PERSISTENT DOUBLE LIMITS AND FLEXIBLE WEIGHTED LIMITS

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Abstract. This is the sequel of an article on persistent double limits in weak double categories. Here we consider their links with flexible weighted limits in 2-categories.

Introduction

Our recent article 'Persistent double limits' [GP3] continues our study of double limits in a weak double category [GP1, GP2]. The main results of [GP3] deal with an invariance property, called *persistence*, which was introduced in 1989 [Pa]. This property is characterised by two Persistence Theorems: essentially, a weak double category $\mathbb{I}$ parametrises persistent (double) limits if and only if every connected component of its ordinary category of objects and horizontal arrows has a natural weak initial object, if and only if $\mathbb{I}$-based limits and pseudo limits coincide up to equivalence.

Here we consider the links of flexible weighted limits in a 2-category $\mathbf{A}$ (defined in [BKPS]) with the persistent double limits in $\mathbf{A}$ (viewed as a double category with trivial vertical arrows). These links were already conjectured in [Pa] and investigated in Verity's thesis [Ve].

In Section 1 we prove that, for a given 2-functor $W\colon \mathbf{I} \to \mathbf{Cat}$ (called the *weight*), the W-weighted limit of a 2-functor $F\colon \mathbf{I} \to \mathbf{A}$ can be obtained as a double limit in $\mathbf{A}$, parametrised over a double category $\mathbb{E}\mathrm{l}(W)$ *of elements of* W (as stated in [Pa]). It is thus a universal double cone, i.e. a *terminal object in a double category* $\mathbb{C}\mathrm{one}_W(F)$ *of weighted cones* of F. The same holds for the pseudo case, concerned with weighted pseudo limits, pseudo double limits and weighted pseudo cones in the double category $\mathrm{Ps}\mathbb{C}\mathrm{one}_W(F)$.

Section 2 is based on results of [BKPS], saying that pseudo W-limits can be reduced to strict W'-limits, with respect to a derived weight $W'\colon \mathbf{I} \to \mathbf{Cat}$. In fact we show that the double categories $\mathrm{Ps}\mathbb{C}\mathrm{one}_W(F)$ and $\mathbb{C}\mathrm{one}_{W'}(F)$ are isomorphic.

In the last Section 3, taking advantage of the first Persistence Theorem of [GP3], we prove that a 2-functor $W\colon \mathbf{I} \to \mathbf{Cat}$ is a flexible

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weight if and only if the associated double category $\mathbb{E}\mathrm{l}(W)$ parametrises persistent double limits.

1. From weighted 2-limits to double limits

After reviewing the definition of W-weighted limits in a 2-category $\mathbf{A}$ [St, K1, K2], we construct the double category of elements $\mathbb{E}\mathrm{l}(W)$ as (a double category version of) the Grothendieck semidirect product construction applied to W. Note that it is not a 2-category but in fact has non-trivial vertical arrows. Then we prove that W-weighted limits can be obtained as double limits in $\mathbf{A}$ (viewed as a horizontal double category), parametrised by $\mathbb{E}\mathrm{l}(W)$. They are thus universal double cones, i.e. *terminal cones based on a double category.* All this works both in the pseudo sense and the strict one.

$\mathbf{I}$ is always a small 2-category equipped with a 2-functor $W\colon \mathbf{I} \to \mathbf{Cat}$, its *weight*. We write as $[\mathbf{I}, \mathbf{Cat}]$ (resp. $[\mathbf{I}, \mathbf{Cat}]_{\mathrm{ps}}$) the 2-category of 2-functors $\mathbf{I} \to \mathbf{Cat}$, their 2-natural (resp. pseudo natural) transformations, and modifications.

1.1. **Weighted limits and pseudo limits.** The *W-weighted pseudo limit* (L, λ), or *pseudo W-limit*, of a 2-functor $F\colon \mathbf{I} \to \mathbf{A}$ is an object $L = \mathrm{psLim}_W F$ of $\mathbf{A}$ equipped with a pseudo natural transformation

$$\lambda\colon W \to \mathbf{A}(L, F(-))\colon \mathbf{I} \to \mathbf{Cat}, \tag{1}$$

that gives, for every A in $\mathbf{A}$, an isomorphism of categories

$$\mathbf{A}(A, L) \;\cong\; [\mathbf{I}, \mathbf{Cat}]_{\mathrm{ps}}(W, \mathbf{A}(A, F)). \tag{2}$$

This means that:

