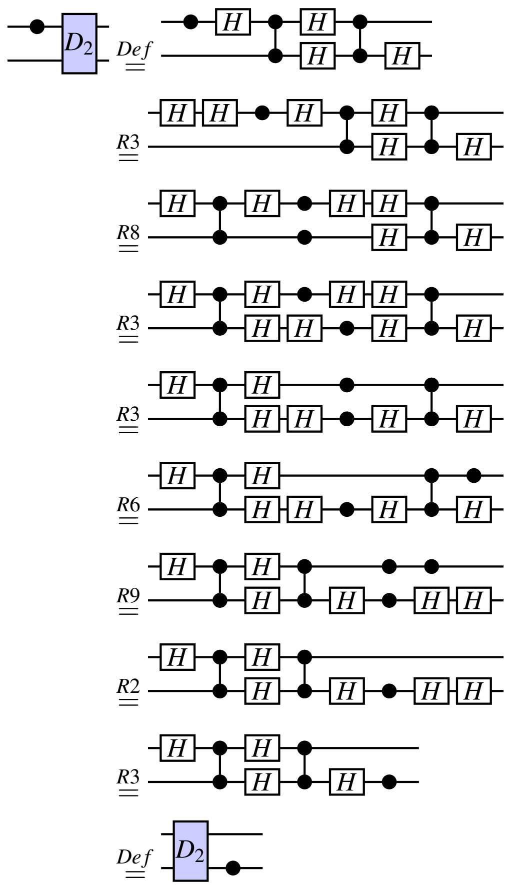
8.1



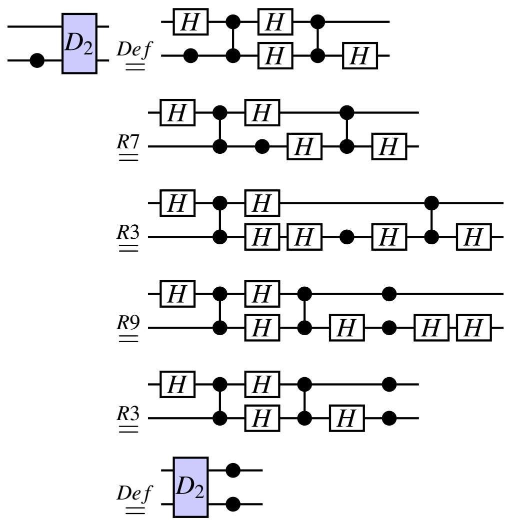
8.2



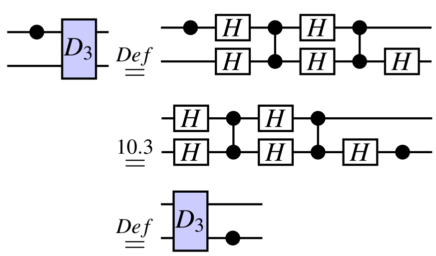
8.3



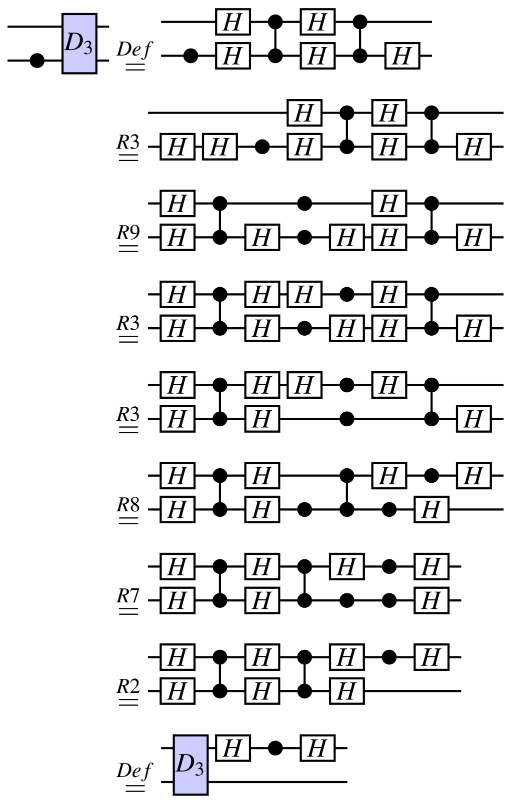
8.4



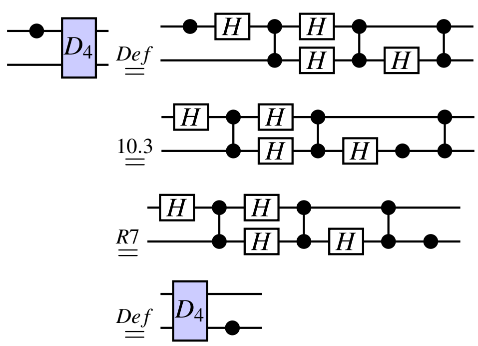
8.5



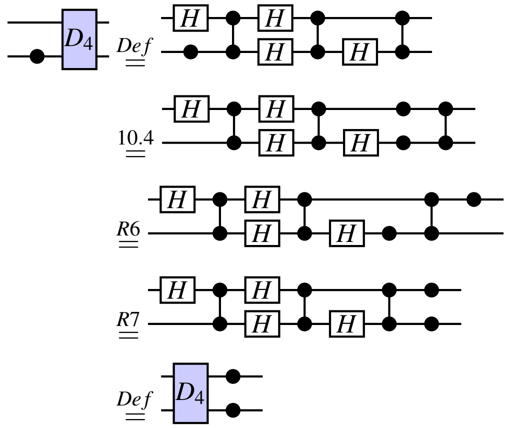
8.6



8.7

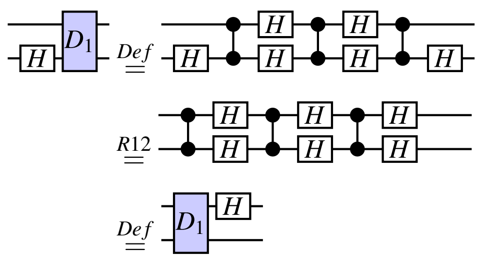


8.8



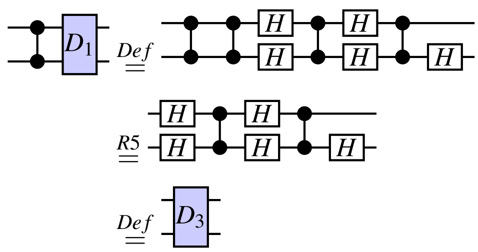
D_4
