The difference calculus for functors on presheaves

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CT 2024 Santiago de Compostela

June 27, 2024

Calculus of differences

- Aim: Categorify Newton's difference operator Δ
 - For $f: \mathbb{R} \longrightarrow \mathbb{R}$, $\Delta[f](x) = f(x+1) f(x)$
 - A discrete version of derivative

Inspired in part by:

- Work on polynomial functors by Kock [6], Niu/Spivak [7], and many others
- Work on analytic functors by Joyal [5] et. al.
- Multivariable analytic functors, e.g. Fiore/Gambino/Hyland/Winskel [4]
- Differential structures, see Cockett/Cruttwell [3]

Likely related to:

- The cartesian difference categories of Alvarez-Picallo/Pacaud-Lemay [1]
- The Goodwillie calculus, see e.g. Bauer/Johnson/Osborne/Riehl/Tebbe [2]

General idea

• For $F: \mathbf{Set} \longrightarrow \mathbf{Set}$, perturb the input and measure the difference in output

$$\Delta[F](X) = F(X+1) \setminus F(X)$$

Example

 $F(X) = X^3$, then F(X+1) has eight kinds of elements:

$$(x_1, x_2, x_3)$$

$$(x_1, x_2, *), (x_1, *, x_3), (*, x_2, x_3)$$

$$(x_1, *, *), (*, x_2, *), (*, *, x_3)$$

$$(*, *, *)$$

$$\Delta[F](X) = 3X^2 + 3X + 1$$

Example

 $F(X) = 2^X$ covariant power set, then F(X+1) has two kinds of elements:

$$A \subseteq X \subseteq X + 1$$

$$A \cup \{*\} \subseteq X + 1 \quad (A \subseteq X)$$

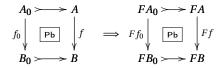
$$\Delta[F](X) = 2^X$$

Tautness

• $F(X+1) \setminus F(X)$ not always functorial

Definition

(Manes 2002) A functor is taut if it preserves inverse images



A natural transformation $t: F \longrightarrow G$ is taut if the naturality squares corresponding to monos are pullbacks



 Get a sub-2-category Taut of Cat whose objects are categories with inverse images and taut functors and taut natural transformations

Limits

Taut functors are closed under limits.

Proposition

- (1) Limits in Cat(Set, Set) of taut functors are taut.
- (2) The inclusion

 $\mathcal{T}aut(\mathbf{Set}, \mathbf{Set}) \longrightarrow \mathscr{C}at(\mathbf{Set}, \mathbf{Set})$

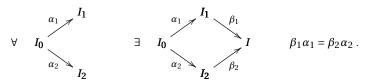
creates non-empty connected limits.

(3) The product of taut functors is taut but the projections are not.

Confluence

Theorem

I colimits commute with inverse images in Set if and only if



Definition

If I satisfies the above conditions we say it's confluent.

Example

Filtered colimits, coproducts, quotients by group actions are all confluent.

Proposition

- (1) Confluent colimits in $\mathscr{C}at(\mathbf{Set},\mathbf{Set})$ of taut functors are taut.
- (2) $\mathcal{F}aut(\mathbf{Set}, \mathbf{Set}) \longrightarrow \mathcal{C}at(\mathbf{Set}, \mathbf{Set})$ creates all colimits.

Examples

- Polynomial functors $P(X) = \sum_{i \in I} X^{A_i}$ are taut
- Analytic functors $\widetilde{F}(X) = \int_{-\infty}^{n} X^n \times F(n) \cong \sum_{n} X^n \times F(n) / S_n$ are taut ($F: \mathbf{Bij} \longrightarrow \mathbf{Set}$ a species)
- Manes: Collection monads are finitary taut monads

The difference operator

Proposition

(1) If $F: \mathbf{Set} \longrightarrow \mathbf{Set}$ is taut then

$$\Delta[F](X) = F(X+1) \setminus F(X)$$

defines a taut subfunctor of F(X+1).

(2) A taut transformation $t: F \longrightarrow G$ restricts to a taut transformation $\Delta[t]: \Delta[F] \longrightarrow \Delta[G]$.

