Retrocells

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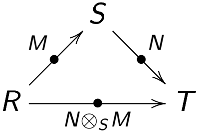
Bimodules

• The bicategory $\mathcal{B}im$ has rings $R, S, T, \ldots$ as objects, bimodules $M : R \longrightarrow\!\!\bullet\!\!\rightarrow S$ as 1-cells, and S-R-linear maps as 2-cells
Composition is $\otimes$



• $\mathcal{B}im$ is biclosed, $\otimes$ has right adjoints in each variable

$$\frac{\dfrac{M \longrightarrow N \obslash_T P}{N \otimes_S M \longrightarrow P}}{N \longrightarrow P \oslash_R M}$$

$N \obslash_T P = Hom_T(N, P)$, $P \oslash_R M = Hom_R(M, P)$

Biclosed

Many bicategories are biclosed

• $\mathcal{B}im$	:	Rings, bimodules, linear maps
• $\mathcal{P}rof$	:	Categories, profunctors, natural transformations
• $\mathbf{V}$-$\mathcal{P}rof$	:	$\mathbf{V}$ – with colimits preserved by $\otimes$ – biclosed – limits
• $\mathcal{S}pan(\mathbf{A})$	:	$\mathbf{A}$ with pullbacks and locally cartesian closed

Scandal

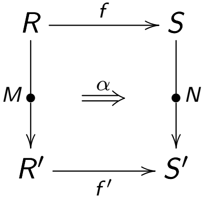
Good bicategories (all of the above) are the vertical part of naturally occurring double categories:

$\mathbb{R}\mathrm{ing}$, $\mathbb{C}\mathrm{at}$, $\mathbf{V}$-$\mathbb{C}\mathrm{at}$, $\mathbb{S}\mathrm{pan}\mathbf{A}$

But the internal homs $\oslash$ and $\obslash$ are not double functors!

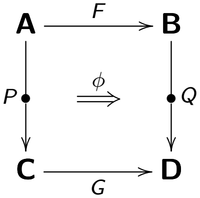
- A *double category* is a "category with two sorts of morphisms"

- **Example:** $\mathbb{R}\mathrm{ing}$



$\mathbb{C}\mathrm{at}$

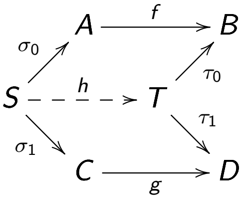
- **Example:** $\mathbb{C}\mathrm{at}$



$$P : \mathbf{A}^{op} \times \mathbf{C} \longrightarrow \mathbf{Set}$$
$$Q : \mathbf{B}^{op} \times \mathbf{D} \longrightarrow \mathbf{Set}$$
$$\phi : P(-,=) \longrightarrow Q(F-, G =)$$

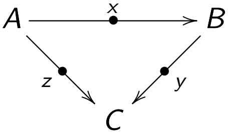
$\mathbb{S}$pan

- **Example:** $\mathbb{S}\text{pan}\, \mathbf{A}$



Left homs

- $\mathbb{A}$ has *left homs* if $y \bullet (\ )$ has a right adjoint $y \mathbin{\backslash\!\!\bullet} (\ )$ in $\mathcal{V}ert\mathbb{A}$



$$\frac{y \bullet x \to z}{x \to y \mathbin{\backslash\!\!\bullet} z} \qquad \text{in } \mathcal{V}ert\mathbb{A}$$

Mike Shulman, “Framed bicategories and monoidal fibrations” (TAC 2008)
Roald Koudenburg, “On pointwise Kan extensions in double categories” (TAC 2014)

Respecting boundaries

- $y \setminus\!\!\!\bullet\, z$ is covariant in z and contravariant in y

$$y' \xrightarrow{\beta} y,\ z \xrightarrow{\gamma} z' \quad \rightsquigarrow \quad y \setminus\!\!\!\bullet\, z \xrightarrow[\bullet]{\beta \setminus\!\!\!\bullet\, \gamma} y' \setminus\!\!\!\bullet\, z'$$

