What's category theory, anyway?

Dedicated to the memory of Dietmar Schumacher (1935-2014)

Robert Paré

November 7, 2014

Many subjects

How many subjects are there in mathematics?

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- Linear algebra

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- Combinatorics

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- Geometry

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- ⋮

Many subjects

How many subjects are there in mathematics?

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- $\vdots$

Answer: 64

AMS Subject Classification (MathSciNet)

00: General
01: History and biography
03: Mathematical logic and foundations
05: Combinatorics
06: Order theory
08: General algebraic systems
11: Number theory
12: Field theory and polynomials
13: Commutative rings and algebras
14: Algebraic geometry
15: Linear and multilinear algebra; matrix theory
16: Associative rings and associative algebras
17: Non-associative rings and non-associative algebras
18: Category theory; homological algebra
19: K-theory
20: Group theory and generalizations
22: Topological groups, Lie groups, and analysis upon them
26: Real functions, including derivatives and integrals
28: Measure and integration
30: Complex functions
31: Potential theory
32: Several complex variables and analytic spaces

33: Special functions
34: Ordinary differential equations
35: Partial differential equations
37: Dynamical systems and ergodic theory
39: Difference equations and functional equations
40: Sequences, series, summability
41: Approximations and expansions
42: Harmonic analysis
43: Abstract harmonic analysis
44: Integral transforms, operational calculus
45: Integral equations
46: Functional analysis
47: Operator theory
49: Calculus of variations and optimal control; optimization
51: Geometry
52: Convex geometry and discrete geometry
53: Differential geometry
54: General topology
55: Algebraic topology
57: Manifolds
58: Global analysis, analysis on manifolds
60: Probability theory, stochastic processes
62: Statistics

65: Numerical analysis
68: Computer science
70: Mechanics
74: Mechanics of deformable solids
76: Fluid mechanics
78: Optics, electromagnetic theory
80: Classical thermodynamics, heat transfer
81: Quantum theory
82: Statistical mechanics, structure of matter
83: Relativity and gravitational theory
85: Astronomy and astrophysics
86: Geophysics
90: Operations research, mathematical programming
91: Game theory, economics, social and behavioral sciences
92: Biology and other natural sciences
93: Systems theory; control
94: Information and communication, circuits
97: Mathematics education

The holy grail

- Have to specialize
- Work is often duplicated
- One subject might benefit from others

The category theorists' holy grail: the unification of mathematics

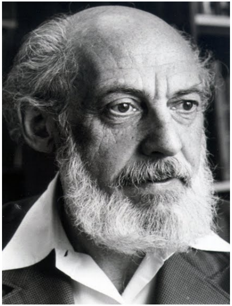
What is category theory?

- Foundations
- Relevant foundations
- Framework in which to compare different subjects – study their similarities and differences
- Need a unifying principle

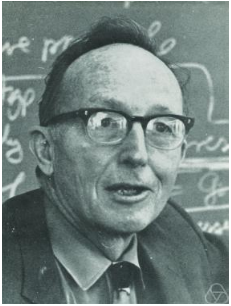
Birth of category theory

S. Eilenberg and S. Mac Lane, General theory of natural equivalences, *Trans. Amer. Math. Soc.*, **58** (1945), 231-294

Eilenberg



Mac Lane



Categories

Many structures in mathematics come with a corresponding notion of function between them

- Vector spaces – linear functions
- Graphs – edge-preserving functions
- Topological spaces – continuous functions
- Groups – homomorphisms

Definition

A category **A** consists of

- A class of *objects* $A, B, C, \ldots$
- For each pair of objects A, B, a set of *morphisms* $\mathbf{A}(A, B)$.
  For $f \in \mathbf{A}(A, B)$ we write

$$f : A \longrightarrow B$$

- For each object A a special morphism

$$1_A : A \longrightarrow A$$

  the *identity* on A
- For all pairs $A \xrightarrow{f} B \xrightarrow{g} C$ a *composite* $gf : A \longrightarrow C$

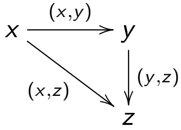
  Satisfying
- For every $f : A \longrightarrow B$, $1_B f = f = f 1_A$
- For every $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, $h(gf) = (hg)f$