(i) for every similar pair $(A, h\colon W \to \mathbf{A}(A, F))$ there is a unique morphism $f\colon A \to L$ in $\mathbf{A}$ such that:

$$h = \mathbf{A}(L, f).\lambda\colon W \to \mathbf{A}(A, F), \tag{3}$$

(ii) for every modification $\xi\colon h \to k\colon W \to \mathbf{A}(A, F)$ there is a unique 2-cell $\alpha\colon f \to g\colon A \to L$ in $\mathbf{A}$ such that:

$$\xi = \mathbf{A}(L, \alpha).\lambda\colon h \to k\colon W \to \mathbf{A}(A, F). \tag{4}$$

The (strict) *W-limit* of F, written as $\mathrm{Lim}_W F$, is similarly defined, replacing 'pseudo natural' by 2-natural and $[\mathbf{I}, \mathbf{Cat}]_{\mathrm{ps}}$ by $[\mathbf{I}, \mathbf{Cat}]$.

The trivial weight $W\colon \mathbf{I} \to \mathbf{Cat}$, constant at the terminal category $\mathbf{1}$, gives the *conical limit* of F (i.e. its ordinary 2-limit). As well known, all conical limits in $\mathbf{A}$ can be constructed from products and equalisers (in the 2-dimensional sense).

Moreover, all weighted limits can be constructed from the conical ones, adding cotensors $2 \pitchfork X$; this is the limit of the functor $X\colon \mathbf{1} \to \mathbf{A}$ weighted by $\mathbf{2}\colon \mathbf{1} \to \mathbf{Cat}$ [St].

1.2. **From weighted 2-categories to double categories.** We want now to show that all W-weighted limits (resp. pseudo limits) in $\mathbf{A}$ can be obtained as double limits (resp. pseudo limits) in $\mathbb{A}$, parametrised over the double category $\mathbb{E}\mathrm{l}(W)$ *of elements* of the 2-functor $W\colon \mathbf{I} \to \mathbf{Cat}$.

The latter is defined as the following double comma $\mathbf{1} \downdownarrows W$ (see [GP2], Subsection 2.5)

$$\begin{array}{ccc} \mathbb{E}\mathrm{l}(W) & \xrightarrow{P} & \mathbf{1} \\ {\scriptstyle Q}\downarrow & \overset{\pi}{\swarrow} & \downarrow \\ \mathbf{I} & \xrightarrow[W]{} & \mathbf{Cat} \end{array} \tag{5}$$

Concretely, an object of $\mathbb{E}\mathrm{l}(W)$ is a pair (I, X) where $I \in \mathrm{Ob}\mathbf{I}$ and $X \in \mathrm{Ob}(WI)$ (viewed as a functor $X\colon \mathbf{1} \to WI$).

A horizontal arrow $i = (i, X)\colon (I, X) \to (I', X')$ 'is' an $\mathbf{I}$-morphism $i\colon I \to I'$ such that $(Wi)(X) = X'$; they compose as in $\mathbf{I}$. A vertical arrow $x = (I, x)\colon (I, X) \to\!\!\!\!\bullet\; (I, Y)$ is a $W(I)$-morphism $x\colon X \to Y$; they compose as in $W(I)$.

A double cell $\xi\colon (x\,{}^{i}_{j}\,y)$

$$\begin{array}{ccc} (I, X) & \xrightarrow{i} & (I', X') \\ {\scriptstyle x}\downarrow\!\bullet & \xi & \bullet\!\downarrow{\scriptstyle y} \\ (I, Y) & \xrightarrow[j]{} & (I, Y') \end{array} \tag{6}$$

comes from a 2-cell $\xi\colon i \to j\colon I \to I'$ of $\mathbf{I}$ such that $(W\xi)(x) = y$, where $(W\xi)(x)$ is the diagonal of the commutative square

$$\begin{array}{ccc} (Wi)X & \xrightarrow{(W\xi)X} & (Wj)X \\ {\scriptstyle (Wi)x}\downarrow & = & \downarrow{\scriptstyle (Wj)Y} \\ (Wi)Y & \xrightarrow[(W\xi)Y]{} & (Wj)Y' \end{array} \qquad \begin{array}{c} (Wx)(x) = y\colon X' \to Y', \\ (Wi)(X) = X', \\ (Wj)(Y) = Y', \end{array} \tag{7}$$

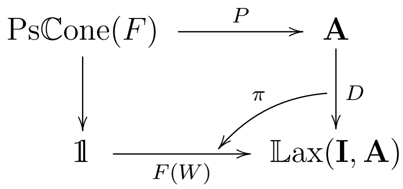
which expresses the naturality of $W\xi$ on the map $x\colon X \to Y$.