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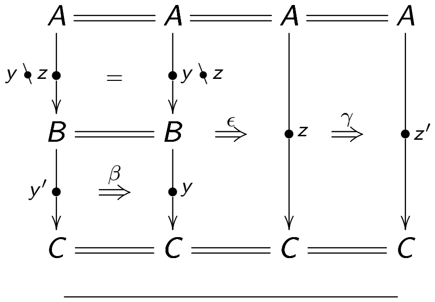
- $y \mathbin{\backslash\!\!\!\bullet} z$ is covariant in z and contravariant in y

$$y' \xrightarrow{\beta} y,\; z \xrightarrow{\gamma} z' \quad \rightsquigarrow \quad y \mathbin{\backslash\!\!\!\bullet} z \xrightarrow[\bullet]{\beta \mathbin{\backslash\!\!\!\bullet} \gamma} y' \mathbin{\backslash\!\!\!\bullet} z'$$

- We have evaluation $\epsilon : y \bullet (y \mathbin{\backslash\!\!\!\bullet} z) \longrightarrow y$



$$y \mathbin{\backslash\!\!\!\bullet} z \longrightarrow y' \mathbin{\backslash\!\!\!\bullet} z'$$

Respecting boundaries

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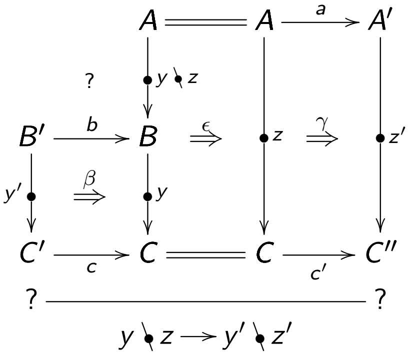
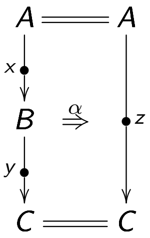
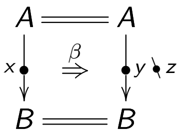


Globular universal

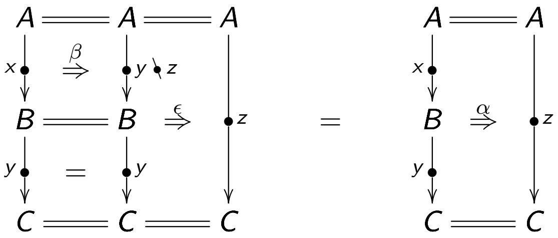
$\forall$



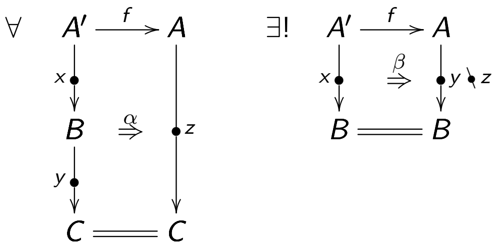
$\exists!$



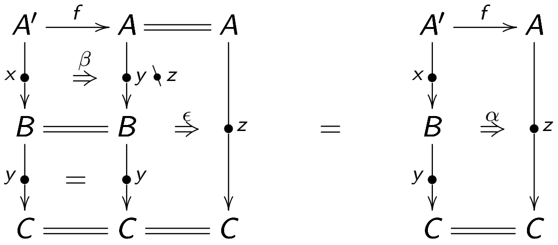
s.t.



More universal

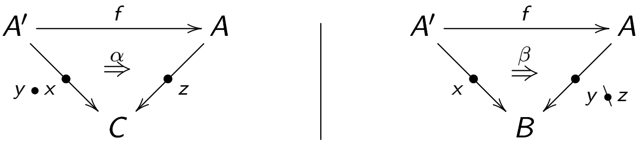


s.t.



