

A double category take on restriction categories

Robert Paré

CT2018
Azores, Portugal

July 13, 2018

The plan

- The theory of restriction categories is a nice, simply axiomatized theory of partial morphisms
- It is well motivated with many examples and has lots of nice results
- But it is somewhat tangential to mainstream category theory
- The plan is to bring it back into the fold by taking a double category perspective
- Every restriction category has a canonically associated double category
- What can double categories tell us about restriction categories?
- What can restriction categories tell us about double categories?
- References

[CL] R. Cockett, S. Lack, Restriction Categories I: Categories of Partial Maps, Theoretical Computer Science 270 (2002) 223-259

[C] R. Cockett, Introduction to Restriction Categories, Estonia Slides (2010)

[DeW] D. DeWolf, Restriction Category Perspectives of Partial Computation and Geometry, Thesis, Dalhousie University, 2017

If you want something done right
you have to do it yourself.
AND, you have to do it right.

Micah McCurdy

Double categories

- There are many instances where we have two kinds of morphism between the same kind of objects:
 - External/internal
 - Total/partial
 - Deterministic/stochastic
 - Classical/quantum
 - Linear/smooth
 - Classical/intuitionistic
 - Lax/oplax
 - Strong/weak
 - Horizontal/vertical

Double categories formalize this

- A double category is a category with two kinds of morphisms, \longrightarrow and $\longrightarrow\bullet\longrightarrow$, and cells \Downarrow relating them

The usual suspects

- $\mathbb{R}el$ – Sets, functions, relations

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 R \downarrow & \leq & \downarrow S \\
 C & \xrightarrow{g} & D
 \end{array}
 \quad a \sim_R c \Rightarrow f(a) \sim_S g(c)$$

If \mathbf{A} is a regular category we can also construct $\mathbb{R}el(\mathbf{A})$

- $\square \mathbf{A}$ – \mathbf{A} any category – the double category of commutative squares in \mathbf{A}

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 h \downarrow & & \downarrow k \\
 C & \xrightarrow{g} & D
 \end{array}$$

There is a *subdouble category* of pullback squares $\mathbb{P}b\square \mathbf{A}$

- $\mathbb{Q}\mathcal{A}$ – \mathcal{A} is a 2-category – the double category of *quintets* in \mathcal{A}

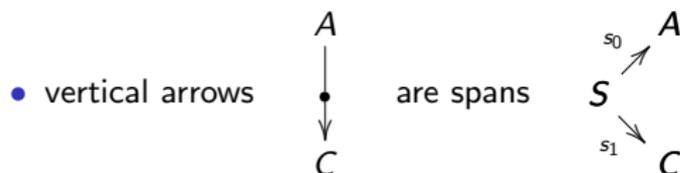
$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 h \downarrow & \alpha \swarrow & \downarrow k \\
 C & \xrightarrow{g} & D
 \end{array}$$

Spans

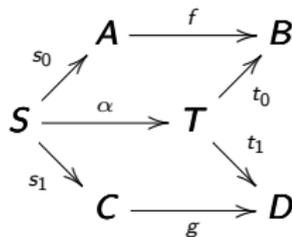
A a category with pullbacks

$\text{Span}(\mathbf{A})$ has same objects as **A**

- horizontal arrows are morphisms of **A**



- cells $\begin{array}{ccc} A & \xrightarrow{f} & B \\ s \downarrow & \Rightarrow & \downarrow T \\ C & \xrightarrow{g} & D \end{array}$ are commutative diagrams



- vertical composition uses pullbacks

$\text{Span}(\mathbf{A})$ is a *weak* double category

Restriction categories

Definition

A *restriction category* is a category equipped with a *restriction operator*

$$A \xrightarrow{f} B \rightsquigarrow A \xrightarrow{\bar{f}} A$$

satisfying

$$\text{R1. } f\bar{f} = f$$

$$\text{R2. } \bar{f}\bar{g} = \bar{g}\bar{f}$$

$$\text{R3. } \overline{gf} = \bar{g}\bar{f}$$

$$\text{R4. } \bar{g}f = f\overline{gf}$$

Example

Let \mathbf{A} be a category. A *stable system of monics* \mathbf{M} is a subcategory such that