A 2-functor $F\colon \mathbf{I} \to \mathbf{A}$ between 2-categories has an associated double functor $F(W)$ with values in the horizontal double category of $\mathbf{A}$

$$\begin{gathered} F(W)\colon \mathbb{E}\mathrm{l}(W) \to \mathbb{A}, \qquad (I, X) \mapsto FI, \\ (i\colon (I, X) \to (I', X')) \mapsto Fi\colon FI \to FI', \\ (x\colon (I, X) \to\!\!\!\!\bullet\; (I, Y)) \mapsto e_{FI}, \\ (\xi\colon (x\,{}^{i}_{j}\,y)) \mapsto (F\xi\colon (FI\,{}^{Fi}_{Fj}\,FI')). \end{gathered} \tag{8}$$

1.3. **Cones and limits.** The double category $\mathrm{Ps}\mathbb{C}\mathrm{one}(F(W))$ of the pseudo cones of the double functor $F(W)\colon \mathbb{E}\mathrm{l}(W) \to \mathbf{A}$ is defined in [GP3], Section 5, as a double comma $D \downdownarrows F(W)$ of the diagonal functor D, where $\mathbb{1}$ is the singleton double category

(9)



It can be analysed as follows.

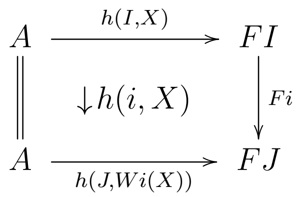
(a) A pseudo cone $(A, h\colon A \to F(W))$ is an object A of $\mathbf{A}$ equipped with:

- a map $h(I,X)\colon A \to FI$, for every I in $\mathbf{I}$ and every $X \in W(I)$,

- a 2-cell $h(I,x)\colon h(I,X) \to h(I,Y)\colon A \to FI$, for every I in $\mathbf{I}$ and every $x\colon X \to Y$ in $W(I)$,

- an invertible 2-cell $h(i,X)\colon Fi.h(I,X) \to h(j, Wi(X))$, for every $i\colon I \to J$ in $\mathbf{I}$ and every $X \in W(I)$

(10)



under the axioms (pht1–5) of naturality and coherence (in [GP3], Subsection 3.2).

It is a *cone* when all the comparison cells $h(i,X)$ are vertical identities.

When speaking of a *consistent pair* (I,X), or (I,x), or (i,X) we will mean one as above.

(b) A *horizontal morphism* $f\colon (A,h) \to (A',h')$ of pseudo cones is a horizontal arrow $f\colon A \to A'$ in $\mathbf{A}$ that commutes with the cone elements (for every consistent pair (I,X), or (I,x), or (i,X)), as follows:

$$(i) \quad h(I,X) = h'(I,X).f\colon A \to FI,$$

$$(ii) \quad h(I,x) = h'(I,x).f\colon A \Rightarrow FI,$$

$$(iii) \quad h(i,X) = h'(i,X).f\colon A \Rightarrow FJ.$$

Horizontal morphisms compose, forming a category.

(c) A vertical morphism $\xi\colon (A,h) \to\!\!\!\!\!\!\bullet\;\, (A,k)$ of pseudo cones is a modification $\xi\colon (A \,{}^{h}_{k}\, F(W))$.

We have thus, for every consistent pair (I,X), a 2-cell

$$\xi(I,X)\colon h(I,X) \to k(I,X)\colon A \to FI$$

in $\mathbf{A}$ that satisfies the conditions (mod1, 2) of [GP3], Definition 4.2.

(d) A double cell of cones

$$\begin{array}{ccc} (A,h) & \xrightarrow{\ f\ } & (A',h') \\ {\scriptstyle\xi}\downarrow\!\!\bullet & \alpha & \bullet\!\!\downarrow{\scriptstyle\zeta} \\ (A,k) & \xrightarrow[\ g\ ]{} & (A',k') \end{array} \tag{11}$$

is a 2-cell $\alpha\colon f \to g\colon A \to A'$ in $\mathbf{A}$ such that, for every pair (I,X)

$$A \;\underset{g}{\overset{f}{\substack{\longrightarrow \\ \downarrow\alpha \\ \longrightarrow}}}\; A' \;\underset{k'(I,Y)}{\overset{h'(I,X)}{\substack{\longrightarrow \\ \downarrow\zeta(I,X) \\ \longrightarrow}}}\; FI \qquad = \qquad \xi(I,X). \tag{12}$$