Def
10.4
R6
R7
Def
D_4

8.9



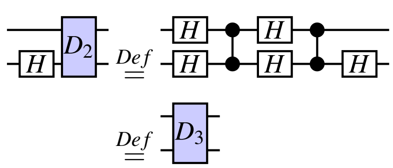
D_1
Def
R12
Def
D_1

8.10



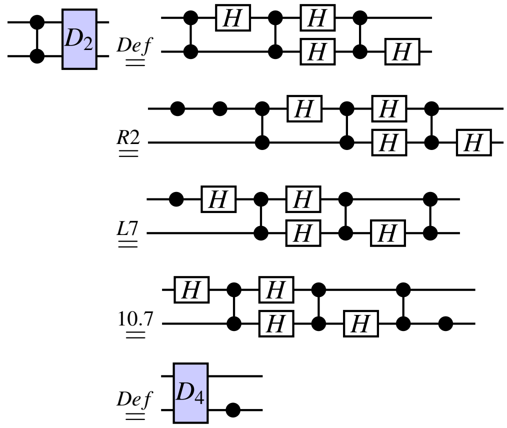
D_1
Def
R5
Def
D_3

8.11

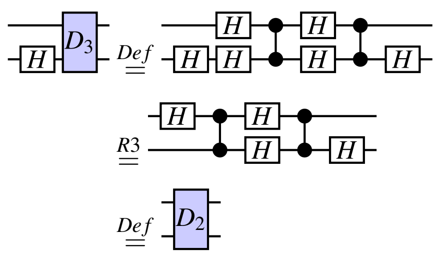


D_2
Def
Def
D_3

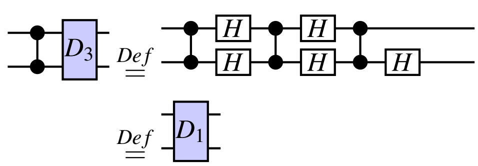
8.12



8.13



8.14



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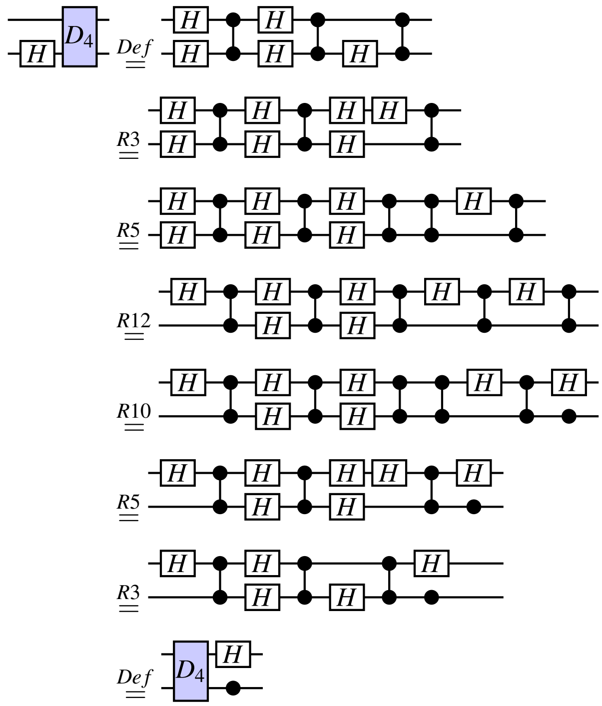