The functor

$$\Delta$$
: $\mathcal{T}aut(\mathbf{Set}, \mathbf{Set}) \longrightarrow \mathcal{T}aut(\mathbf{Set}, \mathbf{Set})$

is called the difference operator.

Example

$$\Delta[C] = 0$$

$$\Delta[X] = 1$$

Colimits

Proposition

 Δ preserves colimits: For Γ : $I \longrightarrow \mathcal{T}aut(\mathbf{Set}, \mathbf{Set})$

$$\Delta[\varinjlim_I \Gamma I] \cong \varinjlim_I \Delta[\Gamma I]$$

Corollary

(1)
$$\Delta[F+G] \cong \Delta[F] + \Delta[G]$$

(2)
$$\Delta[CF] \cong C\Delta[F]$$

Limits

Proposition

$$\Delta[F\times G]\cong (\Delta[F]\times G)+(F\times \Delta[G])+(\Delta[F]\times \Delta[G]).$$

More generally:

Proposition

$$\Delta \left[\prod_{i \in I} F_i \right] \cong \sum_{f \subsetneq I} \left(\prod_{j \in J} F_j \right) \times \left(\prod_{k \notin J} \Delta[F_k] \right).$$

Theorem

 Δ preserves non-empty connected limits

$$\Delta[\varprojlim_I \Gamma I] \cong \varprojlim_I \Delta[\Gamma I] \ .$$

Lax chain rule

Theorem

For taut functors F and G there is a taut natural transformation

$$\gamma_{G,F}: (\Delta[G] \circ F) \times \Delta[F] \longrightarrow \Delta[G \circ F]$$

which is:

- (1) monic,
- (2) natural in F and G,
- (3) associative

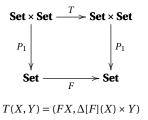
(4) unitary
$$(\Delta[\operatorname{Id}] \circ F) \times \Delta[F] \xrightarrow{\gamma_{\operatorname{Id},F}} \Delta[\operatorname{Id} \circ F] \qquad (\Delta[F] \circ \operatorname{Id}) \times \Delta[\operatorname{Id}] \xrightarrow{\gamma_{F,\operatorname{Id}}} \Delta[F \circ \operatorname{Id}]$$

$$\parallel \qquad \qquad \parallel \qquad \qquad \parallel \qquad \qquad \parallel$$

$$1 \times \Delta[F] \xrightarrow{\cong} \Delta[F] , \qquad \Delta[F] \times 1 \xrightarrow{\cong} \Delta[F] .$$

Tangent structure

For a taut functor F we define



Proposition

 $T: \mathcal{T}aut(\mathbf{Set}, \mathbf{Set}) \longrightarrow \mathcal{T}aut(\mathbf{Set} \times \mathbf{Set}, \mathbf{Set} \times \mathbf{Set})$ is a lax normal monoidal functor

Polynomial functors

Proposition

If $P(X) = \sum_{i \in I} X^{A_i}$ is a polynomial functor, then $\Delta[P](X)$ is again polynomial

$$\Delta[P](X) \quad \cong \sum_{S \subsetneq A_i, \ i \in I} X^S$$

Example

$$\Delta[X^A] = \sum_{S \subsetneq A} X^S$$

Example

$$\Delta[X^n] = \sum_{k=0}^{n-1} \binom{n}{k} X^k$$

Multivariable functors

· Extend the difference calculus to functors

$$F: \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{B}}$$

- B families of functors in A variables
- Partial difference with respect to A:

For Φ in $\mathbf{Set}^{\mathbf{A}}$, perturb it by adding a single element of type A freely, $\Phi \leadsto \Phi + \mathbf{A}(A,-)$

$$\Delta_A[F](\Phi) = F(\Phi + \mathbf{A}(A, -)) \setminus F(\Phi)$$

- The one-variable theory carries over with some modifications
- Based on profunctors

Profunctors (a.k.a. 2-matrices)

- A profunctor P: A →> B is a functor P: A^{op} × B →> Set
 A morphism of profunctors is a natural transformation
- P can be thought of as a B by A matrix of sets
- Composition of $P: \mathbf{A} \longrightarrow \mathbf{B}$ with $Q: \mathbf{B} \longrightarrow \mathbf{C}$ is "matrix multiplication"

$$(Q \otimes P)(A, C) = \int^{B} Q(B, C) \times P(A, B)$$

· Identities are hom functors

$$Id_{\mathbf{A}} = \mathbf{A}(-, -) : \mathbf{A}^{op} \times \mathbf{A} \longrightarrow \mathbf{Set}$$

