Colimits and Profunctors

Robert Paré

Dalhousie University

pare@mathstat.dal.ca

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The Problem

For two diagrams

$$\begin{array}{ccc} \mathbf{I} & & \mathbf{J} \\ & {}_{\Gamma}\searrow \quad \swarrow_{\Phi} & \\ & \mathbf{A} & \end{array}$$

what is the most general kind of morphism $\Gamma \longrightarrow \Phi$ which will produce a morphism

$$\varinjlim \Gamma \longrightarrow \varinjlim \Phi \quad ?$$

Answer: A morphism $\varinjlim \Gamma \longrightarrow \varinjlim \Phi$.
Want something more syntactic? E.g.

$$\begin{array}{ccc} \mathbf{I} & \xrightarrow{F} & \mathbf{J} \\ & {}_{\Gamma}\searrow \quad \swarrow_{\Phi} & \\ & \mathbf{A} & \end{array}$$

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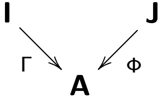

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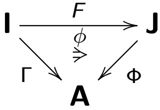
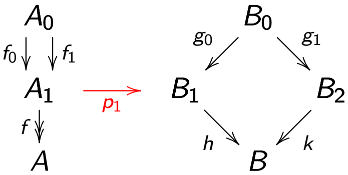


Example



$$\begin{aligned} p_1 f_0 &= g_0 p_2 \\ p_1 f_1 &= g_0 p_3 \\ g_1 p_2 &= g_1 p_3 \end{aligned}$$

Thus we get

$$\begin{aligned} h p_1 f_0 &= h g_0 p_2 \\ &= k g_1 p_2 \\ &= k g_1 p_3 \\ &= h g_0 p_3 \end{aligned}$$

Problems

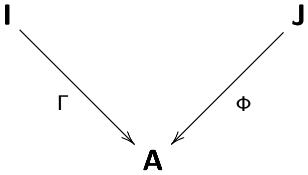
- Different schemes (number of arrows, placement, equations) may give the same p
- It might be difficult to compose such schemes

On the positive side

- It is equational so for any functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ for which the coequalizer and pushout below exist we get an induced morphism q

The Problem (Refined)

For two diagrams



what is the most general kind of morphism $\Gamma \longrightarrow \Phi$ which will produce a morphism

$$\varinjlim F\Gamma \longrightarrow \varinjlim F\Phi$$

for every $F : \mathbf{A} \longrightarrow \mathbf{B}$ for which the $\varinjlim$'s exist?

- Should be natural in F (in a way to be specified)

Take F to be the Yoneda embedding $Y : \mathbf{A} \longrightarrow \mathbf{Set}^{\mathbf{A}^{op}}$. Then we have the bijections

$$
\begin{array}{c}
\varinjlim Y\Gamma \longrightarrow \varinjlim Y\Phi \\
\hline
\varinjlim_I \mathbf{A}(-,\Gamma I) \longrightarrow \varinjlim_J \mathbf{A}(-,\Phi J) \\
\hline
\langle \mathbf{A}(-,\Gamma I) \longrightarrow \varinjlim_J \mathbf{A}(-,\Phi J)\rangle_I \\
\hline
\langle x_I \in \varinjlim_J \mathbf{A}(\Gamma I,\Phi J)\rangle_I
\end{array}
$$

An element of $\varinjlim_J \mathbf{A}(\Gamma I,\Phi J)$ is an equivalence class of morphisms

$$[\Gamma I \xrightarrow{a} \Phi J]_J$$

where $a \sim a'$ iff there is a path of diagrams

$$
\begin{array}{ccc}
\Gamma I & \xrightarrow{a_k} & \Phi J_k \\
\Vert & & \downarrow{\scriptstyle \Phi j_k} \\
\Gamma I & \xrightarrow[a_{k+1}]{} & \Phi J_{k+1}
\end{array}
$$

Thus, to give a compatible family

$$\langle x_I \in \varinjlim_J \mathbf{A}(\Gamma I, \Phi J)\rangle_I$$

we must give:

- For each I, a J_I and a morphism $\Gamma I \xrightarrow{a_I} \Phi J_I$
- For each $I' \xrightarrow{i} I$ a path of J's and a's joining

  $$\Gamma I' \xrightarrow{\Gamma i} \Gamma I \xrightarrow{a_I} \Phi J_I$$

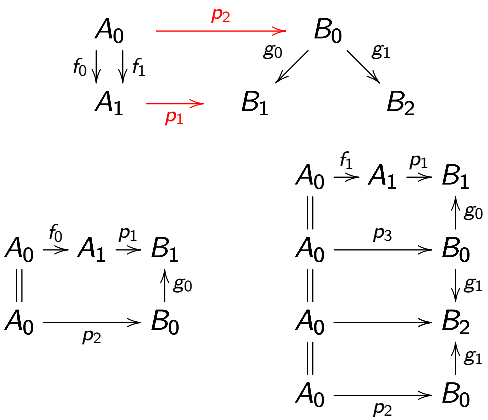
  with

  $$\Gamma I' \xrightarrow[a_{I'}]{} \Phi J_{I'}$$

Two such choices $\langle a_I : \Gamma I \longrightarrow \Phi J_I\rangle$ are $\langle a'_I : \Gamma I \longrightarrow \Phi J'_I\rangle$ are equivalent, if for each J there is a path joining $\Gamma I \xrightarrow{a_I} \Phi J_I$ with $\Gamma I \xrightarrow{a'_I} \Phi J'_I$.

Theorem

The above data induces for every F a morphism $\varinjlim F\Gamma \longrightarrow \varinjlim F\Phi$. Two such sets of data induce the same morphism for all F iff they are equivalent as described above.



Canonization

Recalling our idea of

$$
\begin{array}{ccc}
\mathbf{I} & \xrightarrow{\quad F \quad} & \mathbf{J} \\
 & \overset{\phi}{\Rightarrow} & \\
\Gamma \searrow & & \swarrow \Phi \\
 & \mathbf{A} &
\end{array}
$$

we get for every I, a $J_I = FI$, and a morphism $a_I = \phi I : \Gamma I \longrightarrow \Phi FI$. Naturality of ϕ gives a one-step path

$$
\begin{array}{ccccc}
\Gamma I' & \xrightarrow{\Gamma i} & \Gamma I & \xrightarrow{\phi I} & \Phi FI \\
\Vert & & & & \downarrow{\scriptstyle \Phi Fi} \\
\Gamma I' & & \xrightarrow[\phi I']{\qquad\qquad} & & \Phi FI'
\end{array}
$$

In the general case $I \rightsquigarrow J_I$ is not a functor. There can be several J_I, and for $i : I \longrightarrow I'$ we don't get a morphism $J_I \longrightarrow J_{I'}$ but only a path. This is a kind f "relation between categories". They are called profunctors (distributors, bimodules, modules, relators).

Profunctors

- A *profunctor* $P : \mathbf{A} \xrightarrow{\bullet} \mathbf{B}$ is a functor $P : \mathbf{A}^{op} \times \mathbf{B} \longrightarrow \mathbf{Set}$
- Every functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ gives two profunctors

$$F_* : \mathbf{A} \xrightarrow{\bullet} \mathbf{B}, \quad F_* = \mathbf{B}(F-, -) : \mathbf{A}^{op} \times \mathbf{B} \longrightarrow \mathbf{Set}$$

$$F^* : \mathbf{B} \xrightarrow{\bullet} \mathbf{A}, \quad F^* = \mathbf{B}(-, F-) : \mathbf{B}^{op} \times \mathbf{A} \longrightarrow \mathbf{Set}$$

$$F_* \dashv F^*$$

- Composition $\mathbf{A} \overset{P}{\xrightarrow{\bullet}} \mathbf{B} \overset{Q}{\xrightarrow{\bullet}} \mathbf{C}$

$$Q \otimes P(A, C) = \int^{B} Q(B, C) \times P(A, B)$$

$$= \{[A \underset{P}{\overset{x}{\xrightarrow{\bullet}}} B \underset{Q}{\overset{y}{\xrightarrow{\bullet}}} C]_B\} = \{y \otimes_B x\}$$

- $A \overset{x}{\xrightarrow{\bullet}} B \overset{y}{\xrightarrow{\bullet}} C \sim A \overset{x'}{\xrightarrow{\bullet}} B' \overset{y'}{\xrightarrow{\bullet}} C$ if there is

$$\begin{array}{ccccc} A & \overset{x}{\xrightarrow{\bullet}} & B & \overset{y}{\xrightarrow{\bullet}} & C \\ \| & & \downarrow b & & \| \\ A & \underset{x'}{\xrightarrow{\bullet}} & B' & \underset{y'}{\xrightarrow{\bullet}} & C \end{array} \qquad \begin{aligned} y \otimes x &= y' b \otimes x \\ &= y' \otimes bx \\ &= y' \otimes x' \end{aligned}$$

