SPAN AND COSPAN REPRESENTATIONS OF WEAK DOUBLE CATEGORIES

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Abstract. We prove that many important weak double categories can be 'represented' by spans, using the basic higher limit of the theory: the tabulator. Dually, representations by cospans via cotabulators are also frequent.

Introduction

Strict double categories were introduced and studied by C. Ehresmann [Eh1, Eh2], the weak notion in our series [GP1 - GP4]. The strict case extends the more usual (if historically subsequent) notion of 2-category, while the weak one extends bicategories, priorly established by Bénabou [Be]. The extension is made clear in Section 3.

This note is about weak double categories and the (horizontal) *tabulator* of a vertical arrow. The latter is the 'basic' higher limit of the theory; in fact the main result of [GP1] says that a weak double category has all (horizontal) double limits if and only if it has: double products, double equalisers and tabulators.

We prove here that the existence of tabulators in a weak double category $\mathbb{A}$ produces, under suitable hypotheses, a lax functor $S\colon \mathbb{A} \to \mathbb{S}\mathrm{pan}(\mathbf{C})$ with values in the weak double category *of spans* over the category $\mathbf{C}$ of horizontal arrows of $\mathbb{A}$ (Theorem 6). We say that $\mathbb{A}$ is *span representable* when this functor S is horizontally faithful.

Many important weak double categories can be represented in this sense, by spans or - dually - by cospans, via cotabulators.

Outline. We begin by a brief review of basic notions on weak double categories, from [GP1, GP2], including the weak double categories of spans and cospans, and the (co)tabulator of a vertical arrow.

Sections 6 and 7 give the main definitions and results recalled above, about (co)span representability. Various weak (or strict) double categories are examined in Sections 8 - 12, proving that many of them are both span and cospan representable. Yet the weak double category $\mathbb{S}\mathrm{pan}\mathbf{Set}$, which is trivially span representable, is *not* cospan representable (Section 8), and $\mathbb{C}\mathrm{osp}\mathbf{Set}$ behaves in a dual way.

Finally, some common patterns in the previous proofs of representability are analysed in Section 13.

Work supported by a research grant of Università di Genova.
2000 Mathematics Subject Classification: 18D05,18A30.
Key words and phrases: double category, tabulator, span.

1. Definition

A (strict) *double category* $\mathbb{A}$ consists of the following structure.

(a) A set $\mathrm{Ob}\mathbb{A}$ of *objects* of $\mathbb{A}$.

(b) *Horizontal morphisms* $f\colon X \to X'$ between the previous objects; they form the category $\mathrm{Hor}_0\mathbb{A}$ *of the objects and horizontal maps* of $\mathbb{A}$, with composition written as gf and identities $1_X\colon X \to X$.

(c) *Vertical morphisms* $u\colon X \to\!\!\bullet\; Y$ (often denoted by a dot-marked arrow) between the same objects; they form the category $\mathrm{Ver}_0\mathbb{A}$ *of the objects and vertical maps* of $\mathbb{A}$, with composition written as $v{\bullet}u$ (or $u \otimes v$, in diagrammatic order) and identities written as $e_X\colon X \to\!\!\bullet\; X$ or $1^\bullet_X$.

(d) *Double cells* $a\colon (u \, {f \atop g} \, v)$ with a *boundary* formed of two vertical arrows u, v and two horizontal arrows f, g

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & X' \\ {\scriptstyle u}\,\bullet\!\!\downarrow & a & \downarrow\!\!\bullet\,{\scriptstyle v} \\ Y & \xrightarrow[\;g\;]{} & Y' \end{array} \qquad (1)$$

Writing $a\colon (X \, {X \atop g} \, v)$ or $a\colon (e \, {1 \atop g} \, v)$ we mean that $f = 1_X$ and $u = e_X$. The cell a is also written as $a\colon u \to v$ (with respect to its *horizontal* domain and codomain, which are *vertical* arrows) or as $a\colon f \to\!\!\bullet\; g$ (with respect to its *vertical* domain and codomain, which are *horizontal* arrows).

We refer now to the following diagrams of cells, where the first is called a *consistent matrix* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of cells

$$\begin{array}{ccccc} X & \xrightarrow{\;f\;} & X' & \xrightarrow{\;f'\;} & X'' \\ {\scriptstyle u}\,\bullet\!\!\downarrow & a & \downarrow\!\!\bullet\,{\scriptstyle v} & b & \downarrow\!\!\bullet\,{\scriptstyle w} \\ Y & \xrightarrow{\;g\;} & Y' & \xrightarrow{\;g'\;} & Y'' \\ {\scriptstyle u'}\,\bullet\!\!\downarrow & c & \downarrow\!\!\bullet\,{\scriptstyle v'} & d & \downarrow\!\!\bullet\,{\scriptstyle w'} \\ Z & \xrightarrow[\;h\;]{} & Z' & \xrightarrow[\;h'\;]{} & Z'' \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\;1\;} & X \\ {\scriptstyle u}\,\bullet\!\!\downarrow & 1_u & \downarrow\!\!\bullet\,{\scriptstyle u} \\ Y & \xrightarrow[\;1\;]{} & Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\;f\;} & X' \\ {\scriptstyle e}\,\bullet\!\!\downarrow & e_f & \downarrow\!\!\bullet\,{\scriptstyle e} \\ X & \xrightarrow[\;f\;]{} & X' \end{array} \qquad (2)$$

(e) Cells have a *horizontal composition*, consistent with the horizontal composition of arrows and written as $(a\,|\,b)\colon (u \, {f'f \atop g'g} \, w)$, or $a|b$; this composition gives the category $\mathrm{Hor}_1\mathbb{A}$ *of vertical arrows and cells* $a\colon u \to v$ of $\mathbb{A}$, with identities $1_u\colon (u \, {1 \atop 1} \, u)$.

(f) Cells have also a *vertical composition*, consistent with the vertical composition of arrows and written as $\left(\frac{a}{c}\right)\colon (u'{\bullet}u \, {f \atop h} \, v'{\bullet}v)$, or $\frac{a}{c}$, or $a \otimes c$; this composition gives the category $\mathrm{Ver}_1\mathbb{A}$ *of horizontal arrows and cells* $a\colon f \to\!\!\bullet\; g$ of $\mathbb{A}$, with identities $e_f = 1^\bullet_f\colon (e \, {f \atop f} \, e)$.

(g) The two compositions satisfy the *interchange laws* (for binary and zeroary compositions), which means that we have, in diagram (2):

$$\left(\frac{a\,|\,b}{c\,|\,d}\right) = \left(\frac{a}{c}\,\Big|\,\frac{b}{d}\right), \qquad \left(\frac{1_u}{1_{u'}}\right) = 1_{u'\bullet u}, \qquad (e_f\,|\,e_{f'}) = e_{f'f}, \qquad 1_{e_X} = e_{1_X}. \tag{3}$$

The first condition says that a consistent matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has a precise *pasting*; the last says that an object X has an *identity cell* $\square_X = 1_{e_X} = e_{1_X}$. The expressions $(a\,|\,f')$ and $(f\,|\,b)$ will stand for $(a\,|\,e_{f'})$ and $(e_f\,|\,b)$, when this makes sense.