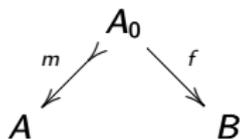
- (1) $m \in \mathbf{M} \Rightarrow m$ monic
- (2) \mathbf{M} contains all isomorphisms
- (3) \mathbf{M} stable under pullback: for every $m \in \mathbf{M}$ and $f \in \mathbf{A}$ as below, the pullback of m along f exists and is in \mathbf{M}

$$\begin{array}{ccc} P & \xrightarrow{g} & B \\ m' \downarrow & \lrcorner & \downarrow m \\ C & \xrightarrow{f} & A \end{array}$$

$$m \in M \Rightarrow m' \in M$$

M-partial morphisms

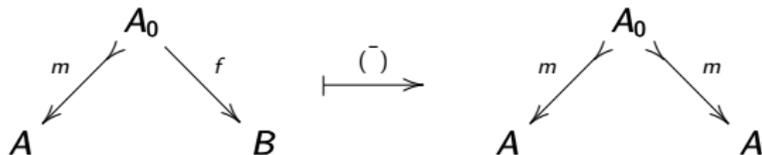
$\text{Par}_M \mathbf{A}$ has the same objects as \mathbf{A} but the morphisms are isomorphism classes of spans



with $m \in M$

Composition is by pullback (like for spans)

The restriction operator is $\overline{(m, f)} = (m, m)$



Example

Let $\mathbf{A} = \mathbf{Top}$ and M given by the open subspaces. Then $\text{Par}_M(\mathbf{Top})$ is the category of topological spaces with continuous functions defined on an open subspace

Properties (cribbed from [CL])

P1. $\bar{f}^2 = \bar{f}$

P2. $\bar{f} \overline{gf} = \overline{gf}$

P3. $\overline{\overline{gf}} = \overline{gf}$

P4. $\overline{\bar{f}} = \bar{f}$

P5. $\overline{\overline{\overline{gf}}} = \overline{\overline{gf}}$

P6. $f\bar{g} = f \Rightarrow \bar{f} = \bar{f}\bar{g}$

Definition

f is *total* if $\bar{f} = 1$

T1. Monos are total

T2. f, g total $\Rightarrow gf$ total

T3. gf total $\Rightarrow f$ total

Order (also from [CL])

Definition

For $f, g : A \rightarrow B$ define $f \leq g$ iff $f = g\bar{f}$

Theorem

\leq is an order relation compatible with composition. This makes \mathbf{A} into a (locally ordered) 2-category

“...seems to be less useful than one might expect” – [CL]

The double category

Let \mathbf{A} be a restriction category

The double category $\mathbb{D}_c(\mathbf{A})$ associated to a restriction category \mathbf{A} has

- the same objects as \mathbf{A}
- total maps as horizontal morphisms
- all maps as vertical morphisms

- There is a unique cell

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow v & \Rightarrow & \downarrow w \\ C & \xrightarrow{g} & D \end{array}$$

if and only if $gv \leq wf$ (iff $gv = wf\bar{v}$)

“Perhaps this will turn out to be more useful than one might expect!” – Me

Companions

Definition

$A \xrightarrow{f} B$ and $A \xrightarrow{v} B$ are *companions* if there are given cells (binding cells)

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 v \downarrow & \epsilon & \downarrow \text{id}_B \\
 B & \xrightarrow{1_B} & B
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 A & \xrightarrow{1_A} & A \\
 \text{id}_A \downarrow & \eta & \downarrow v \\
 A & \xrightarrow{f} & B
 \end{array}$$

such that

$$\begin{array}{ccc}
 A & \xrightarrow{1_A} & A & \xrightarrow{f} & B \\
 \text{id}_A \downarrow & \eta & \downarrow v & \epsilon & \downarrow \text{id}_B \\
 A & \xrightarrow{f} & B & \xrightarrow{1_B} & B
 \end{array}
 =
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \text{id}_A \downarrow & \text{id}_f & \downarrow \text{id}_B \\
 A & \xrightarrow{f} & B
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 A & \xrightarrow{1_A} & A \\
 \text{id}_A \downarrow & \eta & \downarrow v \\
 A & \xrightarrow{f} & B \\
 v \downarrow & \epsilon & \downarrow \text{id}_B \\
 B & \xrightarrow{1_B} & B
 \end{array}
 =
 \begin{array}{ccc}
 A & \xrightarrow{1_A} & A \\
 v \downarrow & 1_v & \downarrow v \\
 B & \xrightarrow{1_B} & B
 \end{array}$$

