COHERENT THEORIES
AS
DOUBLE LAWVERE THEORIES

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GIVEN OPERATION SYMBOLS $\omega, \omega', \dots$ WITH ARITIES

EQUATIONS BETWEEN TERMS $t = t'$

TERM:
- VARIABLE
- IF ω IS AN m-ARY OPERATION SYMB. AND $t_1, \dots, t_m$ TERMS, THEN $\omega(t_1, \dots, t_m)$ IS ALSO A TERM

EX: RINGS $+, \cdot, 0, 1, -$

$(x+y)\cdot(x-y)+0$ IS A TERM

EQUATIONS

$x\cdot(y+z) = x\cdot y + x\cdot z$

ALGEBRA: SET A WITH OPERATIONS

$\omega : A^m \longrightarrow A$

A TERM $t(x_1, \dots, x_m)$ IS

INTERPRETED AS A FUNCTION

$|t| : A^m \longrightarrow A$

TO BE AN ALGEBRA A MUST SATISFY THE GIVEN EQUATIONS $t = t'$

$$\forall a_1, \dots, a_m \in A \left(|t|(a_1, \dots, a_m) = |t'|(a_1, \dots, a_m) \right)$$

HOMOMORPHISMS ARE FUNCTIONS PRESERVING THE OPERATIONS

LAWVERE THEORIES

CATEGORY $\mathbb{T}$: OBJECTS $[0], [1], [2], \dots$

- $[m] = [1] \times [1] \times \cdots \times [1]$
- MORPHISM $[m] \longrightarrow [1]$ IS EQUIVALENCE CLASS OF TERMS $t(x_1, \dots, x_m)$ WHERE TWO TERMS ARE EQUIVALENT IF THEIR EQUALITY FOLLOWS FORMALLY FROM THE GIVEN EQUATIONS.

AN ALG IS A PROD PRES FUNCTOR $\mathbb{T} \to \mathrm{SET}$.

A HOMOMORPHISM IS A NATURAL TRANSF.

FIRST ORDER LOGIC (FINITARY, ONE SORTED)

E.G. THE THEORY OF ORDERED FIELDS

- OPERATION SYMBOLS: $\omega, \omega', \cdots$ WITH ARITIES ≥ 0

  E.G. $+, \cdot, 0, 1, -$

- PREDICATE SYMBOLS: $P, P', \ldots$ WITH ARITIES ≥ 0

  E.G. $\leq$

- TERMS: - VARIABLES $x_0, x_1, x_2, \cdots$

  - IF ω IS AN n-ARY OP SYMBOL AND $t_1, \ldots t_n$ TERMS THEN $\omega(t_1, \ldots t_n)$ TERM

  E.G. $x \cdot (y + x) + 0$

- FORMULAS: - ATOMIC:
  - $t_1 = t_2$
  - $P(t_1, \cdots t_n)$

  E.G. $x(y+z) = xy + xz$

  $0 \leq x \cdot x$

– COMPOSITE: $\Phi \wedge \Psi$, $\Phi \vee \Psi$, $\Phi \to \Psi$, $\neg\Phi$, T, F, $\exists x\, \Phi$, $\forall x\, \Phi$

POSITIVE EXISTENTIAL: ONLY $\wedge, \vee, T, F, \exists$

COHERENT
THEORY: A SET OF SEQUENTS

$\Phi \vdash \Psi$ $\quad \Phi, \Psi$ P.E. FORMULAS

E.G. $T \vdash (xy)z = x(yz)$

$T \vdash x = 0 \vee \exists y\,(xy = 1)$

$0 \leq x \wedge 0 \leq y \vdash 0 \leq xy$

MODEL: – SET M

– FOR ω n-ARY OP. SYMBOL, $M(\omega): M^n \to M$

– FOR P n-ARY PRED. SYMBOL, $M(P) \subseteq M^n$

TERMS $t(x_1 \ldots x_n)$ INTERPRETED AS FNS $M(t): M^n \to M$

FORMULAS $\Phi(x_1 \ldots x_n)$ " " SUBSETS $M\Phi \subseteq M^n$

$\Phi \vdash \Psi$ HOLDS IN M IF $M\Phi \subseteq M\Psi$

M IS A <u>MODEL</u> IF ALL SEQUENTS IN THEORY HOLD IN M.

<u>MORPHISM</u> $f: M \to M'$ PRESERVES $M(\omega)$ & $M(P)$.

