A double category take on restriction categories

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The plan

- The theory of restriction categories is a nice, simply axiomatized theory of partial morphisms
- It is well motivated with many examples and has lots of nice results
- But it is somewhat tangential to mainstream category theory
- The plan is to bring it back into the fold by taking a double category perspective
- Every restriction category has a canonically associated double category
- What can double categories tell us about restriction categories?
- What can restriction categories tell us about double categories?
- References

[CL] R. Cockett, S. Lack, Restriction Categories I: Categories of Partial Maps, Theoretical Computer Science 270 (2002) 223-259

[C] R. Cockett, Introduction to Restriction Categories, Estonia Slides (2010)

[DeW] D. DeWolf, Restriction Category Perspectives of Partial Computation and Geometry, Thesis, Dalhousie University, 2017

Words of wisdom

> If you want something done right
> you have to do it yourself.
> AND, you have to do it right.
>
> Micah McCurdy

Double categories

- There are many instances where we have two kinds of morphism between the same kind of objects:
  - External/internal
  - Total/partial
  - Deterministic/stochastic
  - Classical/quantum
  - Linear/smooth
  - Classical/intuitionistic
  - Lax/oplax
  - Strong/weak
  - Horizontal/vertical

  Double categories formalize this
- A double category is a category with two kinds of morphisms, $\longrightarrow$ and $\xrightarrow{\bullet}$, and cells $\Downarrow$ relating them

The usual suspects

- $\mathbb{R}\mathrm{el}$ – Sets, functions, relations

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "R"', "\bullet" marking] \arrow[dr, phantom, "\leq"] & B \arrow[d, "S", "\bullet" marking] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
\qquad a \sim_R c \Rightarrow f(a) \sim_S g(c)$$

If $\mathbf{A}$ is a regular category we can also construct $\mathbb{R}\mathrm{el}(\mathbf{A})$

- $\Box\mathbf{A}$ – $\mathbf{A}$ any category – the double category of commutative squares in $\mathbf{A}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

There is a *subdouble category* of pullback squares $\mathbb{P}\mathrm{b}\Box\mathbf{A}$

- $\mathbb{Q}\mathcal{A}$ – $\mathcal{A}$ is a 2-category – the double category of *quintets* in $\mathcal{A}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\alpha \swarrow"] & B \arrow[d, "k"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

Spans

A a category with pullbacks
$\mathbb{S}\text{pan}(\mathbf{A})$ has same objects as **A**

- horizontal arrows are morphisms of **A**
- vertical arrows $$\begin{tikzcd} A \arrow[d, "\bullet" marking] \\ C \end{tikzcd}$$ are spans $$\begin{tikzcd} & A \\ S \arrow[ur, "s_0"] \arrow[dr, "s_1"'] & \\ & C \end{tikzcd}$$
- cells $$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\bullet" marking, "S"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "\bullet" marking, "T"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$ are commutative diagrams $$\begin{tikzcd} & A \arrow[rr, "f"] & & B \\ S \arrow[ur, "s_0"] \arrow[dr, "s_1"'] \arrow[rr, "\alpha"] & & T \arrow[ur, "t_0"'] \arrow[dr, "t_1"] & \\ & C \arrow[rr, "g"'] & & D \end{tikzcd}$$
- vertical composition uses pullbacks

$\mathbb{S}\text{pan}(\mathbf{A})$ is a *weak* double category

Restriction categories

Definition

A *restriction category* is a category equipped with a *restriction operator*

$$A \xrightarrow{f} B \rightsquigarrow A \xrightarrow{\overline{f}} A$$

satisfying

R1. $f\overline{f} = f$

R2. $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$

R3. $\overline{g\overline{f}} = \overline{g}\,\overline{f}$

R4. $\overline{g}f = f\,\overline{gf}$

Example

Let **A** be a category. A *stable system of monics* **M** is a subcategory such that