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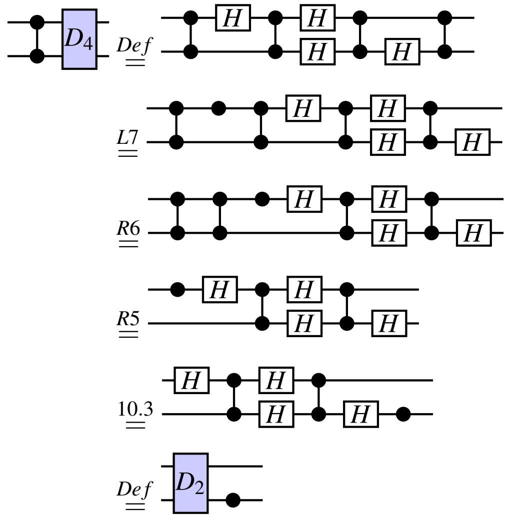

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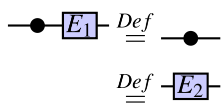

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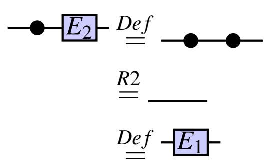
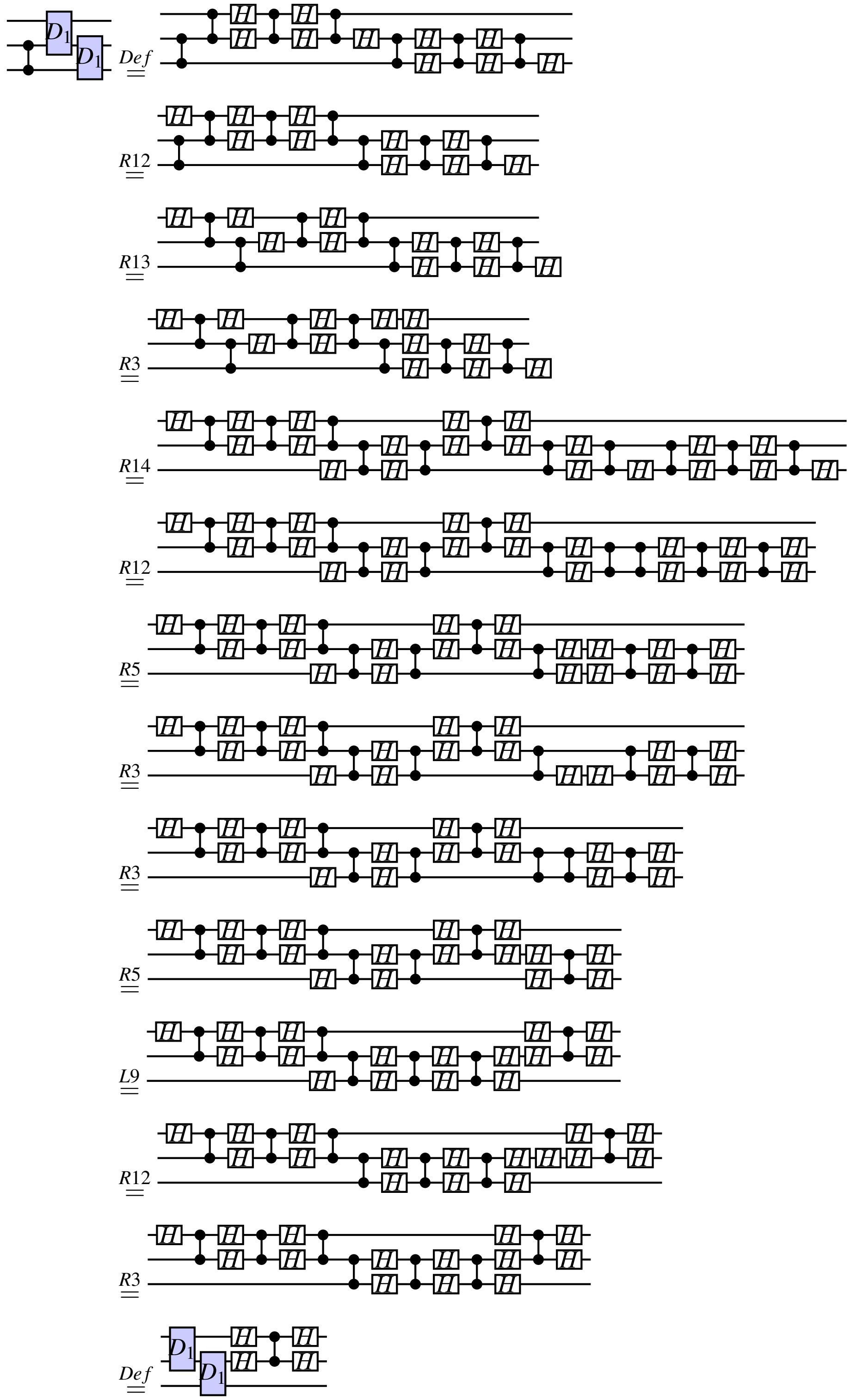