In $\mathbb{D}\mathbf{C}(\mathbf{A})$ every horizontal arrow has a companion, $f_* = f$

Conjoints

Definition

$A \xrightarrow{f} B$ and $B \xrightarrow{u} A$ are *conjoints* if there are given cells (conjunctions)

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \text{id}_A \downarrow \bullet & \alpha & \downarrow \bullet u \\ A & \xrightarrow{1_A} & A \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{1_B} & B \\ u \downarrow \bullet & \beta & \downarrow \bullet \text{id}_B \\ A & \xrightarrow{f} & B \end{array}$$

such that $\beta\alpha = \text{id}_f$ and $\alpha \bullet \beta = 1_u$

Proposition

In $\mathbb{D}c\text{Par}_M(\mathbf{A})$, f has a conjoint if and only if $f \in M$

Conjoints, companions, adjoints

Definition

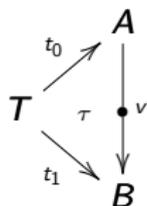
$A \xrightarrow{v} B$ is left adjoint to $B \xrightarrow{u} A$ if it is so in $\mathcal{V}ert \mathbb{A}$

Theorem

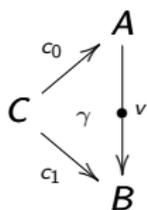
- (1) If f has a companion (conjoint) it is unique up to globular isomorphism
- (2) If f has companion (conjoint) v and g has companion (resp. conjoint) w then gf has companion $w \bullet v$ (resp. conjoint $v \bullet w$)
- (3) Any two of the following conditions imply the third
 - v is a companion for f
 - u is a conjoint for f
 - v is left adjoint to u in $\mathcal{V}ert \mathbb{A}$

Tabulators

Given a vertical arrow $A \xrightarrow{\bullet} B$ in \mathbb{A} its *tabulator*, if it exists, is an object T and a cell τ



such that for any other cell



there exists a unique horizontal morphism $c : C \rightarrow T$ such that $\gamma = \tau c$

The tabulator is *effective* if t_0 has a conjoint t_0^* and t_1 has a companion t_{1*} and the canonical cell induced by τ , $t_{1*} \bullet t_0^* \rightarrow v$ is an isomorphism

Tabulators

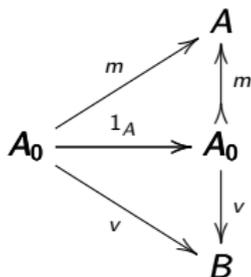
Proposition

In $\mathbb{D}c(\mathbf{A})$, $v : A \dashrightarrow B$ has an effective tabulator if and only if \bar{v} splits

Corollary

$\mathbb{D}c\text{Par}_M(\mathbf{A})$ has tabulators and they are effective

The tabulator of $(m, v) : A \dashrightarrow B$ is:



Double functors and all that

A *double functor* $F : \mathbb{A} \rightarrow \mathbb{B}$ is a function taking elements of \mathbb{A} to similar ones of \mathbb{B}

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow v & \alpha & \downarrow v' \\ C & \xrightarrow{g} & C' \end{array} \quad \mapsto \quad \begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ \downarrow Fv & F\alpha & \downarrow Fv' \\ FC & \xrightarrow{Fg} & FC' \end{array}$$

preserving all compositions and identities

There is a category **Doub** of double categories and double functors

Theorem

Doub is cartesian closed

This tells us that between double functors there are canonically defined *horizontal and vertical transformations* as well as *double modifications* relating them