(1) $m \in \mathbf{M} \Rightarrow m$ monic

(2) **M** contains all isomorphisms

(3) **M** stable under pullback: for every $m \in \mathbf{M}$ and $f \in \mathbf{A}$ as below, the pullback of m along f exists and is in **M**

$$\begin{tikzcd}
P \arrow[r, "g"] \arrow[d, tail, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, tail, "m"] \\
C \arrow[r, "f"'] & A
\end{tikzcd}$$

$$m \in M \Rightarrow m' \in M$$

M-partial morphisms

$\mathrm{Par}_{\mathbf{M}}\mathbf{A}$ has the same objects as **A** but the morphisms are isomorphism classes of spans

$$\begin{tikzcd}
& A_0 \arrow[dl, tail, "m"'] \arrow[dr, "f"] & \\
A & & B
\end{tikzcd}$$

with $m \in M$

Composition is by pullback (like for spans)

The restriction operator is $\overline{(m, f)} = (m, m)$

$$\begin{tikzcd}
& A_0 \arrow[dl, tail, "m"'] \arrow[dr, "f"] & & & A_0 \arrow[dl, tail, "m"'] \arrow[dr, tail, "m"] & \\
A & & B \arrow[r, mapsto, "(\bar{\ })"] & A & & A
\end{tikzcd}$$

Example

Let $\mathbf{A} = \mathbf{Top}$ and **M** given by the open subspaces. Then $\mathrm{Par}_{\mathbf{M}}(\mathbf{Top})$ is the category of topological spaces with continuous functions defined on an open subspace

Properties (cribbed from [CL])

P1. $\overline{f}^2 = \overline{f}$

P2. $\overline{f}\,\overline{gf} = \overline{gf}$

P3. $\overline{\overline{g}f} = \overline{gf}$

P4. $\overline{\overline{f}} = \overline{f}$

P5. $\overline{\overline{g}\,\overline{f}} = \overline{g}\,\overline{f}$

P6. $f\overline{g} = f \Rightarrow \overline{f} = \overline{f}\,\overline{g}$

Definition

f is *total* if $\overline{f} = 1$

T1. Monos are total

T2. f, g total $\Rightarrow$ gf total

T3. gf total $\Rightarrow$ f total

Order (also from [CL])

Definition

For $f, g : A \longrightarrow B$ define $f \leq g$ iff $f = g\overline{f}$

Theorem

$\leq$ is an order relation compatible with composition. This makes **A** *into a (locally ordered) 2-category*

"...seems to be less useful than one might expect" – [CL]

The double category

Let **A** be a restriction category

The double category $\mathbb{D}c(\mathbf{A})$ associated to a restriction category **A** has

- the same objects as **A**
- total maps as horizontal morphisms
- all maps as vertical morphisms
- There is a unique cell $$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "w", "\bullet" marking] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$ if and only if $gv \leq wf$ (iff $gv = wf\overline{v}$)

“Perhaps this will turn out to be more useful than one might expect!” – Me

Companions

Definition

$A \xrightarrow{f} B$ and $\begin{tikzcd} A \arrow[r, "v", "\bullet" marking] & B \end{tikzcd}$ are *companions* if there are given cells (binding cells)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\epsilon"] & B \arrow[d, "\mathrm{id}_B", "\bullet" marking] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "\mathrm{id}_A"', "\bullet" marking] \arrow[dr, phantom, "\eta"] & A \arrow[d, "v", "\bullet" marking] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

such that

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "\mathrm{id}_A"', "\bullet" marking] \arrow[dr, phantom, "\eta"] & A \arrow[r, "f"] \arrow[d, "v", "\bullet" marking] \arrow[dr, phantom, "\epsilon"] & B \arrow[d, "\mathrm{id}_B", "\bullet" marking] \\
A \arrow[r, "f"'] & B \arrow[r, "1_B"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{id}_f"] & B \arrow[d, "\mathrm{id}_B", "\bullet" marking] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "\mathrm{id}_A"', "\bullet" marking] \arrow[dr, phantom, "\eta"] & A \arrow[d, "v", "\bullet" marking] \\
A \arrow[r, "f"'] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\epsilon"] & B \arrow[d, "\mathrm{id}_B", "\bullet" marking] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "1_v"] & A \arrow[d, "v", "\bullet" marking] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}$$