Examples

- **Vect** Objects are vector spaces; morphisms are linear maps
- **Gph** Objects are graphs; morphisms are edge-preserving functions
- **Top** Objects are topological spaces; morphisms are continuous functions
- **Gp** Objects are groups; morphisms are homomorphisms
- **Set** Objects are sets; morphisms are functions

Posets

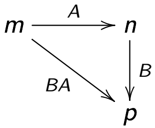
- A poset $(X, \leq)$ gives a category $\mathbf{X}$
  - Objects elements of X
  - $\mathbf{X}(x, y) = \begin{cases} \{(x, y)\} & \text{if} \quad x \leq y \\ \emptyset & \text{ow} \end{cases}$
  - Composition



E.g. $\mathbb{N}$ with “divisibility”, i.e. $m \leq n \Leftrightarrow m|n$

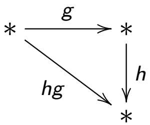
Matrices

- Matrices
  - Objects $0, 1, 2, 3, \ldots$
  - $\mathbf{Mat}(m, n) = \{A | A \text{ is an } n \times m \text{ matrix}\}$



Groups as categories

- A group G gives a category $\mathbf{G}$
  - Objects: a single one $*$
  - Arrows: one for each element of G, $g : * \longrightarrow *$, i.e. $\mathbf{G}(*, *) = G$



Duality

- The opposite of a category **A**
  - Objects are those of **A**
  - $\mathbf{A}^{op}(A, B) = \mathbf{A}(B, A)$
  - Composition is reversed: $\overline{f}\,\overline{g} = \overline{gf}$

Every definition has a dual; every theorem has a dual

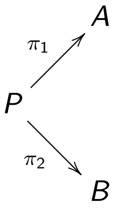
Isomorphism

$f : A \longrightarrow B$ is an *isomorphism* if it has an inverse $g : B \longrightarrow A$, i.e. $gf = 1_A$ *and* $fg = 1_B$

Write $A \cong B$ to mean that there is an iso $f : A \longrightarrow B$, and say A is *isomorphic* to B. A and B are "the same"

Products

A, B objects of $\mathbf{A}$
A *product* of A and B is an object P and morphisms π_1, π_2



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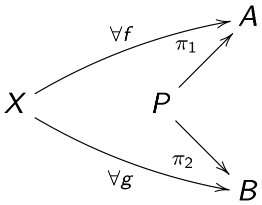

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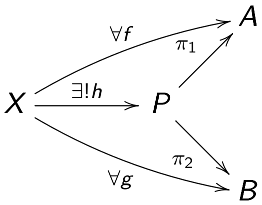
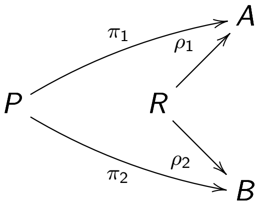


The product may or may not exist

Proposition

If products exist, they are unique up to isomorphism

Proof.



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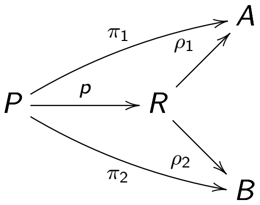

$\rho_1 p = \pi_1$, $\rho_2 p = \pi_2$

□

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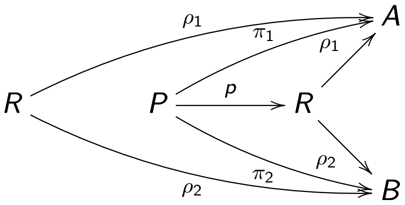

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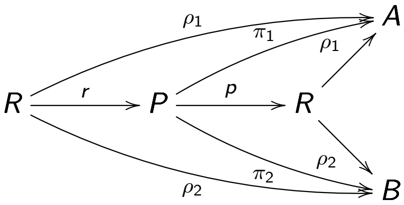

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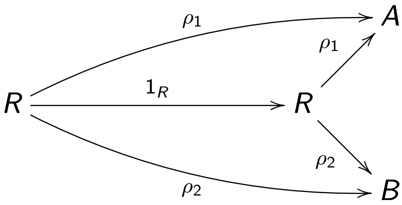

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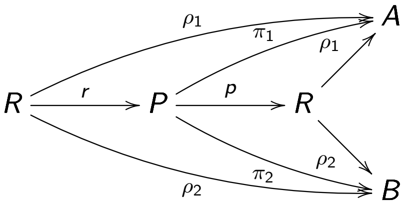