2-vectors (a.k.a. presheaves)

- A profunctor $\mathbb{1} \longrightarrow \mathbf{A}$ is a functor $\mathbb{1}^{op} \times \mathbf{A} \longrightarrow \mathbf{Set}$ which we identify with the presheaf $\Phi \in \mathbf{Set}^{\mathbf{A}}$
- Composing Φ with a profunctor P gives an object $P \otimes \Phi$ of $\mathbf{Set}^{\mathbf{B}}$ and so we get a functor $P \otimes () : \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{B}}$ which is cocontinuous (2-linear)
- Its partial difference with respect to A is

$$\begin{split} \Delta_A[P \otimes (\)](\Phi) &= P \otimes (\Phi + \mathbf{A}(A,-)) \setminus P \otimes \Phi \\ &\cong (P \otimes \Phi + P \otimes \mathbf{A}(A,-)) \setminus P \otimes \Phi \\ &\cong P \otimes \mathbf{A}(A,-) \\ &\cong P(A,-) \end{split}$$

a constant functor (independent of Φ)

$$Set^A \longrightarrow Set^B$$

Tense functors

 $P \otimes ()$ is not taut!

Definition

F is tense if it preserves complemented subobjects and their pullbacks

 $t: F \longrightarrow G$ is tense if the naturality squares corresponding to complemented subobjects are pullbacks

- If F preserves binary coproducts then it's tense, so $P \otimes ($) is tense
- There is a sub-2-category of *Cat*, *Tense*, consisting of presheaf categories, tense functors and tense natural transformations

Limits and colimits

Proposition

- (1) Let $\Gamma: \mathbf{I} \longrightarrow \mathscr{C}\!\mathit{at}(\mathbf{Set}^A, \mathbf{Set}^B)$ be such that $\Gamma(I)$ is tense for every \mathbf{I} . Then $\lim \Gamma$ is also tense. If \mathbf{I} is confluent so is $\lim \Gamma$.

Partial difference

Proposition

Let $F: \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{B}}$ be tense, then

$$\Delta_A[F](\Phi) = F(\Phi + \mathbf{A}(A,-)) \setminus F(\Phi)$$

defines a tense subfunctor

$$\Delta_A[F] > F(-+\mathbf{A}(A,-))$$

functorial in F

$$\Delta_A$$
: $\mathscr{T}ense\ (\mathbf{Set}^{\mathbf{A}}, \mathbf{Set}^{\mathbf{B}}) \longrightarrow \mathscr{T}ense\ (\mathbf{Set}^{\mathbf{A}}, \mathbf{Set}^{\mathbf{B}})$.

Definition

 $\Delta_A[F]$ is the partial difference of F with respect to A.

- $\Delta_A[C] = 0$
- $\Delta_A[P \otimes ()] \cong P(A, -)$ (constant)

Limits and colimits

Proposition

 Δ_A : $\mathscr{T}ense\ (\mathbf{Set^A}, \mathbf{Set^B}) \longrightarrow \mathscr{T}ense\ (\mathbf{Set^A}, \mathbf{Set^B})$ preserves colimits and non-empty connected limits

Corollary

- (1) $\Delta_A[F+G] \cong \Delta_A[F] + \Delta_A[G]$
- (2) $\Delta_A[C \times F] \cong C \times \Delta_A[F]$

Proposition

$$\Delta_A \left[\prod_{i \in I} F_i \right] \cong \sum_{J \subsetneq I} \left(\prod_{j \in J} F_j \right) \times \left(\prod_{k \notin J} \Delta_A[F_k] \right)$$

Corollary

$$\Delta_A[F \times G] \cong (\Delta_A[F] \times G) + (F \times \Delta_A[G]) + (\Delta_A[F] \times \Delta_A[G])$$

(Discrete) Jacobian

For $F: \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{B}}$ a tense functor