In $\mathbb{D}c(\mathbf{A})$ every horizontal arrow has a companion, $f_* = f$

Conjoints

Definition

$A \xrightarrow{f} B$ and $\begin{tikzcd} B \arrow[r, "u", "\bullet" marking] & A \end{tikzcd}$ are *conjoints* if there are given cells (conjunctions)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "u", "\bullet" marking] & B \arrow[r, "1_B"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & B \arrow[d, "\mathrm{id}_B", "\bullet" marking] \\
A \arrow[r, "1_A"'] & A & A \arrow[r, "f"'] & B
\end{tikzcd}$$

such that $\beta\alpha = \mathrm{id}_f$ and $\alpha \bullet \beta = 1_u$

Proposition

In $\mathbb{D}c\mathrm{Par}_{\mathbf{M}}(\mathbf{A})$*,* f *has a conjoint if and only if* $f \in M$

Conjoints, companions, adjoints

Definition

$A \xrightarrow{v} B$ is *left adjoint* to $B \xrightarrow{u} A$ if it is so in $\mathcal{V}ert\,\mathbb{A}$

Theorem

(1) *If f has a companion (conjoint) it is unique up to globular isomorphism*

(2) *If f has companion (conjoint) v and g has companion (resp. conjoint) w then gf has companion $w \bullet v$ (resp. conjoint $v \bullet w$)*

(3) *Any two of the following conditions imply the third*

- *v is a companion for f*
- *u is a conjoint for f*
- *v is left adjoint to u in $\mathcal{V}ert\,\mathbb{A}$*

Tabulators

Given a vertical arrow $A \overset{v}{\longrightarrow\!\!\!\!\!\!\bullet\;\;\;} B$ in $\mathbb{A}$ its *tabulator*, if it exists, is an object T and a cell τ

$$\begin{tikzcd}
 & A \arrow[dd, "\bullet" marking, "v"] \\
T \arrow[ur, "t_0"] \arrow[dr, "t_1"'] \arrow[r, phantom, "\tau"] & {} \\
 & B
\end{tikzcd}$$

such that for any other cell

$$\begin{tikzcd}
 & A \arrow[dd, "\bullet" marking, "v"] \\
C \arrow[ur, "c_0"] \arrow[dr, "c_1"'] \arrow[r, phantom, "\gamma"] & {} \\
 & B
\end{tikzcd}$$

there exists a unique horizontal morphism $c : C \longrightarrow T$ such that $\gamma = \tau c$

The tabulator is *effective* if t_0 has a conjoint t_0^* and t_1 has a companion t_{1*} and the canonical cell induced by τ, $t_{1*} \bullet t_0^* \longrightarrow v$ is an isomorphism

Tabulators

Proposition

In $\mathbb{D}c(\mathbf{A})$, $v : A \mathrel{-\!\!\!\bullet\!\!\!\to} B$ has an effective tabulator if and only if $\overline{v}$ splits

Corollary

$\mathbb{D}c\mathrm{Par}_{\mathbf{M}}(\mathbf{A})$ *has tabulators and they are effective*

The tabulator of $(m, v) : A \mathrel{-\!\!\!\bullet\!\!\!\to} B$ is:

$$\begin{tikzcd}
& A \\
A_0 \arrow[ur, "m"] \arrow[r, "1_A"] \arrow[dr, "v"'] & A_0 \arrow[u, tail, "m"'] \arrow[d, "v"] \\
& B
\end{tikzcd}$$

Double functors and all that

A *double functor* $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a function taking elements of $\mathbb{A}$ to similar ones of $\mathbb{B}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "v'", "\bullet" marking] & & FA \arrow[r, "Ff"] \arrow[d, "Fv"', "\bullet" marking] \arrow[dr, phantom, "F\alpha"] & FA' \arrow[d, "Fv'", "\bullet" marking] \\
C \arrow[r, "g"'] & C' & & FC \arrow[r, "Fg"'] & FC'
\arrow[from=1-2, to=2-4, phantom, "\longmapsto"]
\end{tikzcd}$$

preserving all compositions and identities

There is a category **Doub** of double categories and double functors

Theorem

Doub *is cartesian closed*

This tells us that between double functors there are canonically defined *horizontal and vertical transformations* as well as *double modifications* relating them

Restriction functors

- A *restriction functor* $F : \mathbf{A} \longrightarrow \mathbf{B}$ is a functor that preserves the restriction operator, $F(\overline{f}) = \overline{F(f)}$