Strong universality

Strong universal property:



Companions

- In a double category $\mathbb{A}$, a vertical arrow $v : A \longrightarrow\!\!\!\!\!\bullet\;\, B$ is a *companion* of a horizontal arrow $f : A \longrightarrow B$ if there are *binding cells* α and β such that

$$
\begin{array}{ccccc}
A & \xrightarrow{1_A} & A & \xrightarrow{f} & B \\
{\scriptstyle \mathrm{id}_A}\downarrow\!\bullet & \overset{\alpha}{\Longrightarrow} & \bullet\!\downarrow{\scriptstyle v} & \overset{\beta}{\Longrightarrow} & \bullet\!\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[f]{} & B & \xrightarrow[1_A]{} & B
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \mathrm{id}_A}\downarrow\!\bullet & \overset{\mathrm{id}_f}{\Longrightarrow} & \bullet\!\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[f]{} & B
\end{array}
\qquad \beta\alpha = \mathrm{id}_f
$$

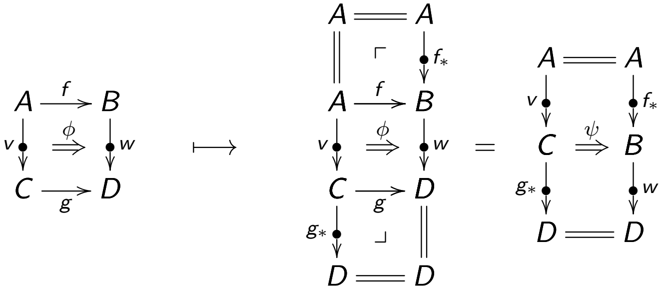
$$
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
{\scriptstyle \mathrm{id}_A}\downarrow\!\bullet & \overset{\alpha}{\Longrightarrow} & \bullet\!\downarrow{\scriptstyle v} \\
A & \xrightarrow[f]{} & B \\
{\scriptstyle v}\downarrow\!\bullet & \overset{\beta}{\Longrightarrow} & \bullet\!\downarrow{\scriptstyle \mathrm{id}_B} \\
B & \xrightarrow[1_A]{} & B
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
{\scriptstyle v}\downarrow\!\bullet & \overset{1_v}{\Longrightarrow} & \bullet\!\downarrow{\scriptstyle v} \\
B & \xrightarrow[1_B]{} & B
\end{array}
\qquad \beta \bullet \alpha = 1_v
$$

Properties

• Companions, when they exist, are unique up to globular isomorphism
We make a choice of companion f_* and, following Ronnie Brown, denote the binding cells by corner brackets

• We have $(1_A)_* \cong \mathrm{id}_A$ and $(gf)_* \cong g_* f_*$

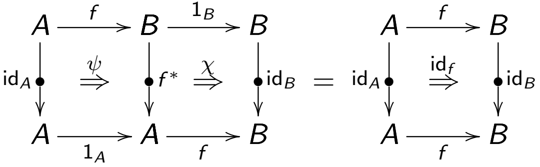
•



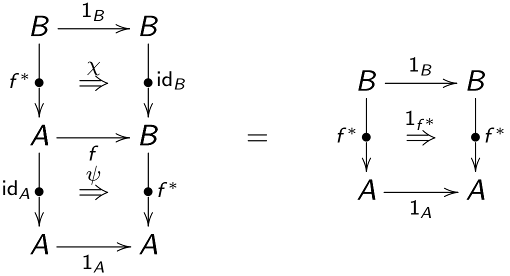
gives a bijection between ϕ's and ψ's

Conjoints

There is a dual notion of *conjoint* f^*



$\chi\psi = \mathrm{id}_f$



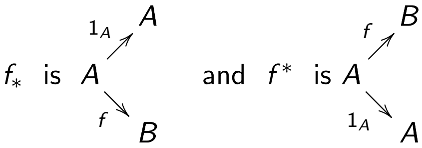
$\psi \bullet \chi = 1_{f^*}$

Examples

- In $\mathbb{R}\mathrm{ing}$, $f : R \longrightarrow S$
  f_* is S considered as an S-R bimodule
  f^* is S considered as an R-S bimodule