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Similarly $rp = 1_P$

□

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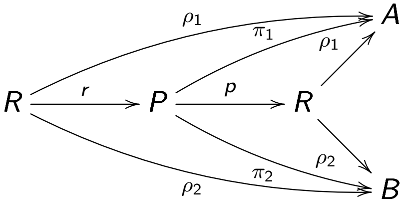

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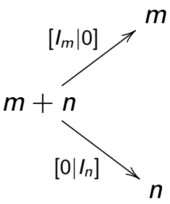
□

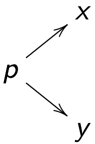
Choose one and call it $A \times B$

Examples of products

- In **Vect** $V \times W$ is $V \oplus W = \{(v, w) | v \in V, w \in W\}$
- In **Gp** $G \times H = \{(g, h) | g \in G, h \in H\}$ with component-wise multiplication
- In **Set** $X \times Y = \{(x, y) | x \in X, y \in Y\}$
- In **Top** $X \times Y = \{(x, y) | x \in X, y \in Y\}$ with the product topology
- In **Mat** the product of m and n is $m + n$ with projections

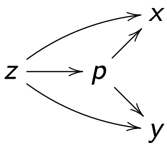


- $(X, \leq)$ a poset and **X** the corresponding category $x, y \in X$



$$\Leftrightarrow \quad p \leq x \quad \& \quad p \leq y$$

so p is a lower bound
Universal property

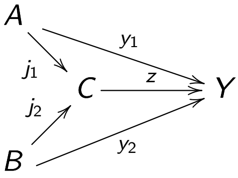


$$z \leq x \quad \& \quad z \leq y$$
$$\Rightarrow z \leq p$$

so p is a *greatest lower bound* (if it exists)
The product of x and y is $x \wedge y$
E.g., for $\mathbb{N}$ with divisibility, the product is g.c.d.

Coproducts

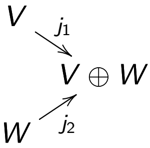
Dual to products



Choose one and write $A + B$

Example

Vect is $V \oplus W$



$j_1(v) = (v, 0)$
$j_2(w) = (0, w)$

Example

Set coproduct is disjoint union

$$X + Y = X \times \{1\} \cup Y \times \{2\}$$

Top coproduct is also disjoint union

Example

Gr coproduct of G and H is the "free product"

$$G \times H = \{(g_1 h_1 g_2 h_2 \ldots g_n h_n) | g_i \in G, h_i \in H\} / \sim$$

Example

Poset **X** coproduct is $x \vee y$ (least upper bound)

Functors

Morphisms of categories

$$\begin{array}{rcl} F : \mathbf{A} & \longrightarrow & \mathbf{B} \\ A & \longmapsto & FA \\ (A \xrightarrow{a} A') & \longmapsto & (FA \xrightarrow{Fa} FA') \end{array}$$

such that

- $F(1_A) = 1_{FA}$
- $F(a'a) = F(a')F(a)$

Examples

- Forgetful functor $U : \mathbf{Vect} \longrightarrow \mathbf{Set}$
- $$\begin{array}{rcl} \mathbf{Mat} & \longrightarrow & \mathbf{Vect} \\ n & \longmapsto & \mathbb{R}^n \\ (m \xrightarrow{A} n) & \longmapsto & (\mathbb{R}^m \xrightarrow{T_A} \mathbb{R}^n) \end{array}$$

Examples

(continued)

- $\textbf{Ring} \xrightarrow{U} \textbf{Gr}$
  $U(R) =$ group of units (invertible elements)
- $\textbf{Gr} \xrightarrow{F} \textbf{Ring}$
  $F(G) = \mathbb{Z}G$ the group ring

Remark

U and F are adjoint functors
$\textbf{Gr}(G, UR) \cong \textbf{Ring}(\mathbb{Z}G, R)$

Eilenberg & Mac Lane

They were studying algebraic topology; homology and homotopy
$H_n : \mathbf{Top} \longrightarrow \mathbf{Ab}$ (Abelian groups)
$\pi_n : \mathbf{Top} \longrightarrow \mathbf{Gr}$

Proposition

Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be a functor. If $A \cong A'$ then $FA \cong FA'$

Proof.