Spelling out the conditions of [GP3], Subsection 5.7, for a pseudo cone (L,λ) of $F(W)$ to be its pseudo limit, we have

(lim1) for every pseudo cone $(A, h\colon A \to F(W))$ there is a unique morphism $f\colon A \to L$ in $\mathbf{A}$ such that

$$h(I,X) = \lambda(I,X).f\colon A \to FI \qquad \text{(for } I \text{ in } \mathbf{I},\ X \text{ in } WI), \tag{13}$$

$$h(I,x) = \lambda(I,x).f\colon A \Rightarrow FI \qquad \text{(for } I \text{ in } \mathbf{I},\ x\colon X \to Y \text{ in } WI), \tag{14}$$

(lim2) for every vertical morphism $\xi\colon (A,h) \to\!\!\!\!\bullet\ (A,k)$ of pseudo cones there is a unique 2-cell $\alpha\colon f \to g\colon A \to L$ in $\mathbf{A}$ such that

$$A \;\underset{g}{\overset{f}{\substack{\longrightarrow \\ \downarrow\alpha \\ \longrightarrow}}}\; L \xrightarrow{\lambda(I,X)} FI \quad = \xi(I,X) \quad \text{(for } I \text{ in } \mathbf{I},\ X \text{ in } WI). \tag{15}$$

1.4. **Proposition** (From weighted 2-limits to double limits)**.** *For every 2-functor* $F\colon \mathbf{I} \to \mathbf{A}$*, the weighted limit* $(\mathrm{Lim}_W F, \lambda)$ *is the same as the double limit of the associated double functor* $F(W)\colon \mathbb{E}\mathrm{l}(W) \to \mathbf{A}$ *(i.e. they solve the same universal problem).*

Similarly the weighted pseudo limit $(\mathrm{psLim}_W F, \lambda)$ *is the same as the pseudo limit of* $F(W)$*.*

Proof. The analytic descriptions of these 'limits', in 1.1 and 1.3, amount to the same. □

1.5. **Definition** (Weighted cones)**.** This result allows us to define the *double* categories of W-weighted pseudo cones and strict cones of the 2-functor $F\colon \mathbf{I} \to \mathbf{A}$ as

$$\begin{gathered} \mathrm{Ps}\mathbb{C}\mathrm{one}_W(F) = \mathrm{Ps}\mathbb{C}\mathrm{one}(F(W)), \\ \mathbb{C}\mathrm{one}_W(F) = \mathbb{C}\mathrm{one}(F(W)). \end{gathered} \tag{16}$$

The terminal objects of these double categories give the W-weighted limit of F, pseudo or strict, respectively.

On the other hand, there seems to be no natural way of expressing the 2-dimensional universal property of weighted (strict or pseudo) limits by terminality in a 2-category.

1.6. **A direct construction of weighted cones.** Let V be the 2-functor

(17) $$V\colon \mathbf{A} \to [\mathbf{I}, \mathbf{Cat}], \qquad V(A) = \mathbf{A}(A, F(-)).$$

Without going through $\mathbb{E}\mathrm{l}(W)$ and $F(W)$, the double categories of weighted (pseudo) cones can be constructed, up to isomorphism, as the following double commas ([GP2], Subsection 2.5)

(18) $$\begin{array}{ccc} \mathrm{Ps}\mathbb{C}\mathrm{one}_W(F) & \longrightarrow & \mathbf{1} \\ \downarrow & \pi & \downarrow W \\ \mathbf{A} & \xrightarrow[V]{} & [\mathbf{I}, \mathbf{Cat}]_{\mathrm{ps}} \end{array} \qquad \begin{array}{ccc} \mathbb{C}\mathrm{one}_W(F) & \longrightarrow & \mathbf{1} \\ \downarrow & \pi' & \downarrow W \\ \mathbf{A} & \xrightarrow[V]{} & [\mathbf{I}, \mathbf{Cat}] \end{array}$$

In fact all the items (including compositions) of these double categories amount to the corresponding ones in the double categories analysed in 1.3.

1.7. **Comments.** We have already recalled in [GP3], Subsection 5.7, that the existence of all weighted limits in a 2-category $\mathbf{A}$ amounts to that of all double limits in the associated horizontal double category.

We will prove in Theorem 3.3 that a 2-functor $W\colon \mathbf{I} \to \mathbf{Cat}$ is a flexible weight if and only if the double category $\mathbb{E}\mathrm{l}(W)$ parametrises persistent limits in $\mathbf{Cat}$ (or equivalently in every weak double category).