- In $\mathbb{C}\mathrm{at}$, $F : \mathbf{A} \longrightarrow \mathbf{B}$
  $F_* = \mathbf{B}(F-, =)$ and $F^* = \mathbf{B}(-, F=)$

- In $\mathbb{S}\mathrm{pan}(\mathbf{A})$, $f : A \longrightarrow B$



What strong means

• The strong universal property is equivalent to the globular one plus the stability property

$$y \mathbin{\backslash\!\!\!\bullet} (z \bullet f_*) \cong (y \mathbin{\backslash\!\!\!\bullet} z) \bullet f_*$$

• If every horizontal arrow has a conjoint, then the strong universal property is equivalent to the globular one

Left duals

- Suppose $\mathbb{A}$ left closed
- For $v : A \xrightarrow{\bullet} B$ we can define its *left dual* ${}^{\bullet}v = v \mathbin{\backslash\!\!\bullet} \mathrm{id}_B : B \xrightarrow{\bullet} A$

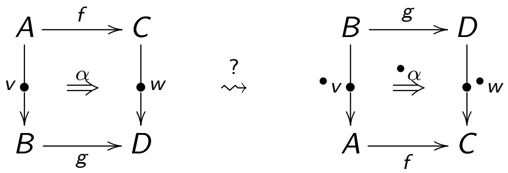
We have

$$ {}^{\bullet}\,\mathrm{id}_B \cong \mathrm{id}_B $$

$$ {}^{\bullet}v \bullet {}^{\bullet}w \longrightarrow {}^{\bullet}(w \bullet v) $$

So perhaps we get a lax normal

$$ \mathbb{A}^{co} \longrightarrow \mathbb{A} $$



A *retrocell*

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & C \\
{\scriptstyle v}\big\downarrow\!\bullet & \overset{\alpha}{\Longleftarrow} & \bullet\!\big\downarrow{\scriptstyle w} \\
B & \xrightarrow[\;g\;]{} & D
\end{array}
\quad \text{is a cell} \quad
\begin{array}{ccc}
A & = & A \\
{\scriptstyle f_*}\big\downarrow\!\bullet & & \bullet\!\big\downarrow{\scriptstyle v} \\
C & \overset{\alpha}{\Longrightarrow} & B \\
{\scriptstyle w}\big\downarrow\!\bullet & & \bullet\!\big\downarrow{\scriptstyle g_*} \\
D & = & D
\end{array}
\quad \text{in} \quad \mathbb{A}
$$

Quintets

- **Example:** In $\mathbb{Q}(\mathcal{A})$, a cell is a quintet

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle h}\big\downarrow & \swarrow & \big\downarrow{\scriptstyle k} \\ C & \xrightarrow[\ g\ ]{} & D \end{array}$$

and a retrocell is a coquintet

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle h}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle k} \\ C & \xrightarrow[\ g\ ]{} & C \end{array}$$

Mates

Proposition

(1) If v and w as below have right adjoints v' and w' in $\mathcal{V}ert\mathbb{A}$, then retrocells α are in bijection with standard cells β:

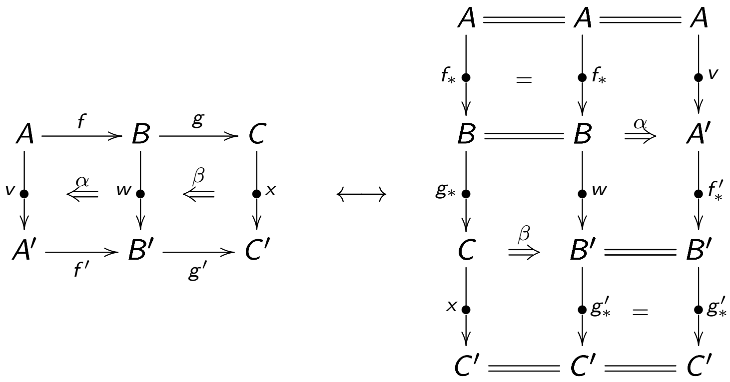
$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle v}\downarrow\!\bullet & \stackrel{\alpha}{\Leftarrow} & \bullet\!\downarrow{\scriptstyle w} \\
C & \xrightarrow[g]{} & D
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{ccc}
C & \xrightarrow{g} & D \\
{\scriptstyle v'}\downarrow\!\bullet & \stackrel{\beta}{\Rightarrow} & \bullet\!\downarrow{\scriptstyle w'} \\
A & \xrightarrow[f]{} & B
\end{array}
$$

(2) If f and g have right adjoints h and k in $\mathcal{H}or\mathbb{A}$, then retrocells α are in bijection with standard cells γ:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle v}\downarrow\!\bullet & \stackrel{\alpha}{\Leftarrow} & \bullet\!\downarrow{\scriptstyle w} \\
C & \xrightarrow[g]{} & D
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{ccc}
B & \xrightarrow{h} & A \\
{\scriptstyle w}\downarrow\!\bullet & \stackrel{\gamma}{\Rightarrow} & \bullet\!\downarrow{\scriptstyle v} \\
D & \xrightarrow[k]{} & C
\end{array}
$$

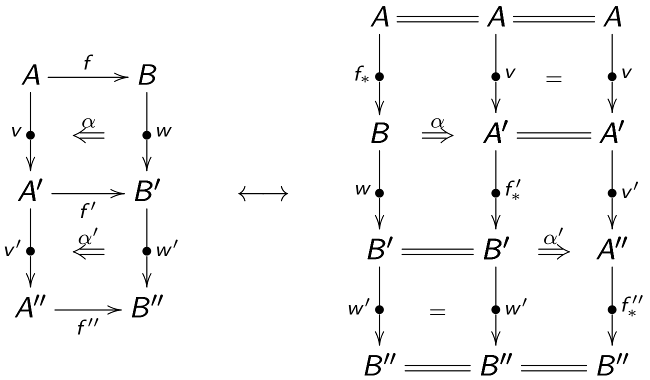
Composition

Retrocells can be composed horizontally



Composition

and vertically

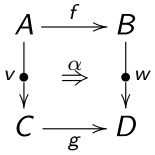


Theorem

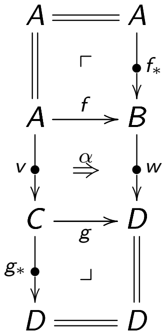
This gives a double category $\mathbb{A}^{ret}$. $\mathbb{A}^{ret}$ has companions and $(\mathbb{A}^{ret})^{ret} \cong \mathbb{A}$

Commuter cells

• In M. Grandis, R. Paré, Kan extensions in double categories, TAC 2008, we introduced *commutative cells* to express the universal property of comma double categories



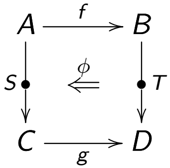
is a *commuter cell* if



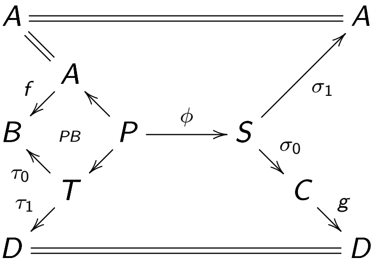
is horizontally invertible

• The inverse would be a retrocell

• In $\mathbb{S}\mathrm{pan}(\mathbf{A})$



is



• In $\mathbb{S}\mathrm{et} = \mathbb{S}\mathrm{pan}(\mathbf{Set})$

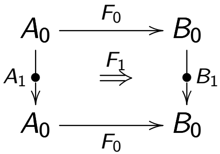
Denote an element of S by $s : a \longrightarrow\!\!\!\!\!\!\bullet\!\!\!\!\!\!\longrightarrow c \quad (\sigma_0 s = a,\ \sigma_1 s = c)$