DOUBLE CATEGORIES

- OBJECTS, HORIZONTAL ARROWS
  VERTICAL ARROWS, CELLS

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "w"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

HORIZ COMPOSITION — CATEGORIES

VERT COMPOSITION — COHERENT ASSOCIATIVITY

Ex: $\mathbb{R}$ING — RINGS, HOMOMORPHISM, BIMODULES
VERTICAL COMPOSITION $= \otimes$

Ex: $\mathbb{S}\mathrm{ET}_s$ — SETS, FUNCTIONS, SPANS

SPAN $A \overset{S}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} C$ is

$$\begin{tikzcd} & S \arrow[dl, "\sigma_0"'] \arrow[dr, "\sigma_1"] & \\ A & & C \end{tikzcd}$$

CELL

$$\begin{tikzcd} & A \arrow[r, "f"] & B & \\ S \arrow[ur, "\sigma_0"] \arrow[rrr] \arrow[dr, "\sigma_1"'] & & & T \arrow[ul, "\tau_0"'] \arrow[dl, "\tau_1"] \\ & C \arrow[r, "g"'] & D & \end{tikzcd}$$

VERT COMP
IS BY PULLBACK

Ex: $\mathbb{S}\mathrm{ET}_r$ SETS, FUNCTION, RELATIONS

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "R"'] \arrow[dr, phantom, "!"] & B \arrow[d, "S"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

IF $a \underset{R}{\sim} c \Rightarrow fa \underset{S}{\sim} gc$.

MAIN EXAMPLE:

LET $\mathbb{T}$ BE A FIRST ORDER COHERENT THEORY

DOUBLE CAT $\mathbb{T}$: OBJECTS $[0], [1], [2], \ldots$

- HORIZ. ARR. $[m] \longrightarrow [n]$ n-TUPLE $(t_1, \ldots, t_n)$ OF EQUIVALENCE CLASSES OF m-ARY TERMS. TERMS t, t' ARE EQUIVALENT IF $t = t'$ FOLLOWS FROM THE EQUATIONS IN $\mathbb{T}$.

- VERT. ARR. $[m] \to [p]$ P.E. FORMULA $\Phi(x_1, \ldots, x_m; z_1, \ldots, z_p)$

- CELL

$$\begin{tikzcd} {[m]} \arrow[r, "(t_i)"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\vdash"] & {[n]} \arrow[d, "\Psi"] \\ {[p]} \arrow[r, "(u_k)"'] & {[q]} \end{tikzcd}$$

UNIQUE ONE IF FROM $\mathbb{T}$ WE CAN PROVE $\Phi \vdash \Psi((t_i); (u_k))$

- VERT COMPOSITION $[m] \xrightarrow{\Phi} [p] \xrightarrow{\Theta} [r]$

$$\Theta \circ \Phi (x_1, \ldots, x_m; z_1, \ldots z_r)$$

$$\equiv \exists y_1, \ldots, y_p \left(\Phi(x_1, \ldots, x_m; y_1, \ldots y_p) \wedge \Theta(y_1, \ldots, y_p; z_1, \ldots, z_r) \right)$$

DOUBLE FUNCTORS

$F : \mathbb{A} \longrightarrow \mathbb{X}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "w"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \longmapsto \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "Fv"'] \arrow[dr, phantom, "F\alpha"] & FB \arrow[d, "Fw"] \\ FC \arrow[r, "Fg"'] & FD \end{tikzcd}$$

PRESERVES HORIZONTAL COMPOSITION

VERTICAL COMPOSITION

$$\begin{tikzcd} A \arrow[d, "v"'] \\ C \arrow[d, "x"'] \\ E \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, equal] \arrow[d, "Fv"'] & FA \arrow[dd, "F(x \cdot v)"] \\ FC \arrow[d, "Fx"'] \arrow[r, phantom, "\varphi_{x,v}"] & {} \\ FE \arrow[r, equal] & FE \end{tikzcd}$$

φ SATISFIES COHERENCE CONDITIONS

F IS <u>STRONG</u> IF THE φ ARE HORIZONTAL ISOS

OTHERWISE IT IS <u>LAX</u>

EX: $\text{INC} : \mathbb{S}\text{ET}_r \longrightarrow \mathbb{S}\text{ET}_s$ A LAX DOUB. FUNCT.

