Comonoids in **Rel**

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Introduction

- [GP] Families Parametrized by Coalgebras, *J. Alg.* 107 (1987), 316-375.
- **CCoalg** is a good category: set-like
  - complete and cocomplete
  - sums are disjoint and universal
  - cartesian-closed
- Can be used to parametrize families of "linear structures"
  - Vector spaces
  - Algebras
  - Coalgebras
  - Hopf algebras
  - etc.
- Gives a deeper understanding of various concepts
  - Cotensor of coalgebras
  - Measurings
  - **Alg** is not enriched in **Vect** but it is in **CCoalg**
    - Homomorphisms $\equiv$ group-likes
    - Derivations $\equiv$ primitives

- Cocommutative coalgebras are considered “trivial” by Hopf algebra people
  - Good - can consider them completely understood
  - But they are not as trivial as sets
- This point of view has, by and large, not been taken up either by the Hopf algebra people or the category theorists. Still much to be done:
  - Decomposition theorems for algebras, coalgebras, Hopf algebras, etc.
  - Category objects in **CCoalg** are Hopf algebroids, should be explored further
  - Understand the categorical meaning of the non-commutative coalgebras
- **CCoalg** is the set theory of linear structures: it embodies a logic specific to them

Rel

- Objects are sets, $A, B, C, \ldots$
- Morphisms $R : A \longrightarrow\!\!\!\!\!\bullet\;\; B$ are relations $R \subseteq A \times B$
- Composition $S \circ R = \{(a, c) | \exists b((b, c) \in S \;\&\; (a, b) \in R)\}$
- Identity $I : A \longrightarrow\!\!\!\!\!\bullet\;\; A$ is the diagonal

- Has a $\otimes$
  - $A \otimes A' = A \times A'$
  - $R \otimes R'$ is the image of

    $R \times R' \subseteq (A \times B) \times (A' \times B') \xrightarrow{\cong} (A \times A') \times (B \times B')$
  - Unit is $1 = \{0\}$
  - Symmetric
  - Has associated hom

$$\mathbf{Rel}(A \times B, C) \cong \mathbf{Rel}(A, B \times C)$$

$$[B, C] = B \times C$$

- Is self-dual

$$(\ )^\circ : \mathbf{Rel}^{op} \xrightarrow{\simeq} \mathbf{Rel}$$

$$A^\circ = A$$

$$R^\circ : B^\circ \longrightarrow A^\circ$$

Rel (continued)

- Coproducts - disjoint union
- Products - same
- "Inclusion" $(\ )_*$: **Set** $\longrightarrow$ **Rel**
  takes a function to its graph
- Has a right adjoint P : **Rel** $\longrightarrow$ **Set**

$$PA = \text{ power set of } A = \{X | X \subseteq A\}$$

$$PR : PA \longrightarrow PB$$

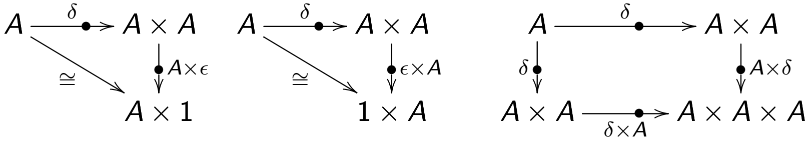
$$X \mapsto \{b \in B | \exists a \in X((a, b) \in R)\}$$

- $(\ )_*$ preserves colimits - **Rel** has colimits of diagrams of functions
- Equivalence relation $R : A \longrightarrow\!\!\!\bullet\!\!\!\longrightarrow A$
  - $I_A \subseteq R$
  - $R^\circ \subseteq R \qquad (R^\circ = R)$
  - $R \circ R \subseteq R \quad (R \circ R = R)$
  - $q : A \longrightarrow A/R$ gives a splitting of R

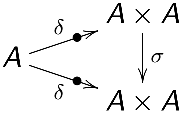
$$R = q^* \cdot q_* \ \& \ q_* \cdot q^* = I_{A/R} \quad (q^* = q_*^\circ)$$

Comonoids

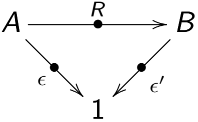
A *comonoid* in **Rel** is a set A with relations $\epsilon : A \to 1$, $\delta : A \to A \times A$ satisfying



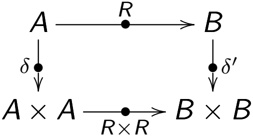
It is *cocommutative* if it also satisfies



A *morphism* of comonoids $R : (A, \epsilon, \delta) \to (B, \epsilon', \delta')$ is a relation $R : A \to B$ such that



and



Examples (Small)

Initial and Terminal

The empty set 0 and the one-point set 1 each have unique comonoid structures. These are the initial and terminal objects of **Comon** (and **CComon**).