$\mathbb{A}$ is said to be *flat* if every double cell $a\colon (u\ {}^{f}_{g}\ v)$ is determined by its boundary - namely the arrows f, g, u, v. A standard example is the double category $\mathbb{R}\mathrm{el}\mathbf{Set}$ of sets, mappings and relations, recalled below in Section 8(c).

2. Hints at weak double categories

More generally, in a *weak double category* $\mathbb{A}$ the horizontal composition behaves categorically (and we still have ordinary categories $\mathrm{Hor}_0\mathbb{A}$ and $\mathrm{Hor}_1\mathbb{A}$), while the composition of vertical arrows is categorical up to *comparison cells*:

- for a vertical arrow $u\colon X \mathrel{-\!\!\bullet\!\!\to} Y$ we have a *left unitor* and a *right unitor*

$$\lambda u\colon e_X \otimes u \to u, \qquad\qquad \rho u\colon u \otimes e_Y \to u,$$

- for three consecutive vertical arrows $u\colon X \mathrel{-\!\!\bullet\!\!\to} Y$, $v\colon Y \mathrel{-\!\!\bullet\!\!\to} Z$ and $w\colon Z \mathrel{-\!\!\bullet\!\!\to} T$ we have an *associator*

$$\kappa(u, v, w)\colon u \otimes (v \otimes w) \to (u \otimes v) \otimes w.$$

Interchange holds strictly, as above. The comparison cells are *special* (which means that their horizontal arrows are identities) and horizontally invertible. Moreover they are assumed to be *natural* and *coherent*, in a sense made precise in [GP1], Section 7; after stating naturality with respect to double cells, the coherence axioms are similar to those of bicategories.

$\mathbb{A}$ is said to be *unitary* if the unitors are identities, so that the vertical identities behave as strict units - a constraint which in concrete cases can often be easily met. The terminology of the strict case is extended to the present one, as far as possible.

A *lax (double) functor* $F\colon \mathbb{X} \to \mathbb{A}$ between weak double categories amounts to assigning:

(a) two functors Hor_0F and Hor_1F, consistent with domain and codomain

$$\begin{array}{ccc} \mathrm{Hor}_1\mathbb{X} & \xrightarrow{\mathrm{Hor}_1F} & \mathrm{Hor}_1\mathbb{A} \\ {\scriptstyle\mathrm{Dom}}\downarrow & & \downarrow{\scriptstyle\mathrm{Dom}} \\ \mathrm{Hor}_0\mathbb{X} & \xrightarrow[\mathrm{Hor}_0F]{} & \mathrm{Hor}_0\mathbb{A} \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{Hor}_1\mathbb{X} & \xrightarrow{\mathrm{Hor}_1F} & \mathrm{Hor}_1\mathbb{A} \\ {\scriptstyle\mathrm{Cod}}\downarrow & & \downarrow{\scriptstyle\mathrm{Cod}} \\ \mathrm{Hor}_0\mathbb{X} & \xrightarrow[\mathrm{Hor}_0F]{} & \mathrm{Hor}_0\mathbb{A} \end{array} \tag{4}$$

(b) for any object X in $\mathbb{X}$, a special cell, the *identity comparison* of F

$$\underline{F}(X)\colon e_{FX} \to Fe_X\colon FX \mathrel{-\!\!\!\bullet\!\!\!\to} FX,$$

(c) for any vertical composite $u \otimes v\colon X \mathrel{-\!\!\!\bullet\!\!\!\to} Y \mathrel{-\!\!\!\bullet\!\!\!\to} Z$ in $\mathbb{X}$, a special cell, the *composition comparison* of F

$$\underline{F}(u,v)\colon Fu \otimes Fv \to F(u \otimes v)\colon FX \mathrel{-\!\!\!\bullet\!\!\!\to} FZ.$$

Again, these comparisons must satisfy axioms of naturality and coherence with the comparisons of $\mathbb{X}$ and $\mathbb{A}$ [GP2].

3. Dualities

A weak double category has a *horizontal opposite* $\mathbb{A}^h$ (reversing the horizontal direction) and a *vertical opposite* $\mathbb{A}^v$ (reversing the vertical direction); a strict structure also has a *transpose* $\mathbb{A}^t$ (interchanging the horizontal and vertical issues).

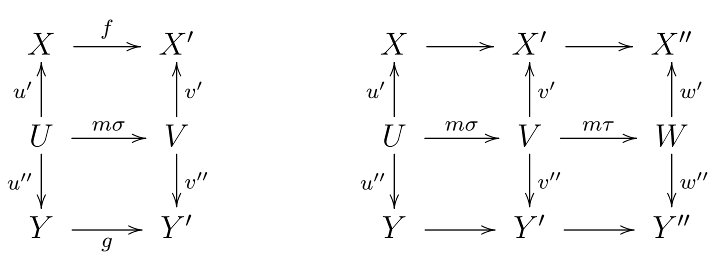
The prefix 'co', as in *colimit*, *coequaliser* or *colax double functor*, refers to horizontal duality, the main one. Let us note that a weak double category whose horizontal arrows are identities is the same as a *bicategory written in vertical*, i.e. with arrows and weak composition in the vertical direction and strict composition in the horizontal one. This is why *the* op*lax functors of bicategories correspond here to* co*lax double functors*. (Transposing the theory of double categories, as is done in some papers, would avoid this conflict of terminology, but would produce other conflicts at a more basic level: for instance, colimits in **Set** would become 'op-limits' in $\mathbb{R}$el**Set** and $\mathbb{S}$pan**Set**.)

4. Spans and cospans

For a category **C** with (a fixed choice of) pullbacks there is a weak double category $\mathbb{S}$pan(**C**) *of spans* over **C**, which will play here an important role.

Objects, horizontal arrows and their composition come from **C**, so that $\mathrm{Hor}_0(\mathbb{S}\mathrm{pan}\mathbf{C}) = \mathbf{C}$.

A vertical arrow $u\colon X \mathrel{-\!\!\!\bullet\!\!\!\to} Y$ is a span $u = (u', u'')$, i.e. a diagram $X \leftarrow U \to Y$ in **C**, or equivalently a functor $u\colon \vee \to \mathbf{C}$ defined on the formal-span category $\bullet \leftarrow \bullet \to \bullet$. A vertical identity is a pair $e_X = (1_X, 1_X)$. A cell $\sigma\colon (u \,{}^{f}_{g}\, v)$ is a natural transformation $u \to v$ of such functors and amounts to the left commutative diagram below



(5)

We say that the cell σ is *represented* by its middle arrow $m\sigma\colon U \to V$, which determines it together with the boundary (the present structure is not flat).