Trivial

□

More functors

- If $(X, \leq)$, $(Y, \leq)$ are posets, then a functor $F : \mathbf{X} \longrightarrow \mathbf{Y}$ is the same as an order-preserving function
- If G, H are groups, then a functor $F : \mathbf{G} \longrightarrow \mathbf{H}$ is the same as a group homomorphism
- If G is a group, then a functor $F : \mathbf{G} \longrightarrow \mathbf{Vect}$ is a *representation* of G

Fibonacci

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	$\cdots$
0	1	1	2	3	5	8	13	21	34	55	89	$\cdots$

Fact: $m|n \Rightarrow F_m|F_n$
If **N** is the natural numbers ordered by divisibility, then

$$F : \mathbf{N} \longrightarrow \mathbf{N}$$

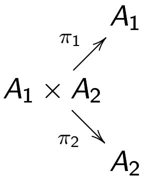
is a functor

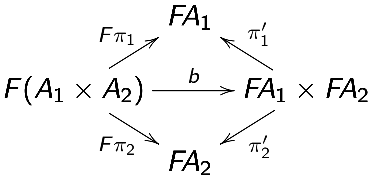
Amazing fact: $gcd(F_m, F_n) = F_{gcd(m,n)}$

Recall that $gcd(m, n)$ is the categorical product of m and n in **N**, so F preserves products

Product-preserving functors

Suppose **A** and **B** have (binary) products. Let





be products in **A** and **B**

F preserves the product $A_1 \times A_2$ if b is an isomorphism

$$F(A_1 \times A_2) \cong FA_1 \times FA_2$$

Lawvere theories

In 1962 Bill Lawvere introduced *algebraic theories* in his thesis, summarized in "Functorial semantics of algebraic theories", Proc. Nat. Acad. Sci., 50 (1963), pp. 869-872

Definition

An *algebraic theory* is a category $\mathbf{T}$ whose objects are (in bijection with) natural numbers

$$[0], [1], [2], \cdots$$

such that $[n] \cong [1] \times [1] \times \cdots \times [1]$ (n times)
A $\mathbf{T}$-*algebra* is a product preserving functor $F : \mathbf{T} \longrightarrow \mathbf{Set}$

Example

Mat is the theory of vector spaces

F : **Mat** $\longrightarrow$ **Set**

$F[1] = X$

$F[n] = X \times \cdots X = X^n$

$([2] \xrightarrow{[1,1]} [1]) \overset{F}{\longmapsto} (X^2 \xrightarrow{+} X)$

$([1] \xrightarrow{[\alpha]} [1]) \overset{F}{\longmapsto} (X \xrightarrow{\alpha\cdot(\ )} X)$

If $A = [1, 1]$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ then $AB = AC$

so

$$\begin{array}{ccc} [3] & \xrightarrow{B} & [2] \\ {\scriptstyle C}\downarrow & & \downarrow{\scriptstyle A} \\ [2] & \xrightarrow[A]{} & [1] \end{array} \quad \overset{F}{\longmapsto} \quad \begin{array}{ccc} X^3 & \xrightarrow{(+)\times(1_X)} & X^2 \\ {\scriptstyle (1_X)\times(+)}\downarrow & & \downarrow{\scriptstyle +} \\ X^2 & \xrightarrow[+]{} & X \end{array}$$

The theory of commutative rings

Objects $[0], [1], \cdots, [n], \cdots$

A morphism $[m] \longrightarrow [n]$ is an n-tuple of polynomials with integer coefficients in the variables $x_1, \cdots, x_m$

E.g. $[3] \xrightarrow{(x_1 + x_2 x_3,\ x_1^2 + 3)} [2]$

Composition is "substitution"

A product-preserving functor is "the same as" a ring

But where are the homomorphisms?