Theorem

- *A double functor* $F : \mathbb{D}c(\mathbf{A}) \longrightarrow \mathbb{D}c(\mathbf{B})$ *is determined by a unique functor* $F : \mathbf{A} \longrightarrow \mathbf{B}$ *which preserves the order and totality*
- *Every restriction functor* F *gives a double functor* $F : \mathbb{D}c(\mathbf{A}) \longrightarrow \mathbb{D}c(\mathbf{B})$
- *Every double functor* $F : \mathbb{D}c(\mathbf{A}) \longrightarrow \mathbb{D}c(\mathbf{B})$ *gives a functor* $F : \mathbf{A} \longrightarrow \mathbf{B}$ *such that* $F(\overline{f}) \geq \overline{F(f)}$

Restriction functors

Question: Does every double functor $F : \mathbb{D}c(\mathbf{A}) \longrightarrow \mathbb{D}c(\mathbf{B})$ come from a restriction functor $F : \mathbf{A} \longrightarrow \mathbf{B}$, i.e., do we get equality $F(\overline{f}) = \overline{F(f)}$?

Theorem

- *If $F : \mathbb{D}c(\mathbf{A}) \longrightarrow \mathbb{D}c(\mathbf{B})$ has a right adjoint, then F comes from a restriction functor*
- *A double functor $\mathbb{D}c\mathrm{Par}_{\mathbf{M}}\mathbf{A} \longrightarrow \mathbb{D}c\mathrm{Par}_{\mathbf{N}}\mathbf{B}$ comes from a unique functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ which restricts to $\mathbf{M} \longrightarrow \mathbf{N}$ and preserves pullbacks of $m \in M$ by arbitrary $f \in \mathbf{A}$ Thus it does come from a restriction functor*

Corollary

If $F : \mathbb{D}c(\mathbf{A}) \longrightarrow \mathbb{D}c(\mathbf{B})$ is an isomorphism, then $F : \mathbf{A} \longrightarrow \mathbf{B}$ is an isomorphism of restriction categories, i.e., $\mathbb{D}c$ is conservative

Taming $\overline{(\ )}$

The corollary tells us that the double category $\mathbb{D}c\mathbf{A}$ completely determines the restriction structure. So how can we recover $\overline{(\ )}$?

Proposition

For any $v : A \longrightarrow\!\!\!\!\!\!\bullet\!\!\!\!\rightarrow B$, $\overline{v}$ *is the least* $e : A \longrightarrow\!\!\!\!\!\!\bullet\!\!\!\!\rightarrow A$ *such that*

(1) $e \leq 1_A$

(2) $v = v \bullet e$

Proof.

(1) $e \leq 1_A \Leftrightarrow e = \overline{e}$

(2) $\overline{v} = \overline{v \bullet e} = \overline{v \bullet \overline{e}} = \overline{v} \bullet \overline{e} \leq \overline{e} = e$

□

Comonads

This proposition suggest that $\overline{(\ )}$ is some kind of left adjoint

Condition (1) $e \leq 1_A$ is equivalent to $e = \overline{e}$ which is idempotent, and so is equivalent to being a comonad in $\mathbb{D}c\mathbf{A}$

In the presence of (1), condition (2) $v = v \bullet e$ is equivalent to $v \leq v \bullet e$, i.e. v has a coaction by e

For any double category $\mathbb{A}$ we have
(a) A category of comonads $\mathbf{CoMon}\mathbb{A}$

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "e"', "\bullet" marking] \arrow[rd, phantom, "\epsilon"] & A \arrow[d, "\mathrm{id}", "\bullet" marking] \\
A \arrow[r, equal] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[dd, "e"', "\bullet" marking] & A \arrow[d, "e", "\bullet" marking] \\
\arrow[r, phantom, "\delta"] & A \arrow[d, "e", "\bullet" marking] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

with unit and associativity laws

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "e"', "\bullet" marking] \arrow[rd, phantom, "\phi"] & A' \arrow[d, "e'", "\bullet" marking] \\
A \arrow[r, "f"'] & A'
\end{tikzcd}$$

compatible with unit and comultiplication

(b) A category of vertical arrows $\mathbf{VArr}\mathbb{A}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[rd, phantom, "\gamma"] & A' \arrow[d, "v'", "\bullet" marking] \\
B \arrow[r, "g"'] & B'
\end{tikzcd}$$