Then

$$\phi : (a, fa \xrightarrow{t} d) \longmapsto (a \xrightarrow{\phi t} c_t), \quad g(c_t) = d$$

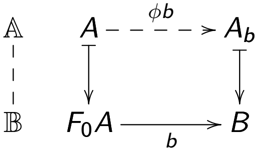
Category objects

- A category object in **A** is a vertical monad in $\mathbb{S}\mathrm{pan}(\mathbf{A})$
- An internal functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a cell



respecting composition and identities

- A retrocell ϕ is an object function F_0 together with a lifting operation



- If ϕ respects composition and identities, then this is exactly a *cofunctor* $\mathbb{B} \longrightarrow \mathbb{A}$ in the sense of Aguiar

Discrete opfibrations

• ϕ looks like the lifting property for opfibrations without the projection functor

• If F is also a functor and F_1 and ϕ are companions in a certain double category of cells and retrocells, then F is a discrete opfibration. In fact F is a discrete opfibration if and only if F_1 is a commuter cell

Lax functors

• If $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a double functor, we get $F^{ret} : \mathbb{A}^{ret} \longrightarrow \mathbb{B}^{ret}$

• If $F : \mathbb{A} \longrightarrow \mathbb{B}$ is just lax, it doesn't extend to $\mathbb{A}^{ret}$; it should properly respect companions

• If F is lax normal, then F preserves companions and also composites of the form $A \xrightarrow{f_*} B \xrightarrow{v} C$ (proarrows, drawn with a bullet on the arrow)

$$\phi(v, f_*) : F(v) \bullet F(f_*) \longrightarrow F(v \bullet f_*) \quad \text{iso}$$

[Dawson, Paré, Pronk, The Span Construction, TAC 2010]

Paranormal

Definition

F is *paranormal* if it is normal and also preserves compositions of the form $g_* \bullet v$

$$\phi(g_*, v) : F(g_*) \bullet F(v) \longrightarrow F(g_* \bullet v) \quad \text{iso}$$

Theorem

If F is lax paranormal, then it extends to $F^{ret} : \mathbb{A}^{ret} \longrightarrow \mathbb{B}^{ret}$, oplax paranormal

Back to duals

Theorem

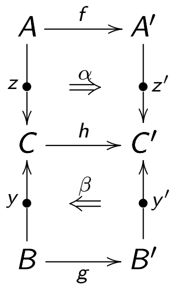
If $\mathbb{A}$ has companions and left duals, the left dual is a lax normal double functor which is the identity on objects and horizontal arrows

$$^{\bullet}(\ ) : \mathbb{A}^{ret\ co} \longrightarrow \mathbb{A}$$

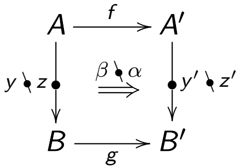
- The proof uses strong universality

Functoriality of $\setminus\!\!\!\bullet$

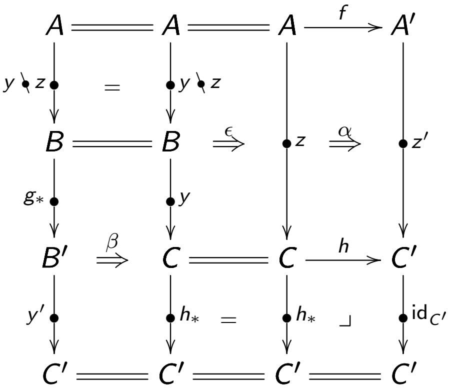
A cell α and a retrocell β as in



produce a cell



given by



Theorem

$\backslash\!\!\bullet$ is functorial in both variables, covariant in the top variable and retrovariant in the bottom one