Restriction functors

- A *restriction functor* $F : \mathbf{A} \rightarrow \mathbf{B}$ is a functor that preserves the restriction operator, $F(\bar{f}) = \overline{F(f)}$

Theorem

- A double functor $F : \mathbb{D}\mathbf{C}(\mathbf{A}) \rightarrow \mathbb{D}\mathbf{C}(\mathbf{B})$ is determined by a unique functor $F : \mathbf{A} \rightarrow \mathbf{B}$ which preserves the order and totality
- Every restriction functor F gives a double functor $F : \mathbb{D}\mathbf{C}(\mathbf{A}) \rightarrow \mathbb{D}\mathbf{C}(\mathbf{B})$
- Every double functor $F : \mathbb{D}\mathbf{C}(\mathbf{A}) \rightarrow \mathbb{D}\mathbf{C}(\mathbf{B})$ gives a functor $F : \mathbf{A} \rightarrow \mathbf{B}$ such that $F(\bar{f}) \geq \overline{F(f)}$

Restriction functors

Question: Does every double functor $F : \mathbb{D}c(\mathbf{A}) \longrightarrow \mathbb{D}c(\mathbf{B})$ come from a restriction functor $F : \mathbf{A} \longrightarrow \mathbf{B}$, i.e., do we get equality $F(\bar{f}) = \overline{F(f)}$?

Theorem

- If $F : \mathbb{D}c(\mathbf{A}) \longrightarrow \mathbb{D}c(\mathbf{B})$ has a right adjoint, then F comes from a restriction functor
- A double functor $\mathbb{D}c\text{Par}_M \mathbf{A} \longrightarrow \mathbb{D}c\text{Par}_N \mathbf{B}$ comes from a unique functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ which restricts to $\mathbf{M} \longrightarrow \mathbf{N}$ and preserves pullbacks of $m \in M$ by arbitrary $f \in \mathbf{A}$. Thus it does come from a restriction functor

Corollary

If $F : \mathbb{D}c(\mathbf{A}) \longrightarrow \mathbb{D}c(\mathbf{B})$ is an isomorphism, then $F : \mathbf{A} \longrightarrow \mathbf{B}$ is an isomorphism of restriction categories, i.e., $\mathbb{D}c$ is conservative

Taming $\overline{(\)}$

The corollary tells us that the double category $\mathbb{D}c\mathbf{A}$ completely determines the restriction structure. So how can we recover $\overline{(\)}$?

Proposition

For any $v : A \dashrightarrow B$, \bar{v} is the least $e : A \dashrightarrow A$ such that

- (1) $e \leq 1_A$
- (2) $v = v \bullet e$

Proof.

- (1) $e \leq 1_A \Leftrightarrow e = \bar{e}$
- (2) $\bar{v} = \overline{v \bullet e} = \overline{v \bullet \bar{e}} = \bar{v} \bullet \bar{e} \leq \bar{e} = e$



This proposition suggest that $\overline{(\)}$ is some kind of left adjoint

Condition (1) $e \leq 1_A$ is equivalent to $e = \bar{e}$ which is idempotent, and so is equivalent to being a comonad in $\mathbb{D}c\mathbf{A}$

In the presence of (1), condition (2) $v = v \bullet e$ is equivalent to $v \leq v \bullet e$, i.e. v has a coaction by e

For any double category \mathbb{A} we have

(a) A category of comonads $\mathbf{CoMon}\mathbb{A}$

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 e \downarrow & \epsilon & \downarrow \text{id} \\
 A & \xlongequal{\quad} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 \downarrow & & \downarrow e \\
 e \bullet & \delta & A \\
 \downarrow & & \downarrow e \\
 A & \xlongequal{\quad} & A
 \end{array}
 \quad \text{with unit and associativity laws}$$

$$\begin{array}{ccc}
 A & \xrightarrow{f} & A' \\
 e \downarrow & \phi & \downarrow e' \\
 A & \xrightarrow{f} & A'
 \end{array}
 \quad \text{compatible with unit and comultiplication}$$