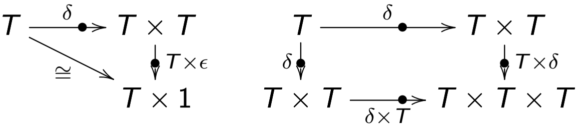
Trigonometric

$T = \{s, c\}$

$\epsilon(s) = \emptyset = 0, \quad \epsilon(c) = \{0\} = 1$

$\delta(s) = \{(s, c), (c, s)\}$

$\delta(c) = \{(s, s), (c, c)\}$



$$s \mapsto \{(s, c), (c, s)\} \mapsto \{s\} \times 1 \cup \{c\} \times 0 = \{(s, 0)\}$$

$$c \mapsto \{(s, s), (c, c)\} \mapsto \{(s, s, c), (s, c, s)\} \cup \{(c, s, s), (c, c, c)\}$$

$$c \mapsto \{(s, s), (c, c)\} \mapsto \{(s, c, s), (c, s, s)\} \cup \{(s, s, c), (c, c, c)\}$$

Example (Discrete)

Every set A admits a discrete comonoid structure, ΔA:

$$\epsilon : A \longrightarrow\!\!\!\!\!\!\bullet\;\; 1, \quad \epsilon(a) = 1$$

$$\delta : A \longrightarrow\!\!\!\!\!\!\bullet\;\; A \times A, \quad \delta(a) = \{(a, a)\}.$$

Proposition

A relation $R : A \longrightarrow\!\!\!\!\!\!\bullet\;\; B$ is a comonoid morphism $\Delta A \longrightarrow\!\!\!\!\!\!\bullet\;\; \Delta B$ iff it is (the graph of) a function $A \longrightarrow B$.

Proof.

R is single-valued if and only if it preserves δ, and it is everywhere defined if and only if it preserves ϵ. □

Corollary

$\Delta : \mathbf{Set} \longrightarrow \mathbf{Comon}$ *is a full and faithful functor.*

Proposition

Δ *has a right adjoint* Γ, *given by* $\Gamma A = \mathbf{Comon}(1, A)$.

Example (Matrices)

For any set A we have a comonoid $MA = (A \times A, \epsilon, \delta)$

$$\epsilon : A \times A \longrightarrow\!\!\!\!\!\!\bullet\;\; 1$$

$$\epsilon(a, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

$$\delta : A \times A \longrightarrow\!\!\!\!\!\!\bullet\;\; (A \times A) \times (A \times A)$$

$$\delta(a, b) = \{((a, x), (x, b)) | x \in A\}$$

MA is not cocommutative.

Example (Power series)

Example

- $X = \{x^0, x^1, x^2, \ldots\}$
  - $\epsilon(x^n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{ow} \end{cases}$
  - $\delta(x^n) = \{(x^p, x^q) | p + q = n\}$
  - Set-like: x^0
  - Primitive: x^1
- $Y = \{x^n y^m | n, m \in \mathbb{N}\}$
  - $\epsilon(x^n y^m) = \begin{cases} 1 & \text{if } n = m = 0 \\ 0 & \text{ow} \end{cases}$
  - $\delta(x^n y^m) = \{(x^p y^r, x^q y^s) | p + q = n \quad \text{and} \quad r + s = m\}$
  - Set-like: $1 (= x^0 y^0)$
  - Primitives: $x (= x^1 y^0) \quad \text{and} \quad y (= x^0 y^1)$.

Example (Dirichlet series)

$D = \{x_0, x_1, x_2, \ldots\}$

- $\epsilon(x_n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{ow} \end{cases}$
- $\delta(x_n) = \{(x_p, x_q) | pq = n\}$
- Set-like: x_1
- Primitives: $\{x_p | p \text{ prime}\}$.

Example (Monoid algebra)

$(\ )_* : \mathbf{Set} \longrightarrow \mathbf{Rel}$ preserves $\otimes$ so preserves monoids. M monoid (in **Set**) is also a monoid in **Rel**. By duality it is also a comonoid in **Rel**.