$\text{IM} : \mathbb{S}\text{ET}_s \longrightarrow \mathbb{S}\text{ET}_r$ A STRONG DOUB. F.

FACT: IM IS LEFT ADJOINT TO INC IN THE APPROPRIATE DOUBLE CATEGORY SENSE.

MAIN EXAMPLE:

A MODEL OF $\mathcal{T}$ GIVES A STRONG FUNCTOR

$M : \mathbb{T} \longrightarrow \mathbb{S}\mathrm{ET}_r$.

$$M[n] = M^n$$

$$M[t_1, \dots t_m] = M^n \xrightarrow{\langle M(t_1), \dots M(t_m)\rangle} M^m \qquad \text{WELL-DEF}$$

$$M\Phi = M\Phi$$

THAT WE GET A STRONG FUNCTOR IS "SOUNDNESS".

A MORPHISM OF MODELS $f : M \longrightarrow M'$ GIVES A HORIZONTAL TRANSFORMATION OF STRONG FUNCTORS $\bar{f} : M \longrightarrow M'$ AND IN FACT ARE IN BIJECTION WITH THEM.

BINARY PRODUCTS

MODELS $M : \mathbb{T} \longrightarrow \mathbb{S}\mathrm{ET}_r$ SHOULD PRESERVE FINITE PRODUCTS

IN A DOUBLE CATEGORY

- $A_1 \times A_2$ HAS THE USUAL UNIVERSAL PROPERTY FOR HORIZONTAL ARROWS

- ALSO NEED $\begin{array}{c} A_1 \times A_2 \\ {\scriptstyle v_1 \times v_2} \big\downarrow \\ B_1 \times B_2 \end{array}$ WITH UNIVERSAL PROPERTY W.R.T CELLS

- REQUIRE $id_{A_1} \times id_{A_2} \cong id_{A_1 \times A_2}$

- $\times$ IS STRONG IF $(w_1 \cdot v_1) \times (w_2 \cdot v_2) \xrightarrow{\cong} (w_1 \times w_2) \cdot (v_1 \times v_2)$

EQUIV: $\mathbb{A} \xrightarrow{\Delta} \mathbb{A} \times \mathbb{A}$ HAS A STRONG RIGHT ADJ

OR: $\underline{A}_2 \rightrightarrows \underline{A}_1 \leftrightarrows \underline{A}_0$ IS IN PROD

Ex: $\mathbb{S}\mathrm{ET}_r$ $\quad R_1 \times R_2 : A_1 \times A_2 \longrightarrow B_1 \times B_2$

FINITE PRODUCTS

GIVEN A DOUBLE CAT $\mathbb{A}$ CONSTRUCT $\mathbb{F}\mathrm{am}^*_f\mathbb{A}$

- OBJECT: $(I, \langle A_i \rangle)$ I FIN SET, $A_i \in \mathbb{A}$

- HORIZ: $$\frac{(I, \langle A_i \rangle) \xrightarrow{(f, \langle h_j \rangle)} (J, \langle B_j \rangle)}{f : J \to I, \quad h_j : A_{f(j)} \longrightarrow B_j}$$

- VERT:

$$\begin{tikzcd}
(I,A) \arrow[d, "{(S,v)}"'] \\
(K,C)
\end{tikzcd}
\qquad \Bigg| \qquad
\begin{tikzcd}
 & I \\
S \arrow[ur, "\sigma_0"] \arrow[dr, "\sigma_1"'] & \\
 & K
\end{tikzcd}
\qquad
\begin{tikzcd}
A_{\sigma_0 s} \arrow[d, "v_s"'] \\
C_{\sigma_1 s}
\end{tikzcd}$$

- CELL:

$$\begin{tikzcd}
(I,A) \arrow[r, "{(f,h)}"] \arrow[d, "{(S,v)}"'] \arrow[dr, phantom, "{(x,\alpha)}"] & (J,B) \arrow[d, "{(T,w)}"] \\
(K,C) \arrow[r, "{f',h'}"'] & (L,D)
\end{tikzcd}
\qquad \Bigg| \qquad
\begin{tikzcd}
I & J \arrow[l, "f"'] \\
S \arrow[u] \arrow[d] & T \arrow[l, "x"'] \arrow[u] \arrow[d] \\
K & L \arrow[l, "f'"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A_{f\tau_0(t)} \arrow[r, "h_{\tau_0 t}"] \arrow[d, "v_{xt}"'] \arrow[dr, phantom, "\alpha_t"] & B_{\tau_0 t} \arrow[d, "w_t"] \\
C_{f'\tau_1(t)} \arrow[r, "h'_{\tau_1 t}"'] & D_{\tau_1 t}
\end{tikzcd}$$

$\mathbb{A}$ HAS STRONG (LAX) FINITE PRODUCTS

IF $\Delta : \mathbb{A} \longrightarrow \mathbb{F}\mathrm{am}^*_f\mathbb{A}$ HAS A STRONG (LAX)

RIGHT ADJOINT.