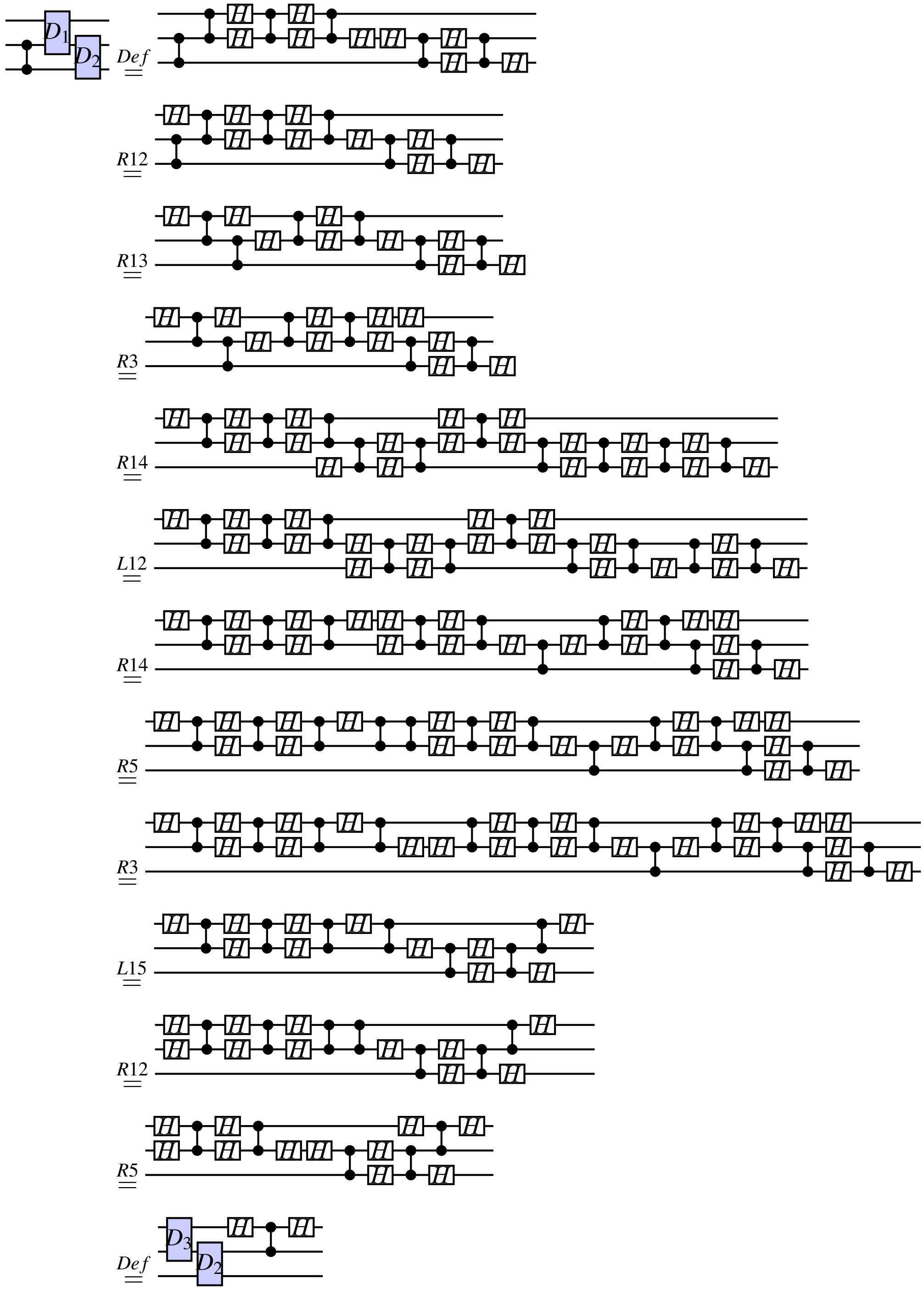
Figure 9:

9.1



D_1
D_1
$\overset{Def}{=}$
$\overset{R12}{=}$
$\overset{R13}{=}$
$\overset{R3}{=}$
$\overset{R14}{=}$
$\overset{R12}{=}$
$\overset{R5}{=}$
$\overset{R3}{=}$
$\overset{R3}{=}$
$\overset{R5}{=}$
$\overset{L9}{=}$
$\overset{R12}{=}$
$\overset{R3}{=}$
$\overset{Def}{=}$