The horizontal composition $\sigma|\tau$ of σ with a second cell $\tau\colon v \to w$ is a composition of natural transformations, as in the right diagram above; it gives the category $\mathrm{Hor}_1(\mathbb{S}\mathrm{pan}\mathbf{C}) = \mathbf{Cat}(\vee, \mathbf{C})$.

The vertical composition $u \otimes v$ of spans is computed by (chosen) pullbacks in $\mathbf{C}$

$$\begin{array}{ccccccc} X & & & Y & & & Z \\ & \nwarrow & \nearrow & & \nwarrow & \nearrow & \\ & & U & & V & & \\ & & \nwarrow & & \nearrow & & \\ & & & W & & & \end{array} \qquad W = U \times_Y V. \tag{6}$$

This is extended to double cells, in the obvious way. For the sake of simplicity *we make* $\mathbb{S}\mathrm{pan}(\mathbf{C})$ *unitary*, by adopting the 'unit constraint' for pullbacks: the chosen pullback of an identity along any morphism is an identity. The associator κ is determined by the universal property of pullbacks.

Dually, for a category $\mathbf{C}$ with (a fixed choice of) pushouts there is a unitary weak double category $\mathbb{C}\mathrm{osp}(\mathbf{C})$ *of cospans* over $\mathbf{C}$, that is horizontally dual to $\mathbb{S}\mathrm{pan}(\mathbf{C}^{\mathrm{op}})$. We have now

$$\mathrm{Hor}_0(\mathbb{C}\mathrm{osp}\mathbf{C}) = \mathbf{C}, \qquad \mathrm{Hor}_1(\mathbb{C}\mathrm{osp}\mathbf{C}) = \mathbf{Cat}(\wedge, \mathbf{C}), \tag{7}$$

where $\wedge = \vee^{\mathrm{op}}$ is the formal-cospan category $\bullet \to \bullet \leftarrow \bullet$.

A vertical arrow $u = (u', u'')\colon \wedge \to \mathbf{C}$ is now a cospan, i.e. a diagram $X \to U \leftarrow Y$ in $\mathbf{C}$, and a cell $\sigma\colon u \to v$ is a natural transformation of such functors. Their vertical composition is computed with pushouts in $\mathbf{C}$; again, we generally follow the 'unit constraint' for pushouts.

5. Tabulators

The (horizontal) *tabulator* of a vertical arrow $u\colon X \to\!\!\!\!\bullet\;\, Y$ in the weak double category $\mathbb{A}$ is an object $T = \top u$ equipped with a double cell $t_u\colon e_T \to u$

$$\begin{array}{ccc} T & \xrightarrow{p} & X \\ {\scriptstyle e_T}\downarrow & t_u & \downarrow{\scriptstyle u} \\ T & \xrightarrow[q]{} & Y \end{array} \qquad\qquad \begin{array}{ccccc} H & \xrightarrow{f} & T & \xrightarrow{p} & X \\ {\scriptstyle e}\downarrow & e_f & {\scriptstyle e}\downarrow & t_u & \downarrow{\scriptstyle u} \\ H & \xrightarrow[f]{} & T & \xrightarrow[q]{} & Y \end{array} \;=\; h, \tag{8}$$

such that the pair $(T, t_u\colon e_T \to u)$ is a universal arrow from the functor $e\colon \mathrm{Hor}_0\mathbb{A} \to \mathrm{Hor}_1\mathbb{A}$ to the object u of $\mathrm{Hor}_1\mathbb{A}$. Explicitly, this means that for every object H and every cell $h\colon e_H \to u$ there is a unique horizontal map $f\colon H \to T$ such that $(e_f \,|\, t_u) = h$, as in the right diagram above.

(In [GP2] we also considered a higher dimensional universal property, which was dropped in later papers and is not used here.) We say that $\mathbb{A}$ *has tabulators* if all of them exist, or equivalently if the degeneracy functor $e\colon \mathrm{Hor}_0\mathbb{A} \to \mathrm{Hor}_1\mathbb{A}$ has a right adjoint

$$\top\colon \mathrm{Hor}_1\mathbb{A} \to \mathrm{Hor}_0\mathbb{A}, \qquad e \dashv \top. \tag{9}$$

In this situation *one can try to represent* $\mathbb{A}$ *as a weak double category of spans*, as we shall see below.

Dually $\mathbb{A}$ *has cotabulators* if the degeneracy functor has a left adjoint

$$\bot\colon \mathrm{Hor}_1\mathbb{A} \to \mathrm{Hor}_0\mathbb{A}, \qquad \bot \dashv e, \tag{10}$$

so that every vertical arrow $u\colon X \to\!\!\!\!\bullet\; Y$ has a cotabulator-object $\bot u$, equipped with two horizontal morphisms $i\colon X \to \bot u$, $j\colon Y \to \bot u$ and a universal cell $\iota\colon (u\,{}^{i}_{j}\,e)$. This *may* allow representing $\mathbb{A}$ as a weak double category of cospans.

For a category $\mathbf{C}$ with pullbacks, the tabulator in $\mathbb{S}\mathrm{pan}(\mathbf{C})$ of a span $u = (u', u'') = (X \leftarrow U \to Y)$ is its central object U, with projections u', u'' and the obvious cell $t_u\colon e_U \to u$. The cotabulator is the pushout of the span in $\mathbf{C}$, provided it exists. All this cannot be formulated within the bicategory $\mathbf{Span}(\mathbf{C})$ (vertically embedded in $\mathbb{S}\mathrm{pan}(\mathbf{C})$ as specified in Section 3).

6. Theorem and Definition (Span representation)

We suppose that:

(a) the weak double category $\mathbb{A}$ *has tabulators,*

(b) the ordinary category $\mathbf{C} = \mathrm{Hor}_0(\mathbb{A})$ *of objects and horizontal arrows has pullbacks.*

There is then a canonical lax functor, which is trivial in degree zero

$$S\colon \mathbb{A} \to \mathbb{S}\mathrm{pan}(\mathbf{C}), \qquad \mathrm{Hor}_0(S) = \mathrm{id}\mathbf{C}, \tag{11}$$

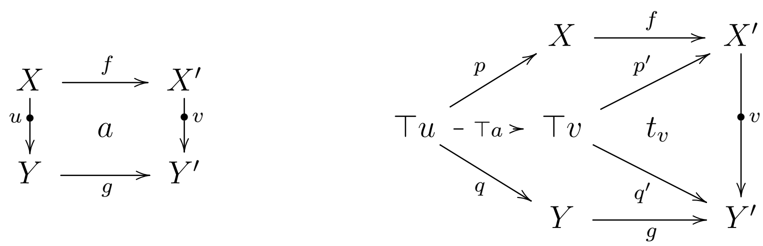
and takes a vertical arrow $u\colon X \to\!\!\!\!\bullet\; Y$ *of* $\mathbb{A}$ *to the span* $Su = (p, q)\colon X \to\!\!\!\!\bullet\; Y$ *determined by the tabulator* $\top u$ *and its projections* $p\colon \top u \to X$, $q\colon \top u \to Y$.