It would be interesting to consider whether any double limit in $\mathbf{A}$, based on a double category $\mathbb{I}$, can be obtained as a single weighted limit for an associated weight $W\colon \mathbf{I} \to \mathbf{Cat}$ (defined on an associated 2-category).

2. Strictifying pseudo limits by derived weights

$\mathbf{I}$ is a fixed small 2-category. We recall from [BKPS] that pseudo W-limits can be reduced to strict W'-limits, with respect to a derived weight $W'\colon \mathbf{I} \to \mathbf{Cat}$. More precisely, we show that the double categories $\mathrm{Ps}\mathbb{C}\mathrm{one}_W(F)$ and $\mathbb{C}\mathrm{one}_{W'}(F)$ are isomorphic.

2.1. **Surjective equivalences.** We recall that, in a 2-category $\mathbf{C}$, a morphism $q\colon X \to A$ is said to be a *surjective equivalence* if it can be completed to an adjoint equivalence $(s, q, \eta, \varepsilon)$ where the unit $\eta\colon 1 \to qs$ is an identity

(19) $$\begin{array}{lll} s\colon A \rightleftarrows X : q & s \dashv q, & \\ \eta\colon 1_A = qs, & \varepsilon\colon sq \cong 1_X, & (\varepsilon s = 1_s,\ q\varepsilon = 1_q). \end{array}$$

In **Cat** this is plainly equivalent to a full and faithful functor $q\colon X \to \mathbf{A}$ which is surjective on objects: then, after choosing a section $s_0\colon \mathrm{Ob}\mathbf{A} \to \mathrm{Ob}X$ for the objects, all the rest is determined.

2.2. **Strictifying pseudo natural transformations.** We recall, from [BKP, BKPS], that the 2-category $[\mathbf{I}, \mathbf{Cat}]$ of 2-functors $\mathbf{I} \to \mathbf{Cat}$, their 2-natural transformations and modifications is 2-reflective in the 2-category $[\mathbf{I}, \mathbf{Cat}]_{\mathrm{ps}}$ of such 2-functors, their pseudo natural transformations and modifications.

The reflector is the *strictifying* 2-functor $(-)'$, right 2-adjoint to the inclusion $(-)$, whose computation will be written out below (in 2.6)

$$(-)'\colon [\mathbf{I}, \mathbf{Cat}]_{\mathrm{ps}} \rightleftarrows [\mathbf{I}, \mathbf{Cat}] : (-), \qquad (-)' \dashv (-). \tag{20}$$

As in [BKPS] we write a *pseudo* natural transformation $\mathbf{k}\colon W \to V$ in bold-face character. The unit and counit of the 2-adjunction are written as

$$\begin{array}{ll} p\colon \mathrm{id}[\mathbf{I}, \mathbf{Cat}]_{\mathrm{ps}} \to (-).(-)', & q\colon (-)'.(-) \to \mathrm{id}[\mathbf{I}, \mathbf{Cat}], \\ \mathbf{p}_W : W \to W', & q_V : V' \to V, \\ q_{W'}.(\mathbf{p}_W)' = 1_{W'}, & q_V.\mathbf{p}_V = 1_V. \end{array} \tag{21}$$

Let us note that the unit p is a 2-natural transformation, whose components $\mathbf{p}_W$ are pseudo natural transformations of 2-functors.

The universal property of the pseudo natural component $\mathbf{p}_W : W \to W'$ says that every pseudo natural $\mathbf{k}\colon W \to V$ can be written as $h\mathbf{p}_W : W \to W' \to V$, for a unique strict $h\colon W' \to V$ (namely $h = q_V.\mathbf{k}'$), yielding an isomorphism of categories

$$\begin{array}{c} [\mathbf{I}, \mathbf{Cat}](W', V) \;\cong\; [\mathbf{I}, \mathbf{Cat}]_{\mathrm{ps}}(W, V), \\ (h\colon W' \to V) \mapsto (h\mathbf{p}_W : W \to V), \\ (\mathbf{k}\colon W \to V) \mapsto (q_V\mathbf{k}' : W' \to V). \end{array} \tag{22}$$

2.3. **The derived weight.** For a fixed weight $W : \mathbf{I} \to \mathbf{Cat}$ we consider its *derived weight* $W' : \mathbf{I} \to \mathbf{Cat}$, with its component of the unit and the counit

$$\begin{array}{ll} \mathbf{p} = \mathbf{p}_W : W \to W', & q = q_W : W' \to W, \\ q\mathbf{p} = 1_W & q_{W'}.(\mathbf{p}_W)' = 1_{W'}. \end{array} \tag{23}$$