9.2



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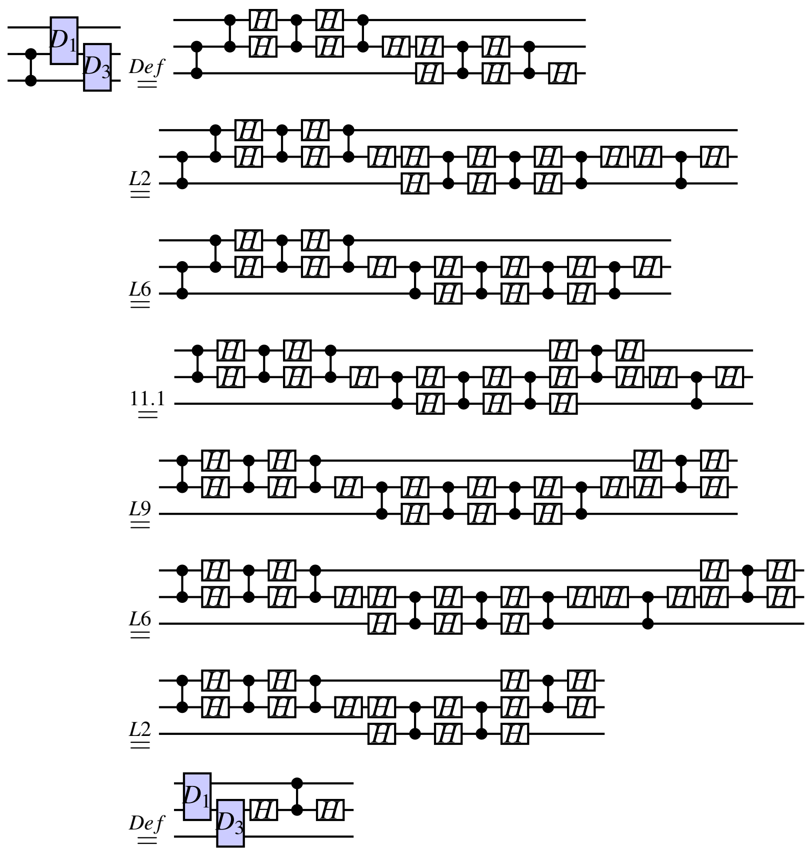
D_1
D_3
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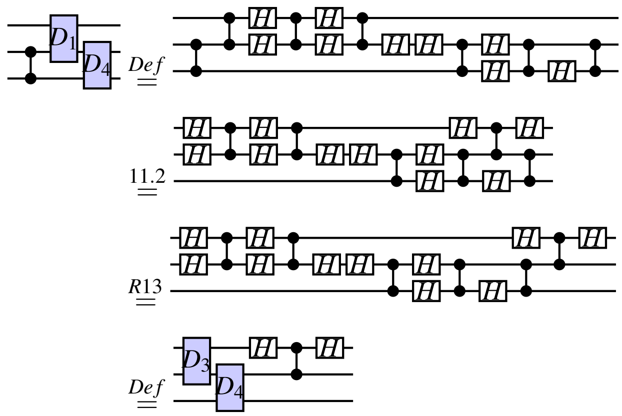
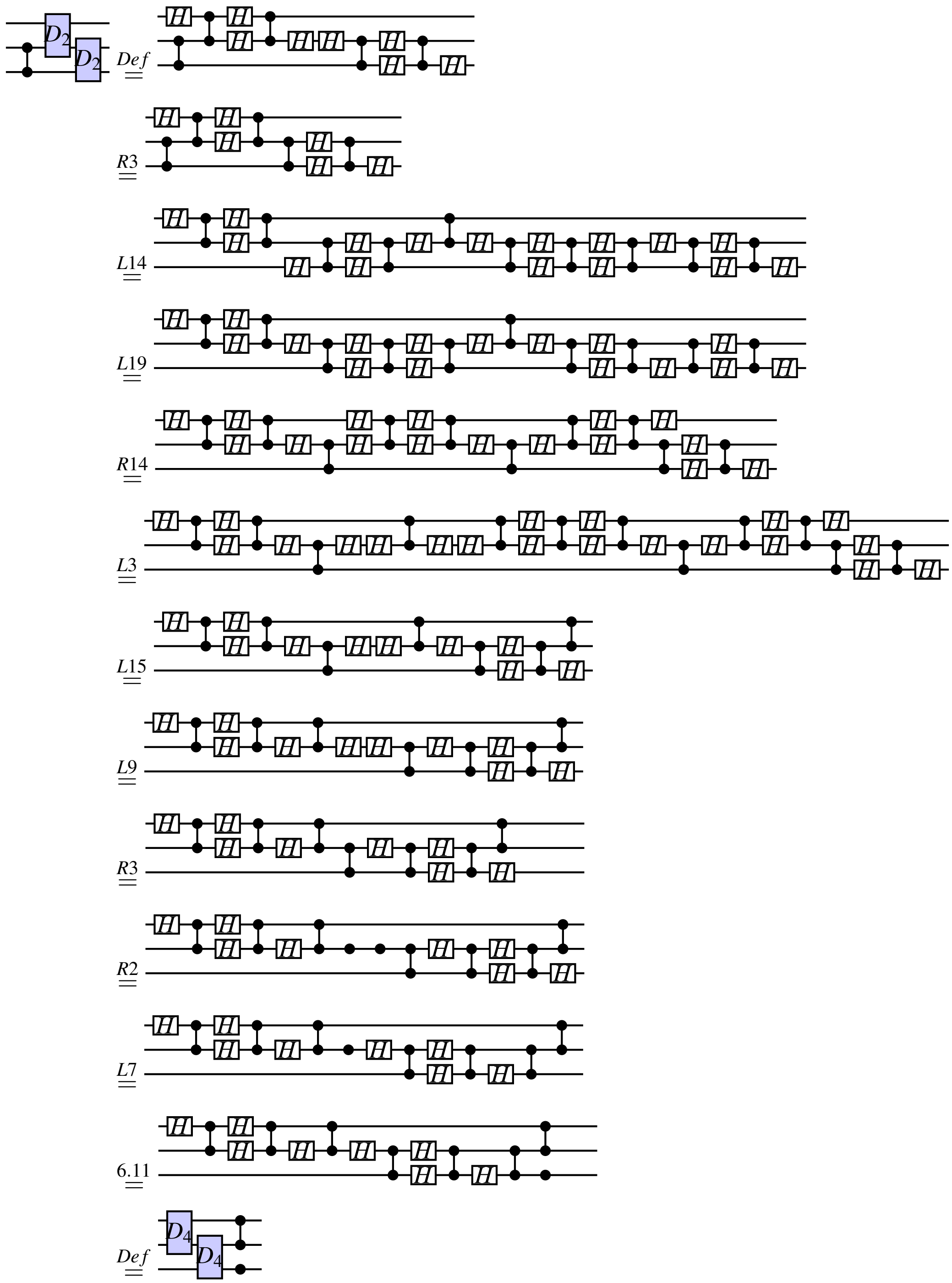
$\stackrel{L2}{=}$
$\stackrel{L6}{=}$
$\stackrel{11.1}{=}$
$\stackrel{L9}{=}$
$\stackrel{L6}{=}$
$\stackrel{L2}{=}$
$\stackrel{Def}{=}$