The lax functor S *will be called the* span representation *of* $\mathbb{A}$.

Note. Related results can be found in Niefield [Ni], for weak double categories with vertical companions and adjoints.

Proof. As in Section 5 we write $t_u\colon (e\,{}^{p}_{q}\,u)$ the universal cell of the tabulator $\top u$.

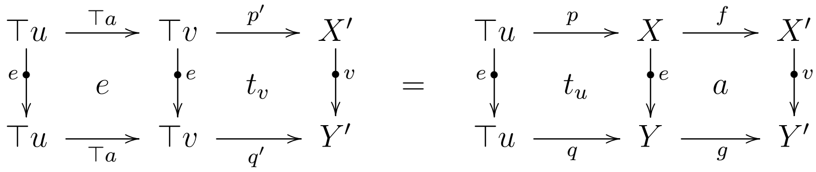
The action of S on a cell a of $\mathbb{A}$ is described by the following diagram



(12)

where the cell $Sa\colon Su \to Sv$ (a morphism of spans) is represented by the coherent morphism $\top a\colon \top u \to \top v$. The latter is determined by the universal property of the universal cell t_v of the tabulator $\top v$

$$(\top a\,|\,t_v) = (t_u\,|\,a) \qquad\qquad (p'.\top a = fp,\ q'.\top a = gq), \tag{13}$$



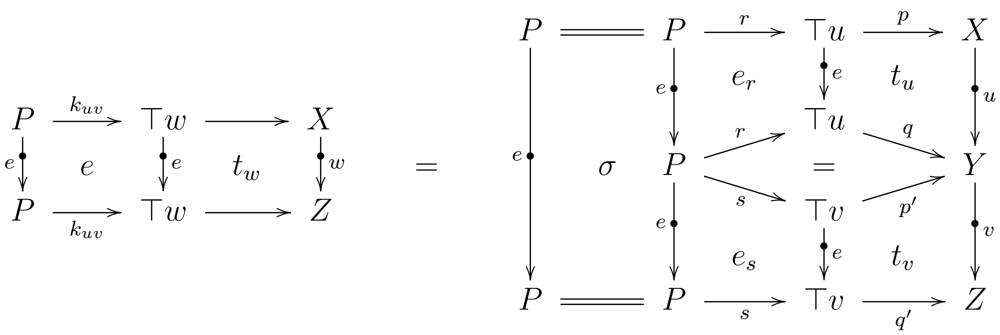
(In the composition $(\top a\,|\,t_v)$ we write $\top a$ for $e_{\top a}$, as already warned at the end of Section 1.)

To define the laxity comparisons, an object X of $\mathbb{A}$ gives a special cell $\underline{S}(X)\colon e_X \to S(e_X)$ represented by the morphism

$$k_X\colon X \to \top e_X, \qquad\qquad (k_X\,|\,t_{eX}) = \Box_X. \tag{14}$$

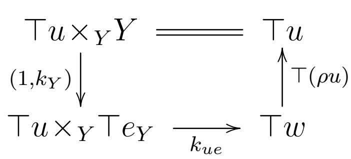
For a vertical composite $w = u \otimes v\colon X \to\!\!\!\bullet\, Y \to\!\!\!\bullet\, Z$, the comparison $\underline{S}(u,v)\colon Su \otimes Sv \to Sw$ is represented by the morphism k_{uv} defined below, where $P = \top u \times_Y \top v$ is a pullback and $\sigma = \lambda(e_P)^{-1} = \rho(e_P)^{-1}$

$$k_{uv}\colon P \to \top w, \qquad\qquad (k_{uv}\,|\,t_w) = \left(\sigma\,|\,\frac{r\,|\,t_u}{s\,|\,t_v}\right), \tag{15}$$



(Note that one cannot apply interchange to $(r\,|\,t_u) \otimes (s\,|\,t_v)$.) Finally we have to verify the coherence conditions of the comparisons of S (see [GP2], Section 2.1), and we only check axiom (iii) for the right unitor.

For a vertical map $u\colon X \to\!\!\!\bullet\, Y$ and $w = u \otimes e_Y$ we have to verify that the following diagram of morphisms of $\mathbf{C}$ commutes



(16)

where the pullback $\top u \times_Y Y$ is realised as $\top u$, by the unit constraint, and the morphism $\top(\rho u)$ is defined by: $(\top(\rho u) \,|\, t_u) = (t_w \,|\, \rho u)$.

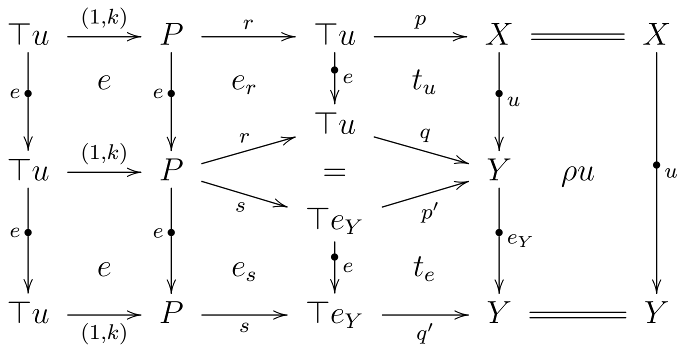
Equivalently, by applying the (cancellable) universal cell t_u and the isocell $\rho = \rho(e_{\top u})$, we show that

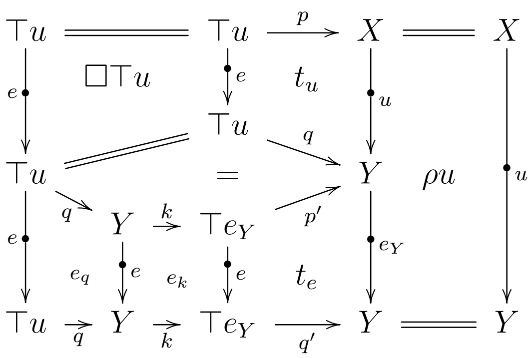
$$(\rho \,|\, (1, k_Y) \,|\, k_{ue} \,|\, \top(\rho u) \,|\, t_u) = (\rho \,|\, t_u).$$