(b) A category of vertical arrows $\mathbf{VArr}\mathbb{A}$

$$\begin{array}{ccc}
 A & \xrightarrow{f} & A' \\
 v \downarrow & \gamma & \downarrow v' \\
 B & \xrightarrow{g} & B'
 \end{array}$$

(c) A category of coactions $\mathbf{CoAct}_{\mathbb{A}}$

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 \downarrow v & \alpha & \downarrow e \\
 \bullet & & \bullet \\
 \downarrow & & \downarrow v \\
 \bullet & & \bullet \\
 \downarrow & & \downarrow \\
 B & \xlongequal{\quad} & B
 \end{array}$$

e comonad

v a vertical arrow

α satisfies unit and associativity laws

$$\begin{array}{ccc}
 A & \xrightarrow{f} & A' \\
 \downarrow e & \phi & \downarrow e' \\
 \bullet & & \bullet \\
 \downarrow & & \downarrow v' \\
 \bullet & & \bullet \\
 \downarrow v & \gamma & \downarrow \\
 \bullet & & \bullet \\
 \downarrow & & \downarrow \\
 B & \xrightarrow{g} & B'
 \end{array}$$

compatible with counits and comultiplication

(d) Functors

$$\mathbf{VArr}_{\mathbb{A}} \begin{array}{c} \xleftarrow{U} \\ \xrightarrow{I} \end{array} \mathbf{CoAct}_{\mathbb{A}} \xrightarrow{V} \mathbf{CoMon}_{\mathbb{A}}$$

U and V are forgetful functors, $I(v) = (1_A, v)$, I is a full and faithful right adjoint to U

Theorem

If \mathbb{A} is $\mathbb{D}c\mathbf{A}$ for a restriction category \mathbf{A} , then U has a left adjoint F given by

$$F(v) = (\bar{v}, v)$$

Remarks :

(1) $\overline{(\quad)} = VF$ so it is a functor

$$\mathbf{VArrD}c\mathbf{A} \longrightarrow \mathbf{ComonD}c\mathbf{A}$$

(2) V has a left adjoint if and only if \mathbb{A} admits Kleisli constructions

Transformations

A horizontal transformation $t : F \longrightarrow G$ between double functors $\mathbb{A} \longrightarrow \mathbb{B}$ consists of assignments:

- (1) for every A in \mathbb{A} a horizontal morphism $tA : FA \longrightarrow GA$
- (2) for every vertical morphism $v : A \longrightarrow C$ a cell

$$\begin{array}{ccc} FA & \xrightarrow{tA} & GA \\ \downarrow Fv & & \downarrow Gv \\ GC & \xrightarrow{tC} & GC \end{array} \quad tv$$

satisfying

- (3) horizontal naturality (for horizontal arrows and cells)
- (4) vertical functoriality (for identities and composition)

Transformations between restriction functors

Let $F, G : \mathbf{A} \rightarrow \mathbf{B}$ be restriction functors. Then a horizontal transformation

$$t : \mathbb{D}c(F) \rightarrow \mathbb{D}c(G)$$

(1) assigns to each A in \mathbf{A} a total morphism

$$tA : FA \rightarrow GA$$

(2) such that for every $f : A \rightarrow \bullet \rightarrow C$ in \mathbf{A} we have

$$\begin{array}{ccc} FA & \xrightarrow{tA} & GA \\ \downarrow Ff \bullet & \leq & \downarrow \bullet Gf \\ FC & \xrightarrow{tC} & GC \end{array}$$

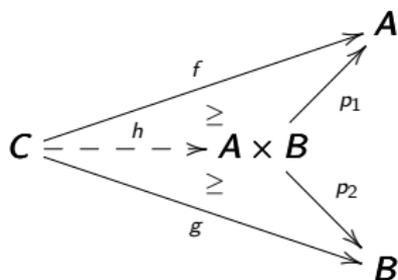
(3) and t is natural for horizontal arrows (i.e. for f total, we have equality in (2))

This is what [CL] call a lax restriction transformation

A vertical transformation $\phi : \mathbb{D}c(F) \rightarrow \mathbb{D}c(G)$ corresponds to an arbitrary natural transformation $F \rightarrow G$