By [BKPS], Proposition 4.1 (or [BKP], Theorem 4.2) there is a unique invertible modification $\varepsilon_W : \mathbf{p}q \cong \mathrm{id}$ that gives an adjoint equivalence in $[\mathbf{I}, \mathbf{Cat}]_{\mathrm{ps}}$

$$\begin{array}{lll} \mathbf{p}\colon W \rightleftarrows W' : q, & \mathbf{p} \dashv q, & \\ \eta\colon 1_W = q\mathbf{p}, & \varepsilon\colon \mathbf{p}q \;\cong\; 1_{W'} & (\varepsilon\mathbf{p} = 1_{\mathbf{p}},\ q\varepsilon = 1_q). \end{array} \tag{24}$$

The retraction q is thus a surjective equivalence in $[\mathbf{I}, \mathbf{Cat}]_{\mathrm{ps}}$.

For every object I in $\mathbf{I}$ we have a surjective equivalence qI of ordinary categories

$$pI\colon W(I) \rightleftarrows W'(I) : qI \qquad pI \dashv qI,$$
$$qI.pI = \mathrm{id}W(I), \qquad \varepsilon_I\colon pI.qI \cong \mathrm{id}W'(I) \tag{25}$$
$$(\varepsilon_I.pI = 1_{pI},\ qI.\varepsilon_I = 1_{qI}),$$

where, for $X = (qI.pI)(X)$ in $W(I)$ and Y in $W'(I)$:

$$W'(I)(pI(X), Y) \cong W(I)(X, qI(Y)),$$
$$(f\colon pI(X) \to Y) \mapsto (qI(f)\colon X \to qI(Y)). \tag{26}$$

As recalled above, this is equivalent to a full and faithful functor $qI\colon W'(I) \to W(I)$ surjective on objects.

2.4. **Theorem.** For every 2-functor $F\colon \mathbf{I} \to \mathbf{A}$ (with values in a 2-category) there is an isomorphism of double categories (writing $V = \mathbf{A}(A, F(-))$)

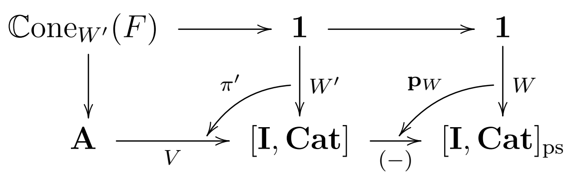
$$\mathbb{C}\mathrm{one}_{W'}(F) \rightleftarrows \mathrm{Ps}\mathbb{C}\mathrm{one}_W(F),$$
$$(h\colon W' \to V) \mapsto (h\mathbf{p}_W\colon W \to V), \tag{27}$$
$$(\mathbf{k}\colon W \to V) \mapsto (q_V.\mathbf{k}'\colon W' \to V).$$

which extends the isomorphism (22) of ordinary categories.

Proof. The double categories $\mathrm{Ps}\mathbb{C}\mathrm{one}_W(F)$ and $\mathbb{C}\mathrm{one}_{W'}(F)$ are defined by the double commas (18).

The pseudo natural transformation $\mathbf{p}_W\colon W \to W'$ gives a diagram of double cells

(28)



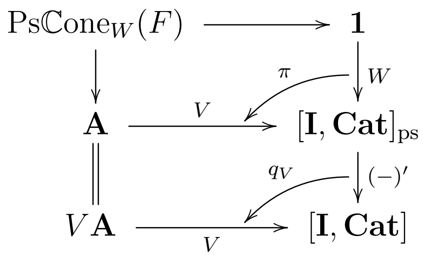
The horizontal universal property of the double comma $\mathrm{Ps}\mathbb{C}\mathrm{one}_W(F)$ (in [GP2], Theorem 2.6) gives a double functor

$$\mathbb{C}\mathrm{one}_{W'}(F) \to \mathrm{Ps}\mathbb{C}\mathrm{one}_W(F),$$

as in (27).

Similarly, the 2-natural transformation $q_V\colon V' \to V$ gives a diagram

(29)



and the vertical universal property of double commas gives a backward double functor, as in (27) which is inverse to the previous one. □

2.5. **Corollary.** For every 2-functor $F\colon \mathbf{I} \to \mathbf{A}$ the pseudo W-limit of F amounts to its W'-limit.

2.6. **Computations.** For a 2-functor $W\colon \mathbf{I} \to \mathbf{Cat}$, the derived weight $W'\colon \mathbf{I} \to \mathbf{Cat}$ works as follows. (This computation was deferred in [BKPS] to an article in preparation, which was not published.)