(c) A category of coactions **CoAct**$\mathbb{A}$

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[dd, "v"'] & A \arrow[d, "e"] \\
{} \arrow[r, phantom, "\alpha"] & A \arrow[d, "v"] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

e comonad

v a vertical arrow

α satisfies unit and associativity laws

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "\phi"] & A' \arrow[d, "e'"] \\
A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\gamma"] & A' \arrow[d, "v'"] \\
B \arrow[r, "g"'] & B'
\end{tikzcd}$$

compatible with counits and comultiplication

(d) Functors

$$\begin{tikzcd}
\mathbf{VArr}\mathbb{A} \arrow[r, shift right, "I"'] & \mathbf{CoAct}\mathbb{A} \arrow[l, shift right, "U"'] \arrow[r, "V"] & \mathbf{CoMon}\mathbb{A}
\end{tikzcd}$$

U and V are forgetful functors, $I(v) = (1_A, v)$, I is a full and faithful right adjoint to U

Theorem

If $\mathbb{A}$ is $\mathbb{D}c\mathbf{A}$ for a restriction category $\mathbf{A}$, then U has a left adjoint F given by

$$F(v) = (\overline{v}, v)$$

Remarks :

(1) $\overline{(\ )} = VF$ so it is a functor

$$\mathbf{VArr}\mathbb{D}c\mathbf{A} \longrightarrow \mathbf{Comon}\mathbb{D}c\mathbf{A}$$

(2) V has a left adjoint if and only if $\mathbb{A}$ admits Kleisli constructions

Transformations

A horizontal transformation $t : F \longrightarrow G$ between double functors $\mathbb{A} \longrightarrow \mathbb{B}$ consists of assignments:

(1) for every A in $\mathbb{A}$ a horizontal morphism $tA : FA \longrightarrow GA$

(2) for every vertical morphism $v : A \xrightarrow{\bullet} C$ a cell

$$\begin{tikzcd}
FA \arrow[r, "tA"] \arrow[d, "\bullet" marking, "Fv"'] \arrow[dr, phantom, "tv"] & GA \arrow[d, "\bullet" marking, "Gv"] \\
GC \arrow[r, "tC"'] & GC
\end{tikzcd}$$

satisfying

(3) horizontal naturality (for horizontal arrows and cells)

(4) vertical functoriality (for identities and composition)

Transformations between restriction functors

Let $F, G : \mathbf{A} \longrightarrow \mathbf{B}$ be restriction functors. Then a horizontal transformation

$$t : \mathbb{D}c(F) \longrightarrow \mathbb{D}c(G)$$

(1) assigns to each A in $\mathbf{A}$ a total morphism

$$tA : FA \longrightarrow GA$$

(2) such that for every $f : A \xrightarrow{\bullet} C$ in $\mathbf{A}$ we have

$$\begin{tikzcd}
FA \arrow[r, "tA"] \arrow[d, "Ff"', "\bullet" marking] \arrow[dr, phantom, "\leq"] & GA \arrow[d, "Gf", "\bullet" marking] \\
FC \arrow[r, "tC"'] & GC
\end{tikzcd}$$

(3) and t is natural for horizontal arrows (i.e. for f total, we have equality in (2))

This is what [CL] call a lax restriction transformation

A vertical transformation $\phi : \mathbb{D}c(F) \longrightarrow \mathbb{D}c(G)$ corresponds to an arbitrary natural transformation $F \longrightarrow G$

Cartesian restriction categories

A restriction category **A** is *cartesian* if for every pair of objects A, B there is an object $A \times B$ and morphisms $p_1 : A \times B \longrightarrow A$, $p_2 : A \times B \longrightarrow B$ with the following universal property

$$\begin{tikzcd}
 & & A \\
C \arrow[rr, dashed, "h"] \arrow[rru, "f"] \arrow[rrd, "g"'] & \arrow[u, phantom, "\geq"] \arrow[d, phantom, "\geq"] & A \times B \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
 & & B
\end{tikzcd}$$