CONCRETELY

- FOR $A_1, \dots, A_n$ HAVE $\prod A_i \xrightarrow{p_i} A_i$ + UNIV. PROP.

- FOR
$$\begin{tikzcd} A_1, \dots A_n \arrow[d, "v_1, \dots v_n"'] \\ C_1, \dots, C_m \end{tikzcd} \qquad \begin{tikzcd} \prod A_i \arrow[r, "p_{\sigma_0 s}"] \arrow[d, "\prod v_s"'] \arrow[dr, phantom, "\pi_s"] & A_{\sigma_0 s} \arrow[d, "v_s"] \\ \prod C_k \arrow[r, "q_{\sigma_1 s}"'] & C_{\sigma_1 s} \end{tikzcd}$$
+ UNIV. PROP.

__Ex__: PARALLEL PRODUCTS

$$\begin{tikzcd} A_1 \arrow[d, "v_1"'] & A_2 \arrow[d, "v_2"'] \\ C_1 & C_2 \end{tikzcd} \rightsquigarrow \begin{tikzcd} A_1 \times A_2 \arrow[d, "v_1 \times v_2"] \\ C_1 \times C_2 \end{tikzcd}$$

SPAN IS
$$\begin{tikzcd} & 2 \\ 2 \arrow[ur, equal] \arrow[dr, equal] & \\ & 2 \end{tikzcd}$$

__Ex__: LOCAL PRODUCTS

$$\begin{tikzcd} A \arrow[d, shift right, "v_1"'] \arrow[d, shift left, "v_2"] \\ C \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[d, "v_1 \times v_2"] \\ C \end{tikzcd}$$

SPAN IS
$$\begin{tikzcd} & 1 \\ 2 \arrow[ur] \arrow[dr] & \\ & 1 \end{tikzcd}$$

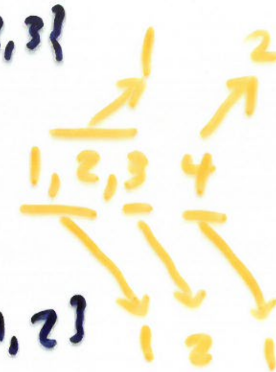
__Ex__: DIAGONALS

$$\begin{tikzcd} & A \arrow[dl, "id_A"'] \arrow[dr, "id_A"] & \\ A & & A \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[d, "\delta"] \\ A \times A \end{tikzcd}$$

SPAN IS
$$\begin{tikzcd} & 1 \\ 2 \arrow[ur] \arrow[dr, equal] & \\ & 2 \end{tikzcd}$$

Ex: TAKE AS SPAN

$$\begin{tikzcd} & \{1,2,3\} \\ \{1,2,3,4\} \arrow[ur] \arrow[dr] & \\ & \{1,2\} \end{tikzcd}$$



$$\begin{tikzcd} A_1 \arrow[d, "v_1"'] \arrow[d, shift left=2, "v_2"] \arrow[dr, "v_3"] & A_2 \arrow[dl, "v_4"] & A_3 \\ C_1 & C_2 & \end{tikzcd}$$

4 PROJECTIONS

$$\begin{tikzcd} A_1 \times A_2 \times A_3 \arrow[r, "p_1"] \arrow[d, "\prod_S v_S"'] \arrow[dr, phantom, "\pi_1"] & A_1 \arrow[d, "v_1"] \\ C_1 \times C_2 \arrow[r, "q_1"'] & C_1 \end{tikzcd}
\qquad
\begin{tikzcd} A_1 \times A_2 \times A_3 \arrow[r, "p_1"] \arrow[d, "\prod_S v_S"'] \arrow[dr, phantom, "\pi_2"] & A_1 \arrow[d, "v_2"] \\ C_1 \times C_2 \arrow[r, "q_1"'] & C_1 \end{tikzcd}$$

