Seeing double

(https://www.mscs.dal.ca/∼pare/FMCS2.pdf)

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Before we start

Double functors

$$\mathbb{S}\mathrm{lice}(\mathbf{A}) \longrightarrow \mathbb{S}\mathrm{lice}(\mathbf{B})$$

are in bijection with natural transformations

$$\mathbf{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathbf{B} \qquad t : F \Rightarrow G$$

The associated double functor is given (on the objects) by

$$\begin{array}{c} A \\ {\scriptstyle f}\downarrow \\ A' \end{array} \quad \longmapsto \quad \begin{array}{c} FA \\ \downarrow{\scriptstyle Ff} \\ FA' \\ \downarrow{\scriptstyle tA'} \\ GA' \end{array}$$

Words of wisdom

If you want something done right
you have to do it yourself.
And, you have to do it right.

Micah McCurdy

The plan

- The theory of restriction categories is a nice, simply axiomatized theory of partial morphisms
- It is well motivated with many examples and has lots of nice results
- But it is somewhat tangential to mainstream category theory
- The plan is to bring it back into the fold by taking a double category perspective
- Every restriction category has a canonically associated double category
- What can double categories tell us about restriction categories?
- What can restriction categories tell us about double categories?
- References

  - R. Cockett, S. Lack, Restriction Categories I: Categories of Partial Maps, Theoretical Computer Science 270 (2002) 223-259
  - R. Cockett, Introduction to Restriction Categories, Estonia Slides (2010)
  - D. DeWolf, Restriction Category Perspectives of Partial Computation and Geometry, Thesis, Dalhousie University, 2017

Restriction categories

Definition

A *restriction category* is a category equipped with a *restriction operator*

$$A \xrightarrow{f} B \rightsquigarrow A \xrightarrow{\bar{f}} A$$

satisfying

R1. $f\bar{f} = f$

R2. $\bar{f}\bar{g} = \bar{g}\bar{f}$

R3. $\overline{g\bar{f}} = \bar{g}\bar{f}$

R4. $\bar{g}f = f\overline{gf}$

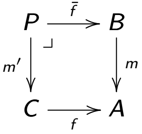
Example

Let **A** be a category and **M** a subcategory such that

(1) $m \in \mathbf{M} \Rightarrow m$ monic

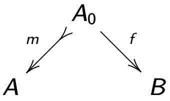
(2) **M** contains all isomorphisms

(3) **M** stable under pullback: for every $m \in \mathbf{M}$ and $f \in \mathbf{A}$ as below, the pullback of m along f exists and is in **M**



$$m \in M \Rightarrow m' \in M$$

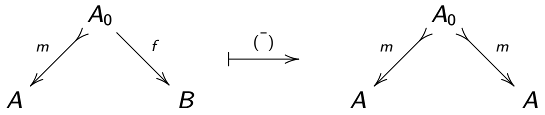
$\mathrm{Par}_{\mathbf{M}}\mathbf{A}$ has the same objects as **A** but the morphisms are isomorphism classes of spans



with $m \in M$

Composition is by pullback

The restriction operator is $\overline{(m, f)} = (m, m)$



The double category

Let **A** be a restriction category

Definition

$f : A \longrightarrow B$ is *total* if $\bar{f} = 1_A$

Proposition

The total morphisms form a subcategory of **A**

The double category $\mathbb{D}c(\mathbf{A})$ *associated to a restriction category* **A** *has*

- *The same objects as* **A**
- *Total maps as horizontal morphisms*
- *All maps as vertical morphisms*
- *There is a unique cell* $\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\big\downarrow & \Rightarrow & \big\downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array}$ *if and only if* $gv = wf\bar{v}$

Theorem

$\mathbb{D}c(\mathbf{A})$ *is a double category*

Remark

C & L define an order relation between $f, g : A \longrightarrow B$, $f \leq g \Leftrightarrow f = g\bar{f}$
Makes $\mathbf{A}$ *into a 2-category. They say "seems to be less useful than one might expect"*