$M^\circ = M$

$\epsilon(x) = 1$ if $x = 1$ and 0 otherwise. $\delta(x) = \{(y, z) | yz = x\}$

- Power series and Dirichlet series are special cases

$$((\mathbb{N}, 0, +), (\mathbb{N}^2, 0, +), (\mathbb{N}, 1, \cdot))$$

- We can get power series in an arbitrary set of variables X by taking the free commutative monoid, $\mathbb{N}^{(X)}$, on X. The free monoid X^* gives power series in non-commuting variables.
- A monoid homomorphism $f : M \longrightarrow N$ in **Set** produces one in **Rel**, $f_* : M \longrightarrow\!\!\!\!\!\!\bullet\ \ N$, and by duality a comonoid homomorphism

$$f^* : N^\circ \longrightarrow M^\circ$$

- Gives an inclusion $\mathbf{Mon}^{op} \longrightarrow \mathbf{Comon}$ which is not full in general.

“Spectrum”

- $\Gamma(M^\circ) = \mathbf{Comon}(1, M^\circ)$

$$1 \longrightarrow\!\!\bullet\!\!\rightarrow M^\circ \quad \leftrightarrow \quad \begin{array}{l} N \subseteq M \\ 1 \in N \\ x \in N \text{ and } y \in N \Leftrightarrow xy \in N \end{array}$$

- The complement $P = M \backslash N$ is prime

$$1 \notin P$$

$$x \in P \;\&\; y \in M \Rightarrow xy \in P$$

$$x \in M \;\&\; y \in P \Rightarrow xy \in P$$

$$xy \in P \Rightarrow x \in P \text{ or } y \in P$$

So Γ is some kind of “spectrum”.

Example (Shuffles)

- Interesting comonoid structure on the free monoid, related to the shuffle algebra.
- A *k-shuffle* on n is $\sigma \in S_n$ which is order-preserving on $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$.
- The free monoid on X is the set of all words

$$X^* = \{x_1 x_2 \ldots x_n | x_i \in X, n \geq 0\}$$

- $\epsilon(w) = \begin{cases} 1 & \text{if } w = 1 \\ 0 & \text{ow} \end{cases}$
  $\delta(x_1 x_2 \ldots x_n) =$
  $\{(x_{\sigma 1} x_{\sigma 2} \ldots x_{\sigma k}, x_{\sigma(k+1)} \ldots x_{\sigma n}) | 0 \leq k \leq n, \sigma \text{ is a } k\text{-shuffle}\}$
- E.g. $\delta(xyx) =$
  $\{(1, xyx), (x, yx), (y, xx), (x, xy), (xy, x), (xx, y), (yx, x), (xyx, 1)\}$

Example ("Lie sets")

- A "*Lie set*" is a set X with a symmetric and reflexive relation $[\ ,\ ]$.
- Think: $[x, y] \Leftrightarrow x$ and y commute.
- Form the "universal enveloping monoid"

$$UX = X^*/\{xy \sim yx | [x, y]\}$$

- If $[\ ,\ ]$ is equality, then UX is the free monoid. If $[\ ,\ ]$ is the total relation, then UX is the free commutative monoid.

Theorem

$\delta : UX \longrightarrow\!\!\!\!\!\bullet\ UX \times UX$ given by $\delta[x_1 \dots x_n] = \{([x_{\sigma 1} \dots x_{\sigma k}], [x_{\sigma(k+1)} \dots x_{\sigma n}]) | 0 \leq k \leq n, \sigma$ a k-shuffle$\}$ is well defined, and makes UX into a cocommutative comonoid.

- UX has just one set-like element, $[1]$. The primitives are the generators $[x]$ determined by words of length 1.

Quotients

Let (A, ϵ, δ) be a comonoid and R an equivalence relation on A.

- $R : A \longrightarrow\!\!\!\!\!\bullet\!\!\!\!\rightarrow A$
  - $I_A \subseteq R$
  - $R^\circ \subseteq R$
  - $R \circ R \subseteq R$
- Let $q : A \longrightarrow \bar{A} = A/R$ be the quotient.

Proposition

$\bar{A}$ is a comonoid and q_* a homomorphism if and only if
(i) $\epsilon \circ R \subseteq \epsilon$
(ii) $\delta \circ R \subseteq (R \times R) \circ \delta$.