$$\begin{tikzcd} A_1 \times A_2 \times A_3 \arrow[r, "p_1"] \arrow[d, "\prod_S v_S"'] \arrow[dr, phantom, "\pi_3"] & A_1 \arrow[d, "v_3"] \\ C_1 \times C_2 \arrow[r, "q_2"'] & C_2 \end{tikzcd}
\qquad
\begin{tikzcd} A_1 \times A_2 \times A_3 \arrow[r, "p_2"] \arrow[d, "\prod_S v_S"'] \arrow[dr, phantom, "\pi_4"] & A_2 \arrow[d, "v_4"] \\ C_1 \times C_2 \arrow[r, "q_1"'] & C_1 \end{tikzcd}$$

& UNIVERSAL

E.G. IN $\mathbb{S}\mathrm{ET}_r$ GIVEN $I \leftarrow S \rightarrow K$, SETS A_i, C_k

RELATIONS $R_s : A_{\sigma_0 s} \rightarrow\!\!\!\!\bullet\;\; C_{\sigma_1 s}$

$\prod R_s : \prod A_i \rightarrow \prod C_k$ IS THE RELATION

$$\langle a_i \rangle \underset{\prod R_s}{\sim} \langle c_k \rangle \iff a_{\sigma_0 s} \underset{R_s}{\sim} c_{\sigma_1 k} \text{ FOR ALL } s \in S$$

$$\iff \bigwedge_{s \in S} a_{\sigma_0 s} \underset{R_s}{\sim} c_{\sigma_1 k}$$

THIS IS <u>LAX</u> FUNCTORIAL.

<u>PROP</u>: $\mathbb{T}$ HAS LAX FINITE PRODUCTS, AND MODELS PRESERVE THEM. $\square$

FINITE COPRODUCTS ARE DEFINED DUALLY

$\mathbb{F}\mathrm{AM}_f \mathbb{A}$ $$\frac{(I, A) \xrightarrow{(f,h)} (J, B)}{f : I \to J, \; h_i : A_i \to B_{fi}}$$

$$\mathbb{F}\mathrm{AM}_f^* \mathbb{A} = (\mathbb{F}\mathrm{AM}_f \mathbb{A}^{op})^{op}$$

LOCAL COPRODUCTS

$\mathbb{F}\mathrm{AM}_{f,l}\mathbb{A}$ FULL SUB DOUBLE CAT OF $\mathbb{F}\mathrm{AM}_{f}\mathbb{A}$ DETERMINED BY OBJECTS $(1, A)$.

$\mathbb{A}$ HAS STRONG (OPLAX) FINITE LOCAL COPRODUCTS IF $\Delta : \mathbb{A} \longrightarrow \mathbb{F}\mathrm{AM}_{f,l}\mathbb{A}$ HAS A STRONG (OPLAX) LEFT ADJOINT.

PROP: $\mathbb{T}$ HAS STRONG LOCAL COPRODUCTS (FINITE) AND MODELS PRESERVE THEM.

THEOREM: LET $\mathcal{T}$ BE A FIRST ORDER COHERENT THEORY AND $\mathbb{T}$ THE ASSOCIATED DOUBLE CATEGORY. THEN THE CATEGORY OF MODELS OF $\mathcal{T}$ IS EQUIVALENT TO THE CATEGORY OF STRONG FUNCTORS $M : \mathbb{T} \longrightarrow \mathbb{S}\mathrm{ET}_r$ THAT PRESERVE PRODUCTS AND LOCAL COPRODUCTS WITH HORIZONTAL TRANSFORMATIONS AS MORPHISMS.

SEMANTICS AS A FUNCTOR

AN IMPORTANT CONSEQUENCE OF FORMULATING A THEORY AS A CATEGORY IS THAT THERE IS A GOOD NOTION OF MORPHISM OF THEORIES.

LET $\mathbb{T}, \mathbb{T}'$ BE DOUBLE CATEGORIES WITH OBJECTS $[0], [1], \ldots$, FINITE PRODUCTS, FINITE LOCAL COPRODUCTS. A MORPHISM

$$F : \mathbb{T} \longrightarrow \mathbb{T}'$$

IS A STRONG DOUBLE FUNCTOR WHICH IS THE IDENTITY ON OBJECTS AND PRESERVES FINITE PRODUCTS AND LOCAL COPRODUCTS.