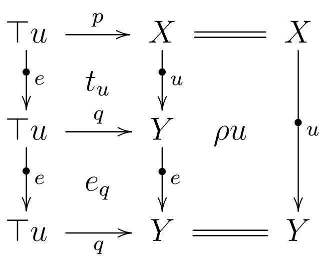
In fact we have

$$\begin{aligned}(\rho \,|\, (1, k_Y) \,|\, k_{ue} \,|\, \top(\rho u) \,|\, t_u) &= (\rho \,|\, (1, k_Y) \,|\, k_{ue} \,|\, t_w \,|\, \rho u) \\ = \left(\rho \,|\, (1, k_Y) \,|\, \sigma \,|\, \frac{r \,|\, t_u}{s \,|\, t_e} \,|\, \rho u\right) &= \left(\rho \,|\, \rho^{-1} \,|\, \frac{(1, k_Y) \,|\, r \,|\, t_u}{(1, k_Y) \,|\, s \,|\, t_e} \,|\, \rho u\right) \\ = \left(\frac{\Box \top u \,|\, t_u}{e_q \,|\, e_k \,|\, t_e} \,|\, \rho u\right) = \left(\frac{\Box \top u \,|\, t_u}{e_q \,|\, \Box Y} \;|\; \rho u\right) &= \left(\frac{t_u}{e_q} \,|\, \rho u\right) = (\rho(e_{\top u}) \,|\, t_u).\end{aligned} \tag{17}$$

The fourth, fifth and seventh terms of these computations are represented below, with $P = \top u \times_Y \top e_Y$ and $k = k_Y$







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7. Span and cospan representability

Let $\mathbb{A}$ be a weak double category.

(a) We say that $\mathbb{A}$ is (horizontally) *span representable* if:

- it has tabulators,
- the ordinary category $\mathbf{C} = \mathrm{Hor}_0(\mathbb{A})$ has pullbacks,
- the span-representation lax functor $S\colon \mathbb{A} \to \mathbb{S}\mathrm{pan}(\mathbf{C})$ of (11) is *horizontally faithful.*

The last condition means that the ordinary functors $\mathrm{Hor}_0 S$ and $\mathrm{Hor}_1 S$ are faithful. This is trivially true for $\mathrm{Hor}_0(S) = \mathrm{id}\mathbf{C}$, and also for $\mathrm{Hor}_1 S$ when $\mathbb{A}$ is flat.

(b) By horizontal duality, if $\mathbb{A}$ has cotabulators and $\mathbf{C} = \mathrm{Hor}_0(\mathbb{A})$ has pushouts we form a colax functor of *cospan representation*

$$C\colon \mathbb{A} \to \mathbb{C}\mathrm{osp}(\mathbf{C}), \qquad \mathrm{Hor}_0(C) = \mathrm{id}\mathbf{C}, \tag{18}$$

that takes a vertical arrow $u\colon X \to\!\!\!\bullet\, Y$ of $\mathbb{A}$ to the cospan $Cu = (i, j)\colon X \to\!\!\!\bullet\, Y$ formed by the cotabulator $\bot u$ and its 'injections' $i\colon A \to \bot u$, $j\colon Y \to \bot u$.

In this situation we say that $\mathbb{A}$ is *cospan representable* if this colax functor is horizontally faithful.

8. Some basic cases

(a) For a category $\mathbf{C}$ with pullbacks, the weak double category $\mathbb{S}\mathrm{pan}(\mathbf{C})$ is span representable, in a strict sense: the functor $S\colon \mathbb{S}\mathrm{pan}(\mathbf{C}) \to \mathbb{S}\mathrm{pan}(\mathbf{C})$ is an isomorphism, and even the identity for the natural choice of the tabulator of a span, namely its central object. Dually, for every category $\mathbf{C}$ with pushouts, $\mathbb{C}\mathrm{osp}(\mathbf{C})$ is 'strictly' cospan representable.

(b) On the other hand it is easy to see that $\mathbb{S}\mathrm{pan}\mathbf{Set}$ is *not* cospan representable, while $\mathbb{C}\mathrm{osp}\mathbf{Set}$ is *not* span representable. For the first fact we consider a morphism of spans $\sigma\colon u \to u$ represented in the left diagram below, where the objects are cardinal sets $(0 = \varnothing, 1 = \{0\}, 2 = \{0, 1\})$

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ \uparrow & & \uparrow \\ 2 & \xrightarrow{m\sigma} & 2 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 \end{array} \qquad\qquad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 2 & \xrightarrow{m\sigma} & 2 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array} \tag{19}$$

All the arrows to 1 are determined but the mapping $m\sigma\colon 2 \to 2$ is arbitrary; the cotabulator pushout is $\bot u = 1$ and $\bot\sigma$ does not determine σ.

The second counterexample is shown in the right diagram above, where again $m\sigma\colon 2 \to 2$ is arbitrary, the tabulator pullback is $\top u = 0$ and $\top\sigma$ does not detect σ.

Similar counterexamples can be given for any category $\mathbf{C}$ with finite limits (or colimits) and some object with at least two endomorphisms.

(c) The (strict) double category $\mathbb{A} = \mathbb{R}\mathrm{el}\mathbf{Set}$ *of sets, mappings and relations* [GP1] has $\mathrm{Hor}_0(\mathbb{A}) = \mathbf{Set}$, relations for vertical arrows and (flat) double cells given by an inequality in the ordered category of relations

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle u}\downarrow & \leqslant & \downarrow{\scriptstyle v} \\ Y & \xrightarrow[g]{} & Y' \end{array} \qquad\qquad gu \leqslant vf. \tag{20}$$

Tabulators and cotabulators exist: $\top u \subset X \times Y$ is the relation itself, as a subset of $X \times Y$, while $\bot u$ is the pushout of the span $Su = (X \leftarrow \top u \rightarrow Y)$, or of any span representing the relation.

Since $\mathbb{R}\mathrm{el}\mathbf{Set}$ is flat it is automatically span and cospan representable. The same holds replacing **Set** with any regular category with pushouts.

In the examples below we examine other strict or weak double categories, referring to their definition in [GP1, GP2], briefly reviewed here.

9. Representing profunctors

The weak double category $\mathbb{C}\mathrm{at}$ *of categories, functors and profunctors* was introduced in [GP1], Section 3.1. Objects are small categories, a horizontal arrow is a functor and a vertical arrow is a profunctor $u \colon X \twoheadrightarrow Y$, defined as a functor $u \colon X^{\mathrm{op}} \times Y \to \mathbf{Set}$. A cell $a \colon (u \,{}^{f}_{g}\, v)$ is a natural transformation $a \colon u \to v(f^{\mathrm{op}} \times g) \colon X^{\mathrm{op}} \times Y \to \mathbf{Set}$. Compositions and comparisons are known or easily defined.