9.4



9.5

9.6


D_2
D_2
$\stackrel{Def}{=}$
$\stackrel{R3}{=}$
$\stackrel{L14}{=}$
$\stackrel{L19}{=}$
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$\stackrel{L3}{=}$
$\stackrel{L15}{=}$
$\stackrel{L9}{=}$
$\stackrel{R3}{=}$
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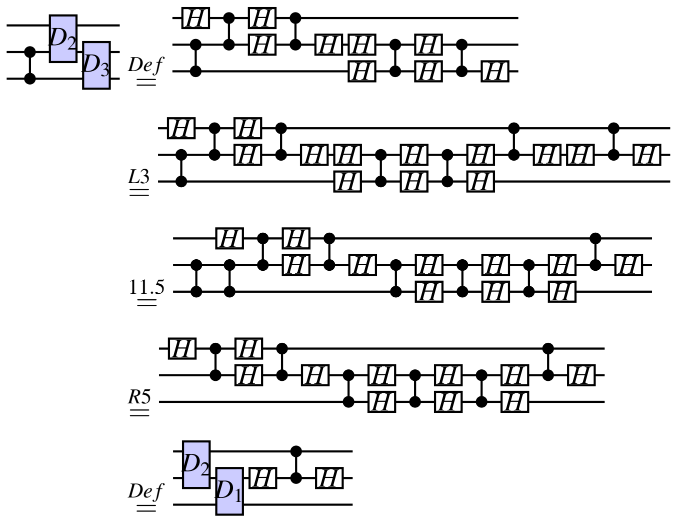
$\stackrel{L7}{=}$
$\stackrel{6.11}{=}$
$\stackrel{Def}{=}$
D_4
D_4

9.7



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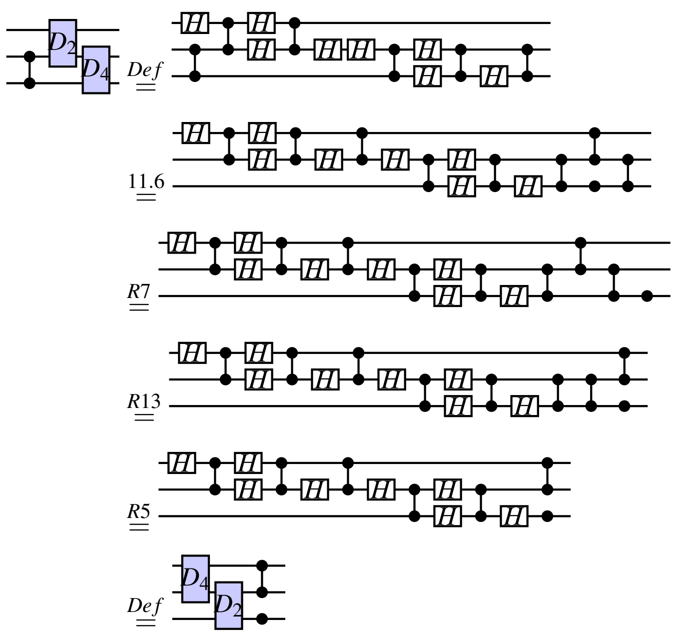