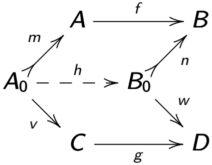
There is a cell

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle v}\,\big\downarrow\!\bullet & \Rightarrow & \bullet\!\big\downarrow\,{\scriptstyle w} \\ C & \xrightarrow[\ g\ ]{} & D \end{array}$$

if and only if $gv \leq wf$. *So our* $\mathbb{D}c(\mathbf{A})$ *is not far from that 2-category. Perhaps it will turn out to be more useful than they might expect!*

Example

In $\mathbb{D}c\mathsf{Par}_{\mathbf{M}}(\mathbf{A})$ there is a cell if and only if there exists a (necessarily unique) morphism h



Companions

Proposition

In $\mathbb{D}c(\mathbf{A})$ every horizontal arrow has a companion, $f_* = f$

Proof.

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle f}\big\downarrow\!\bullet & \Rightarrow & \bullet\!\big\downarrow{\scriptstyle 1} \\
B & \xrightarrow[1]{} & B
\end{array}
\qquad 1 \cdot f = 1 \cdot f \cdot \bar{f}
$$

$$
\begin{array}{ccc}
A & \xrightarrow{1} & A \\
{\scriptstyle 1}\big\downarrow\!\bullet & \Rightarrow & \bullet\!\big\downarrow{\scriptstyle f} \\
A & \xrightarrow[f]{} & B
\end{array}
\qquad f \cdot 1 = f \cdot 1 \cdot \bar{1}
$$

□

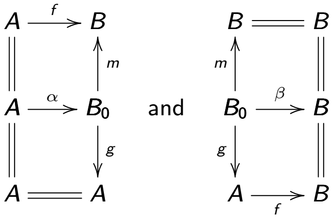
Conjoints

Proposition

In $\mathbb{D}cPar_{\mathbf{M}}(\mathbf{A})$, f has a conjoint if and only if $f \in M$

Proof.

Assume f has conjoint (m, g), then there are α, β



So $m\alpha g = fg = \beta = m$ which implies $\alpha g = 1$
Thus α is an isomorphism and $f = m\alpha \in \mathbf{M}$

□

- If we suspect that $\mathbb{A}$ is of the form $\mathbb{D}c\mathrm{Par}_{\mathbf{M}}(\mathbf{A})$ we can recover $\mathbf{M}$ as those horizontal arrows having a conjoint
- Is the requirement of stability under pullback of conjoints a good double category notion?
- In $\mathbb{D}c(\mathbf{A})$, a horizontal arrow $f : A \longrightarrow B$ always has a companion f_*, and if it also has a conjoint f^* then $f_* \dashv f^*$ so

$$\begin{array}{c} A \\ {\scriptstyle f_* \bullet f^*} \big\downarrow \bullet \\ A \end{array}$$

is a comonad, i.e. an idempotent $\leq \mathrm{id}_A$

Proposition

In $\mathbb{D}c(\mathbf{A})$, $f_* \bullet f^* = \bar{f}^*$

Tabulators

Proposition

$\mathbb{D}cPar_{\mathbf{M}}(\mathbf{A})$ *has tabulators and they are effective*

Proof.

Given $(m, v) : A \longrightarrow\!\!\!\!\!\bullet\!\!\!\!> B$, the tabulator is

$$\begin{array}{ccc} A_0 & \xrightarrow{m} & A \\ \| & & \uparrow m \\ A_0 & = & A_0 \\ \| & & \downarrow v \\ A_0 & \xrightarrow[v]{} & B \end{array}$$

□

Conjecture: In a general $\mathbb{D}c(\mathbf{A})$, $v : A \longrightarrow\!\!\!\!\!\bullet\!\!\!\!> B$ has a tabulator if and only if $\bar{v}$ splits