Proposition

For Φ in $\mathbf{Set}^{\mathbf{A}}$, $\Delta_A[F](\Phi)$ is (contravariantly) functorial in A

$$\Delta[F](\Phi): \mathbf{A}^{op} \longrightarrow \mathbf{Set}^{\mathbf{B}}$$

• $\Delta[F](\Phi)$ is a profunctor $\mathbf{A} \longrightarrow \mathbf{B}$, the *(discrete) Jacobian* of F at Φ

Proposition

 $\Delta[F](\Phi)$ is functorial in Φ giving a tense functor

$$\Delta[F]: \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{A}^{op} \times \mathbf{B}} = \mathscr{P}rof(\mathbf{A}, \mathbf{B})$$

Proposition

 $\Delta[F]$ is functorial in F giving the Jacobian functor

$$\Delta$$
: $\mathscr{T}ense\ (\mathbf{Set}^{\mathbf{A}}, \mathbf{Set}^{\mathbf{B}}) \longrightarrow \mathscr{T}ense\ (\mathbf{Set}^{\mathbf{A}}, \mathbf{Set}^{\mathbf{A}^{op} \times \mathbf{B}})$

Alternate formulations

· Differential operator

$$D[F]: \mathbf{Set}^{\mathbf{A}} \times \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{B}}$$
$$D[F](\Phi, \Psi) = \Delta[F](\Phi) \otimes \Psi$$

D[F] is cocontinuous in the second variable

Tangent functor

$$\begin{array}{c|c} \mathbf{Set^A} \times \mathbf{Set^A} & \xrightarrow{T[F]} & \mathbf{Set^B} \times \mathbf{Set^B} \\ & & \downarrow P_1 \\ & & \downarrow P_1 \\ & & \downarrow P_1 \\ & & \mathbf{Set^A} & \xrightarrow{F} & \mathbf{Set^B} \\ & & & & \\ T[F](\Phi, \Psi) = (F(\Phi), \Delta[F](\Phi) \otimes \Psi) \end{array}$$

T[F] also cocontinuous in the second variable

Lax chain rule

Theorem

For tense functors $F \colon \mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{B}}$, $G \colon \mathbf{Set}^{\mathbf{B}} \longrightarrow \mathbf{Set}^{\mathbf{C}}$ and Φ in $\mathbf{Set}^{\mathbf{A}}$ we have a canonical comparison

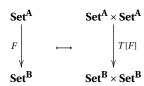
$$\gamma: \Delta[G](F(\Phi)) \otimes_{\mathbf{B}} \Delta[F](\Phi) \longrightarrow \Delta[GF](\Phi)$$

which is

- (1) natural in Φ
- (2) natural in F and G
- (3) associative
- (4) normal

Corollary

 $T: \mathcal{T}ense \longrightarrow \mathcal{T}ense$



is a lax normal functor

Multivariable analytic functors

After Fiore et al. [4]

- !A free symmetric monoidal category generated by A
 - Objects: finite sequences $\langle A_1, \dots, A_n \rangle$
 - Morphisms: $(\sigma, \langle f_1, ..., f_m \rangle)$: $\langle A_1, ..., A_n \rangle \longrightarrow \langle A'_1, ..., A'_m \rangle$ $\sigma \colon m \longrightarrow n$ bijection, $f_i \colon A_{\sigma i} \longrightarrow A'_i$
- A-B symmetric sequence (multivariable species) is a profunctor $P: A \longrightarrow B$
- Defines a multivariable analytic functor

$$\begin{split} \widetilde{P} \colon \mathbf{Set}^{\mathbf{A}} &\longrightarrow \mathbf{Set}^{\mathbf{B}} \\ \widetilde{P}(\Phi)(B) &= \int^{\langle A_1 \dots A_n \rangle \in !\mathbf{A}} P(A_1, \dots, A_n; B) \times \Phi A_1 \times \dots \times \Phi A_n \end{split}$$

Theorem

 \widetilde{P} is tense and $\Delta[\widetilde{P}]$ is an analytic functor $\mathbf{Set}^{\mathbf{A}} \longrightarrow \mathbf{Set}^{\mathbf{A}^{op} \times \mathbf{B}}$