- With $\sim$ from previous example.
  (i) If $w \sim w'$ and $\epsilon(w) = 1$ then $\epsilon(w') = 1$. Holds because equivalence words have the same length.
  (ii) If $x_1 \dots x_n \sim y_1 \dots y_n$ and $\sigma \in S_n$ is a k-shuffle then there exists a k-shuffle $\tau \in S_n$ such that $x_{\sigma 1} \dots x_{\sigma k} \sim y_{\tau 1} \dots y_{\tau k}$ and $x_{\sigma(k+1)} \dots x_{\sigma n} \sim y_{\tau(k+1)} \dots y_{\tau n}$.

Coproducts

- **Comon** has coproducts given by disjoint union.
- ΔA is the coproduct of A copies of the terminal coalgebra 1.
- Nice: Disjoint and universal.
- A is *connected* if it is not the sum of two proper subcomonoids.

Proposition

Every comonoid A can be decomposed uniquely as a sum of connected subcomonoids $A = \sum_{i \in I} A_i$.

- Denote the set of connected components of A by $\pi_0 A$.

Proposition

π_0 *is left adjoint to* $\Delta : \mathbf{Set} \longrightarrow \mathbf{Comon}$.

Binary products

- By general principles **CComon** has $\otimes$ as binary product (i.e. cartesian product as sets).

Proposition

*In **CComon**, binary product distributes over coproduct*

$$B \times (\sum_{i \in I} A_i) \simeq \sum_{i \in I} (B \times A_i)$$

Proposition

A product of two connected cocommutative comonoids is connected.

Corollary

π_0 *preserves binary products of cocommutative comonoids.*

Remark

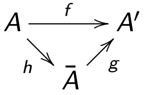
*Similar statements hold for **Comon** with product replaced by* $\otimes$*.*

Categories

- A small category is a sort of partially defined monoid with lots of identities.
- Let **A** be a small category and A its set of morphisms.
  - $$\epsilon : A \longrightarrow\!\!\!\!\!\!\bullet\;\; 1$$

    $$\epsilon(f) = \begin{cases} 1 & \text{if } f = \text{id} \\ 0 & \text{ow} \end{cases}$$
  - $$\delta : A \longrightarrow AxA$$

    $$\delta(f) = \{(g, h) | gh = f\}$$

- (A, ϵ, δ) is a comonoid in **Rel**.

Categories (continued)

Example

- A category with one object is a monoid $\rightsquigarrow$ monoid algebra
- A discrete category is a set $\rightsquigarrow$ ΔA
- The indiscrete category gives matrices MA
- A poset $(X, \leq)$ is a category

$$A = \{(x, y) | x \leq y\}$$

$$\epsilon(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{ow} \end{cases}$$

$$\delta(x, y) = \{(x, z), (z, y) | x \leq z \leq y\}$$

(C.f. Sweedler)

Functors

Functors *don't* give comonoid morphisms!

$$F : \mathbf{A} \longrightarrow \mathbf{B}$$

$$F : A \longrightarrow B$$

Neither $F_* : A \mathrel{-\!\!\bullet\!\!\rightarrow} B$ nor $F^* : B \mathrel{-\!\!\bullet\!\!\rightarrow} A$ are morphisms.

Proposition

$F_* : A \mathrel{-\!\!\bullet\!\!\rightarrow} B$ comes from a functor iff

$$\begin{array}{ccc} A & \xrightarrow{\;F_*\;} & B \\ {}_{\epsilon}\searrow & \subseteq & \swarrow_{\epsilon} \\ & 1 & \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\;F_*\;} & B \\ \delta\downarrow & \subseteq & \downarrow\delta \\ A \times A & \xrightarrow[F_* \times F_*]{} & B \times B \end{array}$$

iff

$$\begin{array}{ccc} B & \xrightarrow{\;F^*\;} & A \\ {}_{\epsilon}\searrow & \supseteq & \swarrow_{\epsilon} \\ & 1 & \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{\;F^*\;} & A \\ \delta\downarrow & \supseteq & \downarrow\delta \\ B \times B & \xrightarrow[F^* \times F^*]{} & A \times A \end{array}$$

Conclusion

- Comonoids in **Rel** seem interesting, and deserve more study. E.g. classification of irreducible ones.
- All of the examples are bicomonoids, but few have antipods.
- Groups give Hopf monoids and we have **Gr** and $\mathbf{Gr}^{op}$ as full subcategories.
- The analogy with coalgebras, bialgebras and Hopf algebras over a field is good. Much like permutation representations of groups as opposed to linear ones. It is strong enough to suggest ideas, but not so strong that the theories are completely parallel.
- Families indexed by comonoids have yet to be explored - but should be.