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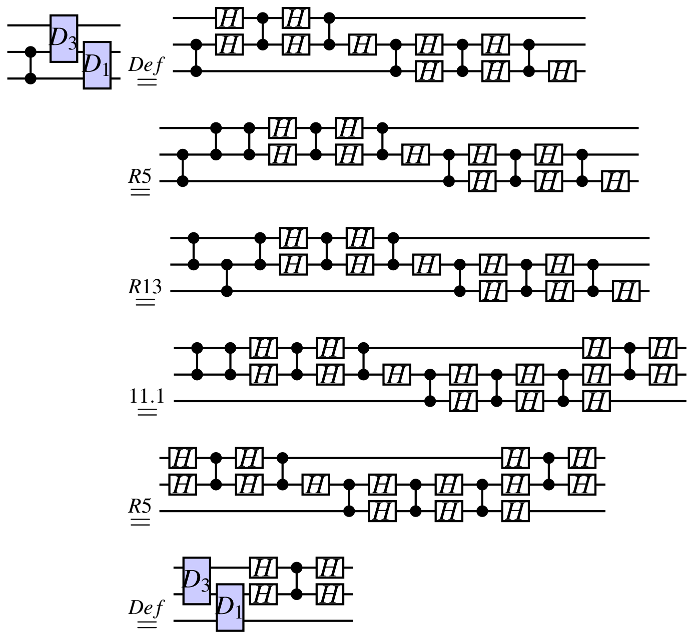

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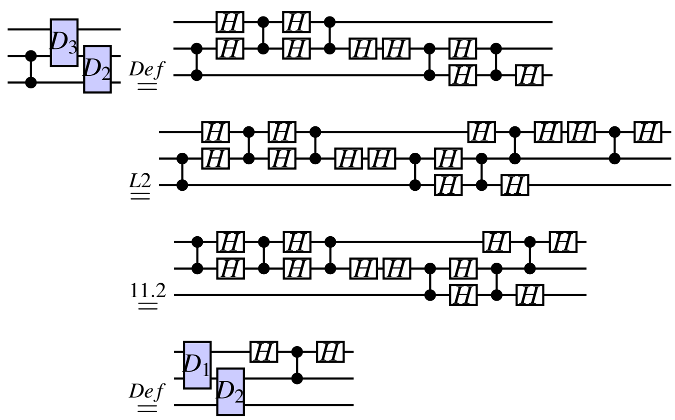

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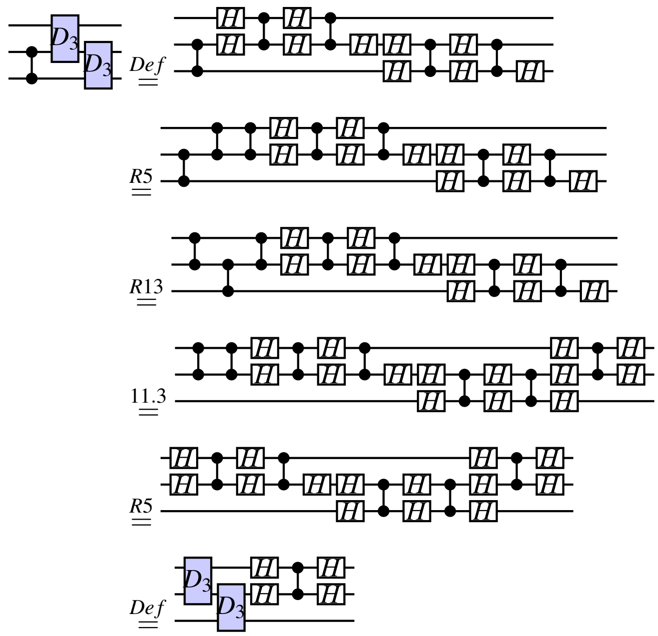

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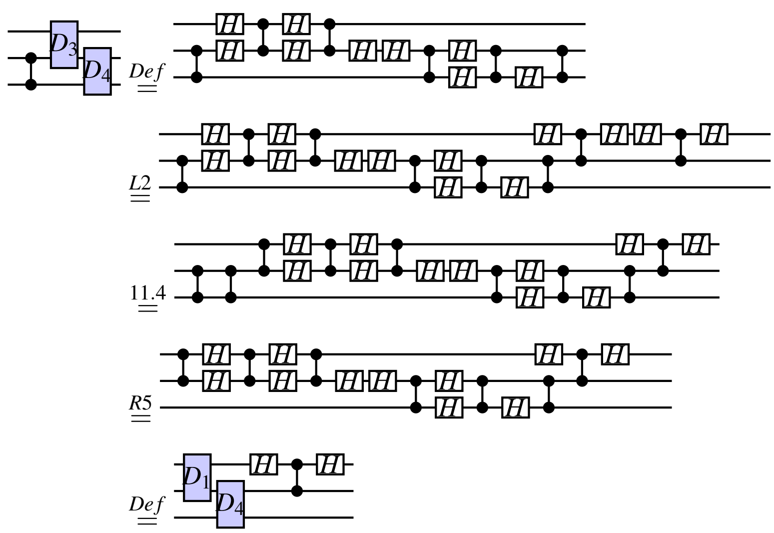