(a) The category $W'(I)$ has objects $(i'\colon I' \to I, X')$, with i' in $\mathbf{I}$ and $X' \in W(I')$.

A morphism is a triple

$$(i', i'', x)\colon (i'\colon I' \to I, X') \to (i''\colon I'' \to I, X''), \tag{30}$$

where $x\colon W(i')(X') \to W(i'')(X'')$ is a map of $W(I)$.

We have thus a forgetful functor

$$q(I)\colon W'(I) \to W(I), \qquad (i', X') \mapsto W(i')(X'), \tag{31}$$

which is a surjective equivalence, with a quasi-inverse section

$$\begin{gathered} s(I)\colon W(I) \to W'(I), \\ X \mapsto (1_I, X), \ \ (x\colon X \to Y) \mapsto (1,1,x)\colon (1_I, X) \to (1_I, Y), \\ \varepsilon\colon sq \cong 1_X, \quad \varepsilon(i', X') = (1, i', 1)\colon (1, Wi'(X')) \to (i', X'). \end{gathered} \tag{32}$$

(b) For $j\colon I \to J$, the functor $W'(j)\colon W'(I) \to W'(J)$ acts as follows:

$$\begin{gathered} (i'\colon I' \to I, X') \mapsto (ji'\colon I' \to J, X'), \\ (i', i'', x) \mapsto (ji', ji'', W(j)(x))\colon W(ji')(X') \to W(ji'')(X''). \end{gathered} \tag{33}$$

(c) For $\alpha\colon j \to k\colon I \to J$, the natural transformation

$$W'(\alpha)\colon W'(j) \to W'(k)\colon W'(I) \to W'(J)$$

has the following component on the object (i', X') (for $i'\colon I' \to I$ and $X' \in W(I')$)

$$\begin{aligned} W'(\alpha)(i', X') = (ji', ki', W(\alpha i')(X'))\colon \\ (ji'\colon I' \to J, X') \to (ki'\colon I' \to J, X'), \end{aligned} \tag{34}$$

where $\alpha i' \colon ji' \to ki' \colon I' \to J$ and

$$W(\alpha i')(X') \colon W(ji')(X') \to W(ki')(X').$$

(d) One proves that:

- the family $q(I)$ is a 2-natural transformation $q = q_W \colon W' \to W$,
- the family $p(I)$ is a pseudo natural transformation $\mathbf{p} = \mathbf{p}_W \colon W \to W'$,

that form the adjoint equivalence (24) in $[\mathbf{I}, \mathbf{Cat}]_{\mathrm{ps}}$.

3. Flexible weights and persistent double limits

We prove here that a 2-functor $W \colon \mathbf{I} \to \mathbf{Cat}$ is a flexible weight, as defined in [BKPS], if and only if the associated double category $\mathbb{E}\mathrm{l}(W)$ parametrises persistent double limits.

3.1. **Definition** (Flexible weight)**.** We know, from section 2.3, that a 2-functor $W \colon \mathbf{I} \to \mathbf{Cat}$ comes with a 2-natural transformation $q = q_W \colon W' \to W$ which is a surjective equivalence in $[\mathbf{I}, \mathbf{Cat}]_{\mathrm{ps}}$, with a weak inverse $p = p_W \colon W \to W'$ which is pseudo natural.

W is said to be a *flexible weight* [BKPS] if $q \colon W' \to W$ is already a surjective equivalence in $[\mathbf{I}, \mathbf{Cat}]$, i.e. it can be completed to an adjoint equivalence $(r, q, \eta, \varepsilon)$ in the 2-category $[\mathbf{I}, \mathbf{Cat}]$ where the unit $\eta \colon 1 \to qr$ is the identity (and the weak inverse $r \colon W \to W'$ is 2-natural).

Then, for every I in $\mathbf{I}$, we have a surjective equivalence qI of ordinary categories

$$\begin{gathered} rI \colon W(I) \rightleftarrows W'(I) : qI, \qquad rI \dashv qI, \\ qI.rI = \mathrm{id}W(I), \qquad \varepsilon_I \colon rI.qI \cong \mathrm{id}W'(I) \\ (\varepsilon_I.rI = 1_{rI},\ qI.\varepsilon_I = 1_{qI}), \end{gathered} \tag{35}$$

where, for $X = (qI.rI)(X)$ in $W(I)$ and Y in $W'(I)$:

$$\begin{gathered} W'(I)(rI(X), Y) \cong W'(I)(X, qI(Y)), \\ (f \colon rI(X) \to Y) \mapsto (qI(f) \colon X \to qI(Y)). \end{gathered} \tag{36}$$

This is equivalent to a full and faithful functor $qI \colon W'(I) \to W(I)$, surjective on objects.