The cotabulator $\bot u = X +_u Y$ of a profunctor $u \colon X \twoheadrightarrow Y$ is the gluing, or collage, of X and Y along u, with new maps given by $(\bot u)(x, y) = u(x, y)$ and no maps 'backwards'; the composition of the new maps with the old ones is defined by the action of u. The inclusions $i \colon X \to \bot u$ and $j \colon Y \to \bot u$ are obvious, as well as the structural cell $\iota \colon (u \,{}^{i}_{j}\, e)$

$$\iota \colon u \to e_{\bot u}(i^{\mathrm{op}} \times j) \colon X^{\mathrm{op}} \times Y \to \mathbf{Set}, \qquad\qquad \iota(x, y) \colon u(x, y) = \bot u(x, y). \tag{21}$$

The tabulator $\top u$ is the *category of elements* of u, or Grothendieck construction. It has objects (x, y, λ) with $x \in \mathrm{Ob}X$, $y \in \mathrm{Ob}Y$, $\lambda \in u(x, y)$ and maps (f, g) of $X \times Y$ which form a commutative square in the collage $X +_u Y$

$$\begin{array}{ll} (f, g) \colon (x, y, \lambda) \to (x', y', \lambda') & (f \colon x \to x',\ g \colon y \to y'), \\ g\lambda = \lambda' f & (u(1_x, g)(\lambda) = u(f, 1_y)(\lambda') \in u(x, y')). \end{array} \tag{22}$$

The functors p, q are obvious, and the structural cell $\tau = t_u \colon e_{\top u} \to u$ is the natural transformation

$$\begin{array}{l} \tau \colon e_{\top u} \to u(p^{\mathrm{op}} \times q) \colon (\top u)^{\mathrm{op}} \times \top u \to \mathbf{Set}, \\ \tau(x, y, \lambda; x', y', \lambda') \colon \top u(x, y, \lambda; x', y', \lambda') \to u(x, y'), \qquad (f, g) \mapsto g\lambda = \lambda' f. \end{array} \tag{23}$$

$\mathbb{C}\mathrm{at}$ is easily seen to be span and cospan representable. Indeed, for a cell $a \colon (u \,{}^{f}_{g}\, v)$, both the functors $\top a$ and $\bot a$ determine every component $a_{xy} \colon u(x, y) \to v(fx, gy)$ of the natural transformation $a\colon u \to v(f^{\mathrm{op}}\times g)\colon X^{\mathrm{op}}\times Y \to \mathbf{Set}$

$$\begin{aligned} &\top a\colon \top u \to \top v, && \top a(x,y,\lambda) = (fx, gy, a_{xy}(\lambda)),\\ &\bot a\colon \bot u \to \bot v, && \bot a(\lambda\colon x \to y) = a_{xy}(\lambda)\colon fx \to gy \qquad (\lambda \in u(x,y)). \end{aligned} \tag{24}$$

10. Representing adjoints.

We prove now that the double category $\mathbb{A}\mathrm{dj}\mathbf{Cat}$ of (small) *categories, functors and adjunctions*, introduced in [GP1], Section 3.5, is also span and cospan representable.

Again $\mathrm{Hor}_0(\mathbb{A}\mathrm{dj}\mathbf{Cat}) = \mathbf{Cat}$. A vertical arrow is now an ordinary adjunction, conventionally directed as the *left* adjoint

$$\begin{aligned} &u = (u_\bullet, u^\bullet, \eta, \varepsilon)\colon X \,\bullet\!\!\to Y, && (u_\bullet\colon X \to Y) \dashv (u^\bullet\colon Y \to X),\\ &\eta\colon 1_X \to u^\bullet u_\bullet, && \varepsilon\colon u_\bullet u^\bullet \to 1_Y. \end{aligned} \tag{25}$$

A double cell $a = (a_\bullet, a^\bullet)\colon u \to v$ is a pair of mate natural transformations, each of them determining the other via the units and counits of the two adjunctions

$$\begin{aligned} &a_\bullet\colon v_\bullet f \to g u_\bullet, \qquad a^\bullet\colon f u^\bullet \to v^\bullet g,\\ &a^\bullet = (f u^\bullet \to v^\bullet v_\bullet f u^\bullet \to v^\bullet g u_\bullet u^\bullet \to v^\bullet g),\\ &a_\bullet = (v_\bullet f \to v_\bullet f u^\bullet u_\bullet \to v_\bullet v^\bullet g u_\bullet \to g u_\bullet). \end{aligned} \tag{26}$$

(a) In $\mathbb{A}\mathrm{dj}\mathbf{Cat}$ the tabulator $\top u$ of an adjunction $u = (u_\bullet, u^\bullet)\colon X \,\bullet\!\!\to Y$ is the 'graph' of the adjunction, namely the following comma category, equipped with the comma-projections p, q and an obvious cell $\tau = t_u\colon (e\,{}^{p}_{q}\,u)$

$$\begin{aligned} &\top u = (u_\bullet{\downarrow}Y) \;\cong\; (X{\downarrow}u^\bullet), && (x,y;c\colon u_\bullet x \to y) \leftrightarrow (x,y;c'\colon x \to u^\bullet y),\\ &p\colon \top u \to X, && q\colon \top u \to Y,\\ &\tau_\bullet\colon u_\bullet p \to q\colon \top u \to Y, && \tau_\bullet(x,y;c) = c\colon u_\bullet x \to y. \end{aligned} \tag{27}$$

The tabulator of a cell $a\colon (u\,{}^{f}_{g}\,v)$, with components $a_\bullet x\colon v_\bullet fx \to g u_\bullet x$, is the following functor

$$\top a\colon \top u \to \top v, \qquad \top a(x,y;c\colon u_\bullet x \to y) = (fx, gy; g(c).a_\bullet x\colon v_\bullet fx \to gy). \tag{28}$$

This proves that $\mathbb{A}\mathrm{dj}\mathbf{Cat}$ *is span representable*: in fact the component $a_\bullet x\colon v_\bullet fx \to g u_\bullet x$ is determined by $\top a(x, u_\bullet x; 1\colon u_\bullet x \to u_\bullet x) = (fx, g u_\bullet x; a_\bullet x\colon v_\bullet fx \to g u_\bullet x)$.

(b) In $\mathbb{A}\mathrm{dj}\mathbf{Cat}$ the cotabulator $C = \bot u = X +_u Y$ is the category consisting of the disjoint union $X + Y$, together with new maps $\hat{c}_x = (x,y;c\colon u_\bullet x \to y)\hat{}\in C(x,y)$ from objects of X to objects of Y that are 'represented' by objects $(x,y;c\colon u_\bullet x \to y)$ of $\top u = (u_\bullet{\downarrow}Y)$; the composition of the new maps with old maps $\varphi \in X(x',x)$, $\psi \in Y(y,y')$ is defined in the obvious way

$$\psi.\hat{c}_x.\varphi = (x',y';\psi.c.u_\bullet(\varphi)\colon u_\bullet x' \to u_\bullet x \to y \to y')\hat{}. \tag{29}$$

The universal cell $\iota_\bullet\colon i \to ju_\bullet\colon X \to \bot u$ is given by $\iota_\bullet x = (1_{u_\bullet x})\hat{} \in C(x, u_\bullet x)$.