The difference symmetric sequence

 $\Delta[\widetilde{P}] \cong \widetilde{Q} \text{ for } Q: !\mathbf{A} \longrightarrow \mathbf{A}^{op} \times \mathbf{B}$

$$Q(A_1,\dots,A_n;A,B) = \sum_{k=1}^{\infty} P(A_1,\dots,A_n,A,\dots,A;B)/\{\mathrm{id}_n\} \times S_k$$

where there are k A's in the k^{th} summand

When A = B = 1, $!A \cong Bij$ and we recover the original definition of species and analytic functor. Then

$$Q: \mathbf{Bij} \longrightarrow \mathbf{Set}$$

$$Q(n) = \sum_{k=1}^{\infty} P(n+k)/\{\mathrm{id}\} \times S_k$$

A Q-structure on n is a positive integer k and an equivalence class of P-structures on n+k, two structures being equivalent if there is a permutation of n+k fixing the first n elements which transforms one into the other

Exponential functors

• How should we categorify $f(x) = a^x$, a > 0?

Example

 $F(X) = 2^X$ covariant power set

If L is a sup-lattice we can make $F(X) = L^X$ into a covariant functor $L^X : \mathbf{Set} \longrightarrow \mathbf{Set}$ by Kan extension. For $f : X \longrightarrow Y$ and $\phi \in L^X$

$$F(f)(\phi)(y) = \bigvee_{f(x)=y} \phi(x) \; .$$

Proposition

 L^X : **Set** \longrightarrow **Set** is taut and

$$\Delta[L^X] \cong L_* \times L^X$$

where $L_* = L \setminus \{\bot\}$.

Example

$$\Delta[3^X] \cong 2 \times 3^X$$

Dirichlet functors?

• A first try might be

$$F(X) = \sum_{i \in I} L_i^X$$

ullet For every positive integer n the ordinal

$$\mathbf{n} = \{1 < 2 < 3 < \dots < n\}$$

is a sup-lattice, but . . .

• For any unbounded sequence $n_1 < n_2 < \dots$

$$\sum_{i\in\mathbb{N}}\mathbf{n}_i^X\cong\sum_{n\in\mathbb{N}}\mathbf{n}^X$$

Normalized exponentials

• L^X is not connected: $\pi_0(L^X) \cong L$

$$L^X \cong \sum_{l \in L} C_l(X) \quad C_l(X) = \left\{ f \colon X {\longrightarrow} L \mid \bigvee f(x) = l \right\}$$

• The normalized exponential

$$L^{[X]} = \{f \colon X \longrightarrow L \mid \bigvee f(x) = \top \}$$

•
$$L^X = \sum_{l \in L} (L/l)^{[X]}$$
 $L/l = \{l' \in L \mid l' \leq l\}$

Proposition

 $L^{[X]}$ is taut and

$$\Delta \left[L^{[X]} \right] \cong \sum_{\substack{l \lor l' = \top \\ l' \neq l}} (L/l)^{[X]}$$

Corollary

If \top is join irreducible (i.e. $l \lor l' = \top \Rightarrow l = \top$ or $l' = \top$) then

$$\Delta \left[L^{[X]} \right] \cong L_* \times L^{[X]} + \sum_{l \neq \top} (L/l)^{[X]}$$

(Covariant) Dirichlet functors

Proposition

If $\langle L_i \rangle_{i \in I}$ and $\langle M_j \rangle_{j \in J}$ are two families of sup-lattices such that

$$\sum_{i \in I} L_i^{[X]} \cong \sum_{j \in J} M_j^{[X]}$$

then there is a bijection $\alpha: I \longrightarrow J$ and lattice isomorphisms

$$L_i \cong M_{\alpha(i)}$$
.

Definition

A (covariant) Dirichlet functor is a functor of the form

$$F(X) = \sum_{i \in I} L_i^{[X]}$$

for $\langle L_i \rangle$ a family of sup-lattices.

Dirichlet difference

Proposition

Dirichlet functors are taut and closed under products and coproducts

Theorem

If $F(X) = \sum_{i \in I} L_i^{[X]}$ is Dirichlet, then so is $\Delta[F](X)$ and

$$\Delta[F](X) = \sum_{i \in I, l \in L_i} C_l \times (L_i/l)^{[X]}$$

where
$$C_l = \{l' \in L_i \mid l' \neq \bot \land l \lor l' = \top\}$$

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