Finally, W is a flexible weight if and only if the 2-natural transformation $q \colon W' \to W$ admits, for every I in $\mathbf{I}$, a section for the objects

$$(rI)_0 \colon \mathrm{Ob}W(I) \to \mathrm{Ob}W'(I),$$

so that the derived 2-functor $rI \colon W(I) \to W'(I)$ is 2-natural in the variable I.

3.2. **Theorem.** *A 2-functor* $W\colon \mathbf{I} \to \mathbf{Cat}$ *is a flexible weight if and only if the double category* $\mathbb{E}\mathrm{l}(W)$ *is grounded, i.e. every connected component of the ordinary category* $\mathrm{Hor}\mathbb{E}\mathrm{l}(W)$ *(of objects and horizontal arrows) has a natural weak initial object (see* [GP3]*, Subsection 6.2).*

Note. We recall that this also amounts to the fact that double limits based on $\mathbb{E}\mathrm{l}(W)$ are persistent.

Proof. Saying that W is a flexible weight means that the 2-natural transformation $q\colon W' \to W$ admits, for every I in $\mathbf{I}$, a section

$$rI\colon \mathrm{Ob}W(I) \to \mathrm{Ob}W'(I)$$

for the objects so that the derived 2-functor $rI\colon W(I) \to W'(I)$ is 2-natural in I.

First, for X in $W(I)$, we have an object of $W'(I)$ (see 2.6)

$$\begin{aligned}(rI)(X) &= (\rho(I,X)\colon r_0(I,X) \to I,\ r_1(I,X)) \\ &\qquad (r_1(I,X) \in W(r_0(I,X)),\end{aligned} \tag{37}$$

satisfying (precisely) the condition of splitting the functor $qI\colon W'(I) \to W(I)$, $(i', X') \mapsto W(i')(X')$

$$W(\rho(I,X))(r_1(I,X)) = X. \tag{38}$$

Note that, on a morphism $x\colon X \to Y$ of $W(I)$, we have:

$$\begin{aligned}(rI)(x\colon X \to Y) &= (\rho(I,X), (\rho(I,Y), x)\colon \\ &\qquad (\rho(I,X), r_1(I,X)) \to (\rho(I,Y), r_1(I,Y)).\end{aligned} \tag{39}$$

Second, we have the condition of 2-naturality on a cell $\alpha\colon j \to k\colon I \to J$

$$WI \xrightarrow{rI} W'I \xRightarrow{W'\alpha} W'J \quad = \quad WI \xRightarrow{W\alpha} WJ \xrightarrow{rJ} W'J, \tag{40}$$

$$\begin{aligned}(W'\alpha.rI)(X) &= W'(\alpha)(\rho(I,X), r_1(I,X)) \\ &= (j\rho(I,X), k\rho(I,X), W(\alpha.\rho(I,X))), \\ rJ(W(\alpha X)) &= (\rho(J,X), \rho(J,X), (W\alpha)X).\end{aligned}$$

$$\begin{array}{ccc} W(r_0(I,X)) & \xrightarrow{W\rho(I,X)} & WI \\ \Vert & & \Downarrow W\alpha \\ W(r_0(J,X)) & \xRightarrow[(W\alpha)X]{} & WJ \end{array}$$

(In particular $r_0(I,X) = r_0(J,X)$, when I, J are in the same connected component of the category of arrows of $\mathbf{I}$.)

Now, equation (38) says that $\rho(I,X)\colon r_0(I,X) \to I$ is a horizontal morphism of the double category $\mathbb{E}\mathrm{l}(W)$

$$\rho(I,X)\colon (r_0(I,X), r_1(I,X)) \to (I,X), \tag{41}$$

and equation (40) says that this family $\rho(I, X)$ is natural with respect to the horizontal morphisms of $\mathbb{E}\mathrm{l}(W)$. In other words, our condition means that $\mathbb{E}\mathrm{l}(W)$ is grounded. □

3.3. **A partial converse.** Verity's thesis gives a partial converse to this result.

As proved in [Ve], Theorem 2.7.1, the class of persistent weighted colimits in the 2-category **Cat** is closed (in the sense of [AK]) and generated by sums, coinserters, coequifiers and idempotent-splittings. It coincides thus with the closed class of (PIES)*-colimits, which precisely amounts to the class of flexible colimits in **Cat**, as proved in [BKPS].

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