Given functors

$$\mathbf{I} \xrightarrow{\Gamma} \mathbf{A} \xleftarrow{\Phi} \mathbf{J}$$

we get a profunctor $\Phi^* \otimes \Gamma_* : \mathbf{I} \longrightarrow\!\!\!\!\!\!\bullet\;\; \mathbf{J}$

$$\Phi^* \otimes \Gamma_*(I, J) = \mathbf{A}(\Gamma I, \Phi J).$$

Proposition

A compatible family $\langle x_I \in \varinjlim_J \mathbf{A}(\Gamma I, \Phi J) \rangle_J$ determines a profunctor $P \subseteq \Phi^* \otimes \Gamma_*$ with the property that for every F and every $a \in P(I, J)$ we have

$$\begin{array}{ccc} F\Gamma I & \xrightarrow{Fa} & F\Phi J \\ {\scriptstyle inj_I}\downarrow & & \downarrow{\scriptstyle inj_J} \\ \varinjlim F\Gamma & \longrightarrow & \varinjlim F\Phi \end{array}$$

for the morphism induced by $\langle x_i \rangle$.

Proof.

$P(I, J) = \{a : \Gamma I \longrightarrow \Phi J | [a] = [x_I]\}.$

□

Total Profunctors

Definition

$P : \mathbf{A} \longrightarrow\!\!\!\!\!\bullet\,\, \mathbf{B}$ is *total* if for every A,

$$\varinjlim_B P(A, B) \cong 1.$$

Let $T : \mathbf{A} \longrightarrow \mathbf{1}$ be the unique functor. Then P is total iff $T_* \otimes P \xrightarrow{\cong} T_*$.

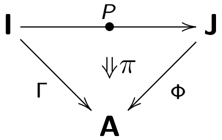
Proposition

(1) Total profunctors are closed under composition.
(2) For any functor $F : \mathbf{A} \longrightarrow \mathbf{B}$, F_* is total. (In particular Id_A is total.)
(3) If P and $P \otimes Q$ are total then Q is total.
(4) Total profunctors are closed under connected colimits and quotients.
(5) F^* is total iff F is final.
(6) For $\mathbf{I} \xleftarrow{\Sigma} \mathbf{K} \xrightarrow{\Theta} \mathbf{J}$, $\Theta_* \otimes \Sigma^*$ is total iff Σ is final.

Profunctors over **A**

Definition

For $\Gamma : \mathbf{I} \longrightarrow \mathbf{A}$ and $\Phi : \mathbf{J} \longrightarrow \mathbf{A}$, a profunctor from Γ to Φ (or a profunctor from $\mathbf{I}$ to $\mathbf{J}$ over $\mathbf{A}$) is



where P is a profunctor $\mathbf{I} \longrightarrow\!\!\bullet\!\!\longrightarrow \mathbf{J}$ and
$\pi : P \longrightarrow \mathbf{A}(\Gamma-, \Phi-) = \Phi^* \otimes \Gamma_*$ is a natural transformation.

Profunctors over **A** compose in the “obvious” way:

$$(Q, \psi) \otimes (P, \pi) = (Q \otimes P, \psi \otimes \pi)$$

$$\psi \otimes \pi(y \otimes x) = (\psi y)(\pi x).$$

Theorem

Let

$$\begin{array}{ccc} \mathbf{I} & \overset{P}{\longrightarrow\!\!\!\!\!\!\!\bullet\;\;} & \mathbf{J} \\ & \Downarrow \pi & \\ \Gamma \searrow & & \swarrow \Phi \\ & \mathbf{A} & \end{array}$$

be a profunctor over $\mathbf{A}$ *with* P *total. Then for every* $F : \mathbf{A} \longrightarrow \mathbf{B}$ *for which* $\varinjlim F\Gamma$ *and* $\varinjlim F\Phi$ *exist, there is a unique morphism* $\varinjlim F\pi : \varinjlim F\Gamma \longrightarrow \varinjlim F\Phi$ *such that for every* $x \in P(I, J)$ *we have*

$$\begin{array}{ccc} F\Gamma I & \xrightarrow{F\pi(x)} & F\Gamma J \\ {\scriptstyle inj_I}\downarrow & & \downarrow{\scriptstyle inj_J} \\ \varinjlim F\Gamma & \xrightarrow[\varinjlim F\phi]{} & \varinjlim F\Phi \end{array}$$