Cartesian restriction categories

A restriction category \mathbf{A} is *cartesian* if for every pair of objects A, B there is an object $A \times B$ and morphisms $p_1 : A \times B \rightarrow A, p_2 : A \times B \rightarrow B$ with the following universal property



For every f, g there exists a unique h such that

$$p_1 h = f \bar{g}$$

$$p_2 h = g \bar{f}$$

(There is also a terminal object condition)

Double products

\mathbb{A} has binary double products if the diagonal $\Delta : \mathbb{A} \longrightarrow \mathbb{A} \times \mathbb{A}$ has a right adjoint in the 2-category of double categories, pseudo-functors and horizontal transformations

This means that:

- (1) for every A, B there is an object $A \times B$ and horizontal arrows $p_1 : A \times B \longrightarrow A$, $p_2 : A \times B \longrightarrow B$ which have the usual universal property with respect to horizontal arrows
- (2) for every pair of vertical arrows $v : A \longrightarrow C$ and $w : B \longrightarrow D$ there is a vertical arrow $v \times w : A \times B \longrightarrow C \times D$ and cells

$$\begin{array}{ccc} A \times B & \xrightarrow{p_1} & A \\ \downarrow v \times w & \pi_1 & \downarrow v \\ C \times D & \xrightarrow{q_2} & C \end{array} \qquad \begin{array}{ccc} A \times B & \xrightarrow{p_2} & B \\ \downarrow v \times w & \pi_2 & \downarrow w \\ C \times D & \xrightarrow{q_2} & D \end{array}$$

with the usual universal property with respect to cells

Conjecture:

\mathbf{A} is a cartesian restriction category if and only if $\mathbb{D}\mathbf{c}(\mathbf{A})$ has finite double products

If \mathbf{A} is a cartesian restriction category, then the universal property of product is the usual one when restricted to total maps, which is the one-dimensional property of double products

Given vertical arrows $v : A \twoheadrightarrow C$, $w : B \twoheadrightarrow D$ we get a unique $v \times w : A \times B \twoheadrightarrow C \times D$ such that

$$q_1 \bullet v \times w = v \bullet p_1 \bullet \overline{w \bullet p_2}$$

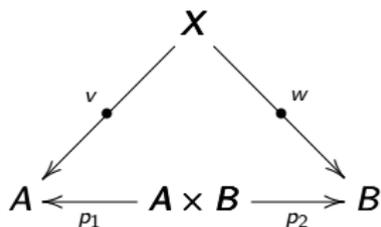
$$q_2 \bullet v \times w = v \bullet p_2 \bullet \overline{w \bullet p_1}$$

Then

$$\begin{array}{ccc} \begin{array}{ccc} X & \xrightarrow{f} & A \\ \downarrow z & \leq & \downarrow v \\ Y & \xrightarrow{g} & C \end{array} & \& & \begin{array}{ccc} X & \xrightarrow{h} & B \\ \downarrow z & \leq & \downarrow w \\ Y & \xrightarrow{k} & D \end{array} & \Leftrightarrow & \begin{array}{ccc} X & \xrightarrow{(f,h)} & A \times B \\ \downarrow z & \leq & \downarrow v \times w \\ Y & \xrightarrow{(g,k)} & C \times D \end{array} \end{array}$$

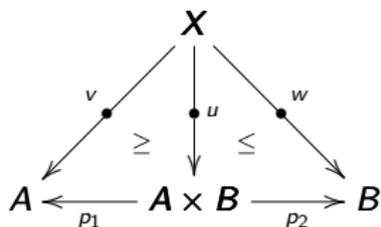
which is the two-dimensional property of double products in $\mathbb{D}\mathbf{c}(\mathbf{A})$

On the other hand if $\mathbb{D}c\mathbb{A}$ has double products, then for



we let $\langle v, w \rangle = X \xrightarrow{\Delta_*} X \times X \xrightarrow{v \times w} A \times B$

For any u as in



with $p_1 u \leq v$ and $p_2 u \leq w$, we have $u \leq \langle v, w \rangle$, i.e. $\langle v, w \rangle$ is the largest such u