Natural transformations

$F, G : \mathbf{A} \longrightarrow \mathbf{B}$ functors, a *natural transformation* $t : F \longrightarrow G$ assigns to each object A of $\mathbf{A}$ a morphism of $\mathbf{B}$

$$tA : FA \longrightarrow GA$$

such that for every $a : A \longrightarrow A'$ we have

$$\begin{array}{ccc} FA & \xrightarrow{tA} & GA \\ {\scriptstyle Fa}\downarrow & & \downarrow{\scriptstyle Ga} \\ FA' & \xrightarrow[tA']{} & GA' \end{array}$$

commutes

Natural transformations (continued)

We have arrived at Eilenberg and Mac Lane's definition of *natural*

- We have *categories* **A**, **B**, ...
- There are morphisms between them $\mathbf{A} \longrightarrow \mathbf{B}$, called *functors*
- There are morphisms between functors $t : F \longrightarrow G$, called *natural transformations*

Their example:

- Let V be a finite dimensional vector space
- V^* is the dual space, i.e. linear functions $\phi : V \longrightarrow K$
- $Dim\ V^* = Dim\ V$ so $V \cong V^*$, but there is no natural isomorphism
- Take dual again V^{**}. Also have $V \cong V^* \cong V^{**}$
  There is a natural map

$$\begin{array}{ccc} V & \longrightarrow & V^{**} \\ v & \longmapsto & \widehat{v} \\ \widehat{v}(\phi) & = & \phi(v) \end{array}$$

Homomorphisms

If $\mathbf{T}$ is an algebraic theory and $F, G : \mathbf{T} \longrightarrow \mathbf{Set}$ two algebras, a *homomorphism* from F to G is a natural transformation $t : F \longrightarrow G$

$F[1] = A$ so $F[n] = A^n$
$G[1] = B$ so $G[n] = B^n$

$t[1] = f : A \longrightarrow B$
$t[n]$ has to be $f^n : A^n \longrightarrow B^n$ (follows from naturality)

Operations are $[n] \xrightarrow{\omega} [1]$
$F(\omega) : A^n \longrightarrow A$

Naturality gives
$$\begin{array}{ccc} A^n & \xrightarrow{f^n} & B^n \\ \omega \downarrow & & \downarrow \omega \\ A & \xrightarrow[f]{} & B \end{array}$$
i.e. f is a homomorphism

A theory is a category

This is what is meant in Lawvere's "Functorial semantics of algebraic theories"

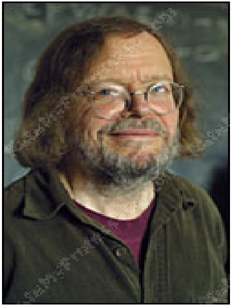
- An algebraic theory is a *category* $\mathbf{T}$ of a certain type
- An algebra is a *functor* $\mathbf{T} \longrightarrow \mathbf{Set}$ with certain properties
- A homomorphism is a *natural* transformation

This opened the flood gates!
Other theories are categories with other properties

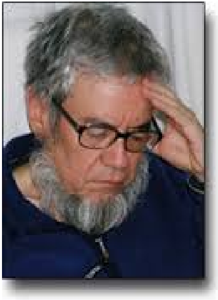
- Categories with finite products give multi-sorted theories
- Categories with finite limits give essentially algebraic theories
- Regular categories give logic with $\exists, \vee, \wedge$
- Pretoposes give full first-order logic $\exists, \forall, \neg, \wedge, \vee$
  - a theory is a category with some structure
  - a model is a functor $\mathbf{T} \longrightarrow \mathbf{Set}$ that preserves the relevant structure
  - a morphism is a natural transformation

Conceptual completeness

Michael Makkai



Gonzalo Reyes



Theorem (Makkai, Reyes)

Pretoposes are conceptually complete

Means: Any "concept" can be defined by formula in the theory
For this they had to say what a *concept* was
This is one of the things category theory can do. It can make precise some intuitive notions. We can now prove theorems that couldn't be expressed before (if we're smart enough)

Grothendieck

Alexandre Grothendieck

- Reformulated algebraic geometry in terms of categories (1960's)
- The new set-up allowed him to prove part of the Weil conjectures
- Deligne proved the general case
    - Grothendieck's framework was essential

Today

- Computer science
  - Semantics of programming languages
  - Design of programming languages
  - Computer verification of programs
  - Quantum computing (Peter Selinger)
- Higher category theory
  - Joyal – Homotopy theory / Spaces and higher categories are "the same"
  - Baez – Quantum gravity / Nature of empty space
  - Makkai – Foundations of mathematics
  - Voivodski – Homotopy type theory

International Conference on Category Theory 2006
White Point, NS