If $(Q, \psi) : \Phi \longrightarrow \Psi$ *is another total profunctor over* $\mathbf{A}$, *we have*

$$\varinjlim F(\psi \otimes \pi) = (\varinjlim F\psi)(\varinjlim F\pi).$$

Saturation

Definition

$P \rightarrowtail Q : \mathbf{I} \longrightarrow\!\!\!\!\!\bullet\;\, \mathbf{J}$ is *saturated* if $x \in Q(I, J)$ and for some j $j : J \longrightarrow J'$, $jx \in P(I, J')$ implies $x \in P(I, J)$.

- P is saturated in Q iff for every I, $P(I, -) \rightarrowtail Q(I, -)$ is complemented in $\mathbf{Set}^{\mathbf{J}}$.
- Every $P \rightarrowtail Q$ has a saturation $\bar{P} \rightarrowtail Q$.

Theorem

Let (P, π) and (P', π') be two total profunctors $\Gamma \longrightarrow\!\!\!\!\!\bullet\;\, \Phi$. Then they induce the same family $\varinjlim F\Gamma \longrightarrow \varinjlim F\Phi$ iff the images of $\pi : P \longrightarrow \Phi^* \otimes \Gamma_*$ and $\pi' : P' \longrightarrow \Phi^* \otimes \Gamma_*$ have the same saturation.

Naturality

Definition

A family of morphisms $b_F : \varinjlim F\Gamma \longrightarrow \varinjlim F\Phi$ is natural if for every G we have

$$\begin{array}{ccc}
\varinjlim GF\Gamma & \xrightarrow{\;b_{GF}\;} & \varinjlim GF\Phi \\
\downarrow & & \downarrow \\
G \varinjlim F\Gamma & \xrightarrow[\;Gb_F\;]{} & G \varinjlim F\Phi
\end{array}$$

Theorem

A total profunctor over $\mathbf{A}$ *induces a natural family as above. Every natural family comes from a total saturated profunctor* $\subseteq \Phi^* \otimes \Gamma_*$. *In fact there is a bijection between natural families and saturated total* $\subseteq \Phi^* \otimes \Gamma_*$.

Cohesive Families

As remarked by Bénabou already in the 70's, a category over $\mathbf{I}$

$$\begin{array}{c}\mathbf{K}\\ \Lambda\big\downarrow\\ \mathbf{I}\end{array}$$

corresponds to a lax normal functor $\mathbf{I} \longrightarrow \mathbf{Prof}$ where an object I is sent to $\mathbf{K}_I$, the fibre over I and a morphism $i : I \longrightarrow I'$ to the profunctor $P_i : \mathbf{K}_I \longrightarrow\!\!\!\!\bullet\!\!\!\!\longrightarrow \mathbf{K}_{I'}$ given by formula

$$P_i(K, K') = \{K \xrightarrow{k} K' | \Lambda k = i\}$$

Definition

$\Lambda : \mathbf{K} \longrightarrow \mathbf{I}$ is a *cohesive* family of categories if each P_i is total.

In elementary terms, for every K in $\mathbf{K}$ and every morphism $i : \Lambda K \longrightarrow I'$, there exists a morphism $k : K \xrightarrow{k} K'$ such that $i = \Lambda k$ and any two such liftings are connected by a path over i.

$$\begin{array}{ccc} K & \overset{k}{\dashrightarrow} & K' \\ & & \\ \Lambda K & \underset{i}{\longrightarrow} & I' \end{array}$$

Proposition

(1) Opfibrations are cohesive families
(2) Cohesive families are stable under pullback
(3) Cohesive families are closed under composition

Definition

A *cohesive* family of diagrams in $\mathbf{A}$ is a span

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{\Gamma} & \mathbf{A} \\ {\scriptstyle\Lambda}\downarrow & & \\ \mathbf{I} & & \end{array}$$

with Λ cohesive.

Let $\Gamma_I = \Gamma|_{\mathbf{K}_I}$.