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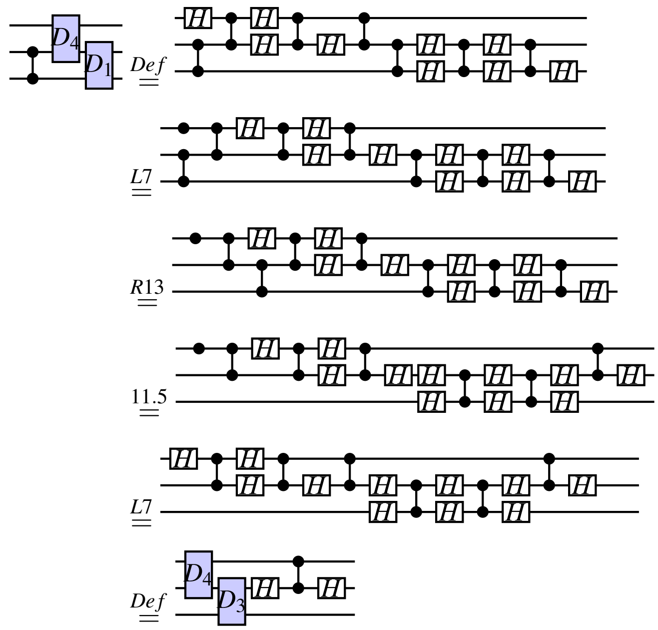

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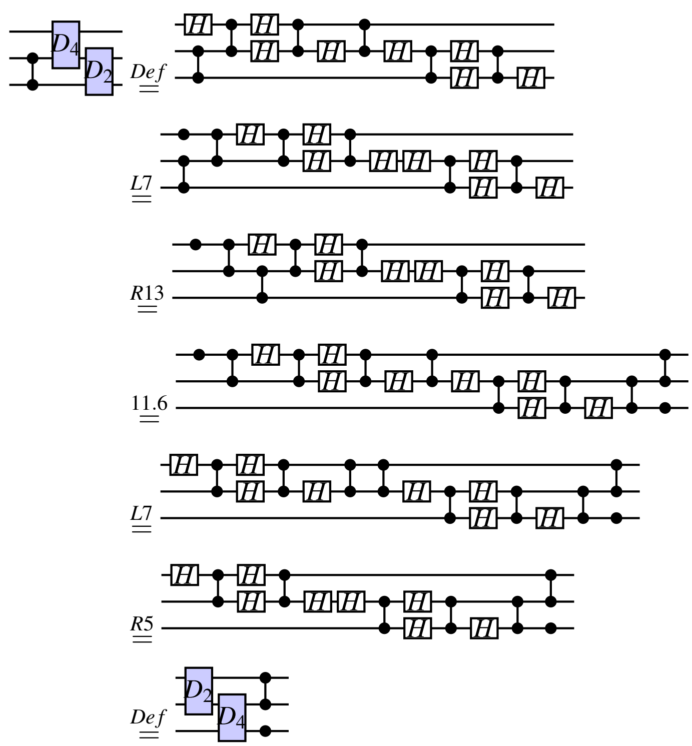

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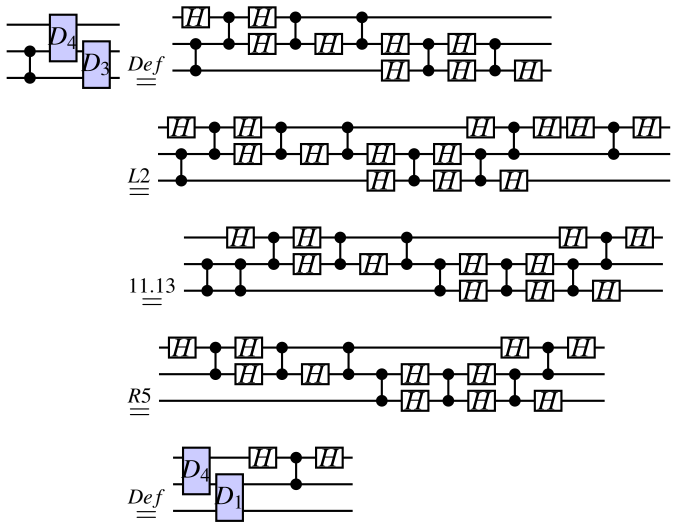

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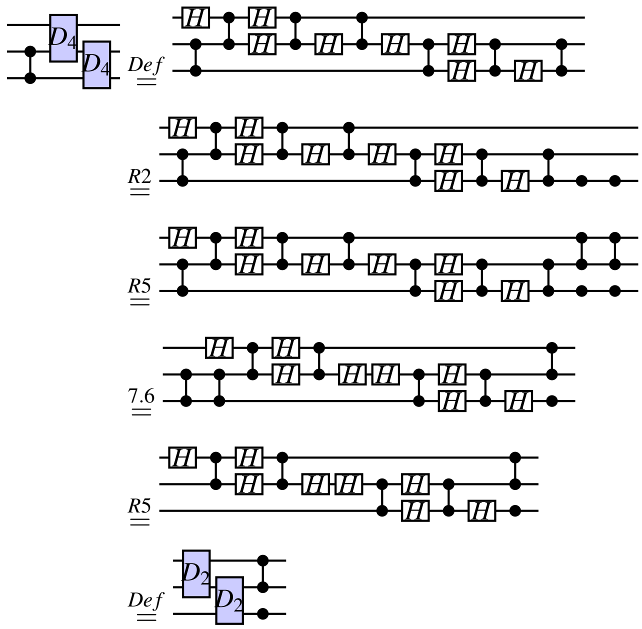

Figure 10:

10.1

10.2