WE GET A CONTRAVARIANT FUNCTOR

$$\mathrm{SEM} : \mathrm{DoubTh} \longrightarrow \mathrm{CAT}/_{\mathrm{SET}}$$

$$\text{SEM}(\mathbb{T}) = \begin{array}{c} \text{MOD}(\mathbb{T}) \\ \big\downarrow U_{\mathbb{T}} \\ \text{SET} \end{array}$$

NEGLECTING SIZE CONSIDERATIONS FOR NOW

WE GET A FUNCTOR IN REVERSE DIRECTION

$$\text{STR} : \text{CAT}/_{\text{SET}} \longrightarrow \text{DOUBTH}$$

$$\text{STR}(A \xrightarrow{V} \text{SET}) = \mathbb{T}_V$$

$\mathbb{T}_V$ OBJ : $[0], [1], \ldots$

HORIZ: $[m] \longrightarrow [n]$ ARE NAT. TRANSF. $V^m \longrightarrow V^n$

VERT: $[m] \longrightarrow\!\!\!\!\!\bullet\;\; [n]$ ARE SUB FUNCTORS $R \rightarrowtail V^m \times V^n$

CELLS: "INCLUSIONS"

$$\begin{array}{ccc} V^m & \xrightarrow{t} & V^n \\ R \big\downarrow\!\!\bullet & \subseteq & \bullet\!\!\big\downarrow S \\ V^p & \xrightarrow[u]{} & V^q \end{array}$$

$\mathbb{T}_V$ IS A FULL SUB DOUBLE CATEGORY

OF $\mathbb{R}\text{EL}(\text{SET}^A)$.

WE HAVE COMPARISONS

$$\begin{tikzcd} A \arrow[rr, "\Phi"] \arrow[dr, "V"'] & & \mathrm{Mod}(\mathbb{T}) \arrow[dl, "U"] \\ & \mathrm{Set} & \end{tikzcd}$$

$$\Phi(A) : \mathbb{T}_V \longrightarrow \mathrm{Set}_r$$

$$\Phi(A)[n] = V(A)^n$$

$$\Phi(A)(t) = t(A) : V^m A \longrightarrow V^n A$$

$$\Phi(A)(R) = R(A) \subseteq V^m A \times V^n A$$

AND $\quad \Psi : \mathbb{T} \longrightarrow \mathbb{T}_U \quad (U : \mathrm{Mod}(\mathbb{T}) \longrightarrow \mathrm{Set})$

$$\Psi[n] = [n]$$

$$\Psi(h : [n] \to [m])(M) = M(h) : M^n \longrightarrow M^m$$

$$\Psi(r : [n] \rightsquigarrow [p])(M) = M(r) \subseteq M^n \times M^p$$

SMALLNESS

THERE MAY BE A PROPER CLASS OF

NATURAL TRANSFS $t : V^m \to V^n$ OR OF $R \rightarrowtail V^m \times V^n$

MOD($\mathbb{T}$) IS ACCESSIBLE, HAS FILT. COLIMS

U PRESERVES FILTERED COLIMITS

LET $\mathcal{ACC}_f$ BE THE CATEGORY OF ACCESSIBLE CATEGORIES WITH FILTERED COLIMITS AND FUNCTORS PRESERVING THESE COLIMITS.

THEN $\quad \text{SEM} : \text{DOUBTH} \longrightarrow \mathcal{ACC}_f/\text{SET}$

AND $\quad \text{STR} : \mathcal{ACC}_f/\text{SET} \longrightarrow \text{DOUBTH}$

DEFINED AS BEFORE EXCEPT $R \rightarrowtail V^m \times V^n$ PRESERVES FILT COLIM.

<u>THEOREM</u>: STR AND SEM ARE ADJOINT ON THE RIGHT WITH UNITS GIVEN BY Φ AND Ψ. $\square$

WHAT'S THE POINT?

- MAKES EXPLICIT THE ANALOGY WITH LAWVERE THYS
- PUTS LOGIC IN THE REALM OF DOUBLE CATEGORIES
  - POSSIBILITIES OF GENERALIZATION
    E.G. MODELS IN $\mathbb{C}\mathrm{AT}_p$
  - $\Phi(x;z) : [m] \rightarrow [n]$ COULD HAVE DIFFERENT CONDITIONS ON x AND z.
  - INFINITARY LOGIC $\mathbb{S}\mathrm{ET}_{\kappa,\lambda}$
- GIVES INSIGHT INTO DOUBLE CATEGORIES