Theorem

$\varinjlim \Gamma_I$ extends to a unique functor $\varinjlim \Gamma_{(\ )} : \mathbf{I} \longrightarrow \mathbf{A}$ such that for all $k : K \longrightarrow K'$ over $i : I \longrightarrow I'$

$$\begin{array}{ccc} \Gamma K & \xrightarrow{\Gamma k} & \Gamma K' \\ {\scriptstyle inj_K}\downarrow & & \downarrow{\scriptstyle inj_{K'}} \\ \varinjlim \Gamma_I & \xrightarrow[\varinjlim \Gamma_i]{} & \varinjlim \Gamma_{I'} \end{array}$$

Kan Extensions

$\varinjlim \Gamma_{(\ )} : \mathbf{I} \longrightarrow \mathbf{A}$ is the left Kan extension and cohesiveness says it is fibrewise. So perhaps a more functorial version of the theorem is:

Theorem

$\Lambda : \mathbf{K} \longrightarrow \mathbf{I}$ *is cohesive iff for every pullback diagram*

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{F} & \mathbf{K} \\ {\scriptstyle \Sigma}\big\downarrow & & \big\downarrow{\scriptstyle \Lambda} \\ \mathbf{J} & \xrightarrow[F]{} & \mathbf{I} \end{array}$$

and every cocomplete $\mathbf{A}$*, the canonical morphism*

$$\begin{array}{ccc} \mathbf{A}^{\mathbf{L}} & \xleftarrow{F^*} & \mathbf{A}^{\mathbf{K}} \\ {\scriptstyle Lan_\Sigma}\big\downarrow & \Downarrow \lambda & \big\downarrow{\scriptstyle Lan_\Lambda} \\ \mathbf{A}^{\mathbf{J}} & \xleftarrow[F^*]{} & \mathbf{A}^{\mathbf{I}} \end{array}$$

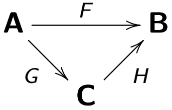
is an isomorphism.

If we take $\mathbb{J} = \mathbf{1}$, $F \rightsquigarrow I \in \mathbf{I}$, we get $(Lan_\Lambda \Gamma) I \cong \varinjlim \Gamma_I$.

The Comprehensive Factorization

Set		**Cat**
Relations	$\leftrightarrow$	Profunctors
Everywhere Defined	$\leftrightarrow$	Total
Single Valued	$\leftrightarrow$	?
Functions	$\leftrightarrow$	Functors

Recall the *comprehensive factorization* on **Cat** (Street & Walters '79). Every functor F factors as



with G final and H a discrete fibration. So the final functors are "epi-like" and the discrete fibrations are "mono-like".

Discrete Valued Profunctors

Definition

P is *discrete valued* if it is of the form $P \cong G_* \otimes F^*$ for some $\mathbf{A} \xleftarrow{F} \mathbf{C} \xrightarrow{G} \mathbf{B}$ with F a discrete fibration.

Theorem

P is discrete valued iff for every A, $P(A, -)$ is multirepresentable (Diers), i.e. a sum of representables. In fact

$$P(A, -) \cong \sum_{FC=A} \mathbf{B}(GC, -).$$

Corollary

The factorization $P \cong G_* \otimes F^*$ is unique up to isomorphism.

Theorem

P is representable iff it is total and discrete valued.

Mealy Morphisms

A small category is a monad in **Span**, which is a lax functor $\mathbf{1} \longrightarrow \mathbf{Span}$.

A lax transformation

$$\mathbf{1} \underset{\mathbf{B}}{\overset{\mathbf{A}}{\rightrightarrows}} \mathbf{Span} \qquad \Downarrow \sigma$$

corresponds to a Mealy morphism (machine)

- For every A, B we have a set $S(A, B)$ of *states*
- Arrows of **A** are the input alphabet
- Arrows of **B** are the output alphabet
- Action

$$A' \xrightarrow{a} A \xrightarrow{s} B \quad \overset{\sigma}{\longmapsto} \quad A' \xrightarrow{s^a} B' \xrightarrow{\sigma(s,a)} B$$

Mealy Profunctors

A Mealy morphism determines a profunctor $P : \mathbf{A} \longrightarrow\!\!\!\!\!\!\bullet\!\!\!\!\rightarrow \mathbf{B}$

$$P(A, B) = \sum_{s : A \longrightarrow\!\!\!\bullet\!\!\!\rightarrow B'} \mathbf{B}(B', B)$$

Theorem

P is a Mealy profunctor iff P is discrete valued.