For every f, g there exists a unique h such that

$$p_1 h = f\overline{g}$$

$$p_2 h = g\overline{f}$$

(There is also a terminal object condition)

Double products

$\mathbb{A}$ has binary double products if the diagonal $\Delta : \mathbb{A} \longrightarrow \mathbb{A} \times \mathbb{A}$ has a right adjoint in the 2-category of double categories, pseudo-functors and horizontal transformations

This means that:

(1) for every A, B there is an object $A \times B$ and horizontal arrows $p_1 : A \times B \longrightarrow A$, $p_2 : A \times B \longrightarrow B$ which have the usual universal property with respect to horizontal arrows

(2) for every pair of vertical arrows $v : A \xrightarrow{\bullet} C$ and $w : B \xrightarrow{\bullet} D$ there is a vertical arrow $v \times w : A \times B \xrightarrow{\bullet} C \times D$ and cells

$$\begin{tikzcd}
A \times B \arrow[r, "p_1"] \arrow[d, "v \times w"', "\bullet" marking] & A \arrow[d, "v", "\bullet" marking] & & A \times B \arrow[r, "p_2"] \arrow[d, "v \times w"', "\bullet" marking] & B \arrow[d, "w", "\bullet" marking] \\
C \times D \arrow[r, "q_2"'] \arrow[ur, phantom, "\pi_1"] & C & & C \times D \arrow[r, "q_2"'] \arrow[ur, phantom, "\pi_2"] & D
\end{tikzcd}$$

with the usual universal property with respect to cells

Conjecture:

A is a cartesian restriction category if and only if $\mathbb{D}c(\mathbf{A})$ has finite double products

If **A** is a cartesian restriction category, then the universal property of product is the usual one when restricted to total maps, which is the one-dimensional property of double products

Given vertical arrows $v : A \mathrel{-\!\!\bullet\!\!\rightarrow} C$, $w : B \mathrel{-\!\!\bullet\!\!\rightarrow} D$ we get a unique $v \times w : A \times B \mathrel{-\!\!\bullet\!\!\rightarrow} C \times D$ such that

$$q_1 \bullet v \times w = v \bullet p_1 \bullet \overline{w \bullet p_2}$$

$$q_2 \bullet v \times w = v \bullet p_2 \bullet \overline{w \bullet p_1}$$

Then

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dd, "\bullet" marking, "z"'] \arrow[ddr, phantom, "\leq"] & A \arrow[dd, "\bullet" marking, "v"] & & X \arrow[r, "h"] \arrow[dd, "\bullet" marking, "z"'] \arrow[ddr, phantom, "\leq"] & B \arrow[dd, "\bullet" marking, "w"] & & X \arrow[r, "{(f,h)}"] \arrow[dd, "\bullet" marking, "z"'] \arrow[ddr, phantom, "\leq"] & A \times B \arrow[dd, "\bullet" marking, "v \times w"] \\
& & \& & & \Leftrightarrow & & \\
Y \arrow[r, "g"'] & C & & Y \arrow[r, "k"'] & D & & Y \arrow[r, "{(g,k)}"'] & C \times D
\end{tikzcd}$$

which is the two-dimensional property of double products in $\mathbb{D}c(\mathbf{A})$

On the other hand if $\mathbb{D}c\mathbb{A}$ has double products, then for

$$\begin{tikzcd}
 & X \arrow[dl, "v"', "\bullet" marking] \arrow[dr, "w", "\bullet" marking] & \\
A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

we let $\langle v, w \rangle = X \xrightarrow{\Delta_*} X \times X \xrightarrow{v \times w} A \times B$

For any u as in

$$\begin{tikzcd}
 & X \arrow[dl, "v"', "\bullet" marking] \arrow[d, "u", "\bullet" marking] \arrow[dr, "w", "\bullet" marking] & \\
A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] \arrow[ul, phantom, "\geq"] \arrow[ur, phantom, "\leq"] & B
\end{tikzcd}$$

with $p_1 u \leq v$ and $p_2 u \leq w$, we have $u \leq \langle v, w \rangle$, i.e. $\langle v, w \rangle$ is the largest such u