The cotabulator of a cell $a\colon (u\,{}^{f}_{g}\,v)$, with components $a_\bullet x\colon v_\bullet fx \to gu_\bullet x$, works as f and g on the old objects and arrows, as $\top a$ on the new arrows

$$\bot a\colon \bot u \to \bot v, \qquad \bot a(x, y; h\colon u_\bullet x \to y)\hat{} = (fx, gy; g(h).a_\bullet x\colon v_\bullet fx \to gy)\hat{}. \tag{30}$$

This determines $a_\bullet x$ as above.

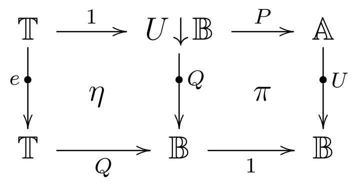
11. Representing $\mathbb{D}$bl

The strict double category $\mathbb{D}$bl of *weak double categories, lax functors and colax functors* is a crucial structure, on which the theory of double adjoints is based. We refer the reader to its introduction in [GP2], Section 2, where the non-obvious point of double cells is dealt with.

We prove now that $\mathbb{D}$bl is span representable, *horizontally* and *vertically.*

(a) First, every colax functor $U\colon \mathbb{A} \to\!\!\bullet\!\!\to \mathbb{B}$ has a *horizontal tabulator* (T, P, Q, τ).

The weak double category $\mathbb{T} = U{\downarrow}\mathbb{B}$ is a 'one-sided' double comma ([GP2], Section 2.5), with strict projections P and Q, which can be used as horizontal *or* vertical arrows. Below the cell η is simply represented by the horizontal transformation $1_Q\colon Q \to Q$ and the tabulator cell $\tau = t_U$ is linked to the comma-cell π by the unit η and counit ε of the *companionship* of Q with 'itself' (see [GP2])



$$\tau = (\eta\,|\,\pi), \quad \pi = \left(\tfrac{\tau}{\varepsilon}\right). \tag{31}$$

To be more explicit, the tabulator $\mathbb{T}$ has objects

$$(A, B, b\colon UA \to B), \tag{32}$$

with A in $\mathbb{A}$ and b horizontal in $\mathbb{B}$. A horizontal arrow of $\mathbb{T}$

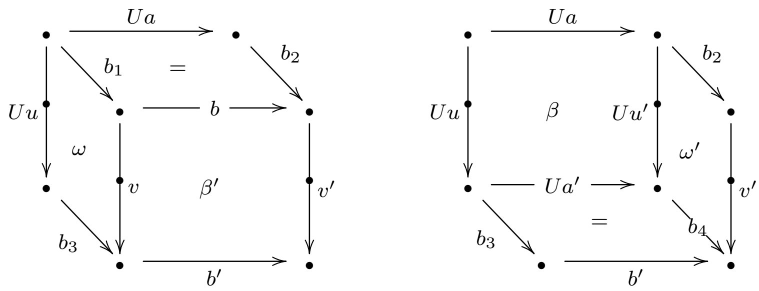
$$(a, b)\colon (A_1, B_1, b_1) \to (A_2, B_2, b_2), \tag{33}$$

'is' a commutative square in $\mathrm{Hor}_0\mathbb{B}$, as in the upper square of diagram (35), below (where the slanting direction must be viewed as horizontal). A vertical arrow of $\mathbb{T}$

$$(u, v, w)\colon (A_1, B_1, b_1) \to (A_3, B_3, b_3), \tag{34}$$

'is' a double cell in $\mathbb{B}$, as in the left square of diagram (35). A double cell (β, β') of $\mathbb{T}$ forms a commutative diagram of double cells of $\mathbb{B}$

$$(\beta, \beta')\colon ((u, v, \omega)\ {}^{(a,b)}_{(a',b')}\ (u', v', \omega')), \qquad (\omega\,|\,\beta') = (\beta\,|\,\omega), \tag{35}$$



The composition laws of $\mathbb{T}$ are obvious, as well as the (strict) double functors P, Q. The double cell τ has components

$$\tau(A, B, b) = b\colon UA \to B, \qquad\qquad \tau(u, v, \omega) = \omega\colon Uu \to v. \tag{36}$$

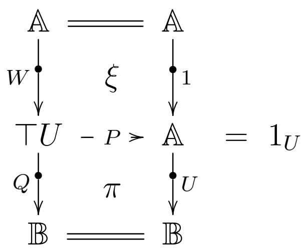
Its universal property follows trivially from that of the double comma, in [GP2], Theorem 2.6(a).

(b) We have thus a span representation

$$S : \mathbb{D}\mathrm{bl} \to \mathbb{S}\mathrm{pan}(\mathrm{Lx}\mathbf{Dbl}), \tag{37}$$

where $\mathrm{Lx}\mathbf{Dbl} = \mathrm{Hor}_0\mathbb{D}\mathrm{bl}$ is the category of weak double categories and lax functors. (Note that, even though the projections P, Q of the double comma $\mathbb{T}$ are strict double functors, a cell $\varphi\colon (U\ {}^{F}_{G}\ V)$ in $\mathbb{D}\mathrm{bl}$ gives a lax functor $\top\varphi\colon \top U \to \top V$.)

To prove that $\mathbb{D}\mathrm{bl}$ is horizontally span representable, we use the *vertical* universal property of the double comma $\mathbb{T} = U \downarrow \mathbb{B}$, in [GP2], Theorem 2.6(b), and deduce the existence of a colax functor $W\colon \mathbb{A} \to \mathbb{T}$ and a cell ξ such that:



$$= \ 1_U \qquad\qquad (QW = U). \tag{38}$$

Now a cell $\varphi\colon (U\ {}^{F}_{G}\ V)$ in $\mathbb{D}\mathrm{bl}$ can be recovered from the lax functor $\top\varphi\colon \top U \to \top V$ as follows

$$\begin{aligned}\varphi &= (1_U \,|\, \varphi) = (\xi \otimes \pi \,|\, e_F \otimes \varphi) = (\xi \otimes t_U \otimes \varepsilon \,|\, e_F \otimes \varphi \otimes e_G)\\ &= (\xi \,|\, e_F) \otimes (t_U \,|\, \varphi) \otimes (\varepsilon \,|\, e_G) = (\xi \,|\, e_F) \otimes (\top\varphi \,|\, t_V) \otimes (\varepsilon \,|\, e_G).\end{aligned}$$

(c) Transpose duality leaves $\mathbb{D}\mathrm{bl}$ invariant up to isomorphism: sending an object $\mathbb{A}$ to the horizontal opposite $\mathbb{A}^h$ and transposing double cells we have an isomorphism $\mathbb{D}\mathrm{bl} \to \mathbb{D}\mathrm{bl}^t$. Therefore $\mathbb{D}\mathrm{bl}$ is also *vertically span representable*, which means that $\mathbb{D}\mathrm{bl}^t$ is span representable by a lax functor

$$S' : \mathbb{D}\mathrm{bl}^t \to \mathbb{S}\mathrm{pan}(\mathrm{Cx}\mathbf{Dbl}) \qquad\qquad (\mathrm{Cx}\mathbf{Dbl} = \mathrm{Hor}_0\mathbb{D}\mathrm{bl}^t = \mathrm{Ver}_0\mathbb{D}\mathrm{bl}). \tag{39}$$

The latter sends a lax functor $F\colon \mathbb{A} \to \mathbb{B}$ to the span $S'(F) = (\mathbb{A} \leftarrow \mathbb{T}' \to \mathbb{B})$ associated to its *vertical tabulator* $(\mathbb{T}, P, Q, \tau)$, where the weak double category $\mathbb{T} = \mathbb{B}{\downarrow}F$ has objects $(A, B, b\colon B \to FA)$, and the cell τ is vertically universal

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{1} & \mathbb{T} \\ {\scriptstyle P}\big\downarrow & \tau & \big\downarrow{\scriptstyle Q} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array} \qquad (40)$$

12. Theorem (Representing quintets)

The 2-category $\mathbf{C}$ *is 2-complete if and only if the associated double category* $\mathbb{Q}\mathbf{C}$ *of quintets has all double limits. In this case the double category* $\mathbb{Q}\mathbf{C}$ *is span representable.*

Proof. Let us recall that the double category $\mathbb{Q}\mathbf{C}$ of quintets (introduced by C. Ehresmann) has for horizontal and vertical maps the morphisms of $\mathbf{C}$, while its double cells are defined by 2-cells of $\mathbf{C}$

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle u}\big\downarrow & \varphi & \big\downarrow{\scriptstyle v} \\ Y & \xrightarrow[g]{} & Y' \end{array} \qquad \varphi\colon vf \to gu\colon X \to Y'. \qquad (41)$$

It is known that $\mathbf{C}$ is 2-complete if and only if it has 2-products, 2-equalisers and cotensors by the arrow-category $\mathbf{2}$ [St]. First, it is easy to see that 2-products (resp. 2-equalisers) in $\mathbf{C}$ are 'the same' as double products (resp. double equalisers) in $\mathbb{Q}\mathbf{C}$. Second, the cotensor $\mathbf{2} * X$ can be obtained as the tabulator of the vertical identity of X: they are defined by the same universal property.

Conversely, if the $\mathbf{C}$-morphism $u\colon X \to Y$ is viewed as vertical in $\mathbb{Q}\mathbf{C}$, its tabulator $(\top u; p, q; \tau)$ can be constructed as the following inserter $(\top u; i, \tau)$

$$\top u \xrightarrow{i} X{\times}Y \underset{p''}{\overset{up'}{\rightrightarrows}} Y, \qquad \tau\colon up'i \to p''i\colon \top u \to Y, \qquad (42)$$

letting $p = p'i\colon \top u \to X$, $q = p''i\colon \top u \to Y$ and viewing τ as a double cell with boundary $(1\, {}^{p}_{q}\, u)$.

If $\mathbf{C}$ is 2-complete, $\mathbb{Q}\mathbf{C}$ is span representable because the lax span representation $S\colon \mathbb{Q}\mathbf{C} \to \mathbb{S}\mathrm{pan}(\mathbf{C})$ operates on a double cell $a\colon vf \to gu$ of $\mathbb{Q}\mathbf{C}$ producing a morphism of spans $Sa\colon Su \to Sv$ whose central map $\top a\colon \top u \to \top v$ is defined as follows

$$\begin{array}{ll} t_u\colon up'i \to p''i\colon \top u \to Y, & t_v\colon vq'j \to q''j\colon \top v \to Y', \\ j.\top a = (f{\times}g)i, & t_v.\top a = gt_u.ap'i\colon vfp'i \to gp''i. \end{array} \qquad (43)$$

$$\begin{array}{ccccc}
\top u & \xrightarrow{\;i\;} & X \times Y & \overset{up'}{\underset{p''}{\rightrightarrows}} & Y \\
{\scriptstyle \top a}\downarrow & & {\scriptstyle f\times g}\downarrow & & \downarrow{\scriptstyle g} \\
\top v & \xrightarrow[j]{} & X' \times Y' & \overset{vq'}{\underset{q''}{\rightrightarrows}} & Y'
\end{array}$$

Now f and g are determined as the vertical faces of the morphism Sa. To recover the 2-cell $a\colon vf \to gu$ of $\mathbf{C}$ from the morphism $\top a$, one uses the map $h\colon X \to \top u$ determined by the conditions $ih = (1, u)\colon X \to X \times Y$ and $t_u.h = 1_u$, so that

$$t_v.\top a.h = (gt_u.ap'i)h = gt_uh.ap'ih = ap'(1, u) = a.$$

■

13. Splitting tabulators

In order to 'explain' how so many double categories are span representable, we observe that the proof for the non-obvious cases above follows a pattern of the following type (as in Section 12), or a vertical version of the same (as in Section 11). However the argument is rather complicated, and - in the examples above - we preferred to give a direct proof, following this guideline.

We are in a weak double category $\mathbb{A}$ with tabulators, and the category $\mathbf{C} = \mathrm{Hor}_0(\mathbb{A})$ has pullbacks. In order that $\mathbb{A}$ be span representable it is sufficient that, for every vertical arrow u, there exist two cells s_u and ε satisfying the following condition:

$$\begin{array}{ccccccc}
X & \xrightarrow{\;h\;} & \top u & \xrightarrow{\;p\;} & X & = = = & X \\
 & & {\scriptstyle e}\downarrow & t_u & \downarrow{\scriptstyle u} & & \\
{\scriptstyle u}\downarrow & s_u & \top u & \xrightarrow{\;q\;} & Y & \rho u & \downarrow{\scriptstyle u} \\
 & & {\scriptstyle q_*}\downarrow & \varepsilon & \downarrow{\scriptstyle e} & & \\
Y & = = = & Y & = = = & Y & = = = & Y
\end{array} \quad = \quad 1_u. \qquad (44)$$

(Typically, q_* is the vertical companion of q and $\varepsilon = \varepsilon_q$ its counit, but this is not needed in the proof. In the strict case ρu is trivial.)

In fact one can recover a cell $a\colon (u\, {}^{f}_{g}\, v)$ from $\top a$ (and Sa), as follows

$$a = (s_u \mid \frac{t_u}{\varepsilon} \mid \rho u \mid a) = (s_u \mid \frac{t_u}{\varepsilon} \mid \frac{a}{e_g} \mid \rho v) = (s_u \mid \frac{t_u \mid a}{\varepsilon \mid e_g} \mid \rho v) = (s_u \mid \frac{\top a \mid t_v}{\varepsilon \mid e_g} \mid \rho v).$$

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