Classification of vertical arrows

- The original definition of elementary topos was in terms of a partial map classifier

$$\frac{B \longrightarrow\!\!\!\!\!\bullet\;\; A}{B \longrightarrow \tilde{A}}$$

- In a topos, relations are classifiable

$$\frac{B \longrightarrow\!\!\!\!\!\bullet\;\; A}{B \longrightarrow \Omega^{A}}$$

- For profunctors

$$\frac{\mathbf{B} \longrightarrow\!\!\!\!\!\bullet\;\; \mathbf{A}}{\mathbf{B} \longrightarrow (\mathbf{Set}^{\mathbf{A}})^{op}}$$

  provided **A** is small
- How do we formalize this in a general double category?

Classification (Beta version)

- The desired bijection

$$\frac{B \xrightarrow{\;v\;} A}{B \xrightarrow{\;\widehat{v}\;} \tilde{A}}$$

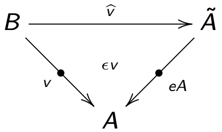
gives $eA : \tilde{A} \longrightarrow A$ and $hA : A \longrightarrow \tilde{A}$

- We express our definition in terms of eA

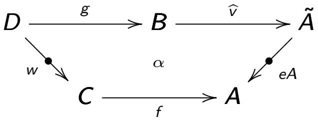
Definition

Let $\mathbb{A}$ be a double category and A an object of $\mathbb{A}$. We say that A is *classifying* if we are given an object $\tilde{A}$ and a vertical morphism $eA : \tilde{A} \longrightarrow A$ with the following universal properties:

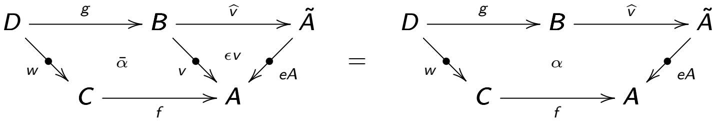
(1) For every vertical arrow $v : B \longrightarrow\!\!\!\bullet\!\!\!\rightarrow A$ there exist a horizontal arrow $\widehat{v} : B \longrightarrow \tilde{A}$ and a cell



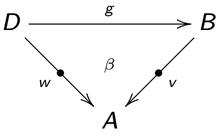
such that for every cell α



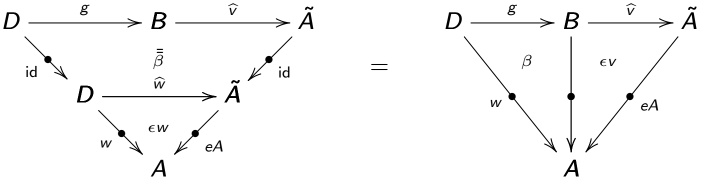
there exists a unique cell $\bar{\alpha}$ such that



(2) For every cell



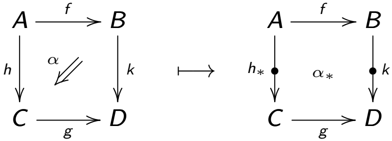
there exists a unique cell $\bar{\bar{\beta}}$ such that



Complete classification

- How do we understand this?
- Take a more global approach
  Assume $\mathbb{A}$ is companionable, i.e. every horizontal arrow f has a companion f_*
  Then we get a (pseudo) double functor

$$(\ )_* : \mathbb{Q}\,\mathcal{H}\mathrm{or}\mathbb{A} \longrightarrow \mathbb{A}$$



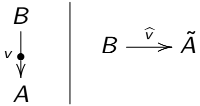
Exercise!

Definition

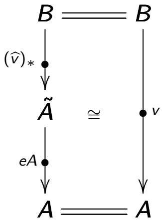
Say that $\mathbb{A}$ is *classifying* if $(\ )_*$ has a *down adjoint* $(\tilde{\ })$,
i.e. a right adjoint in the vertical direction

Bijections

The adjunction can be formalized in terms of bijections

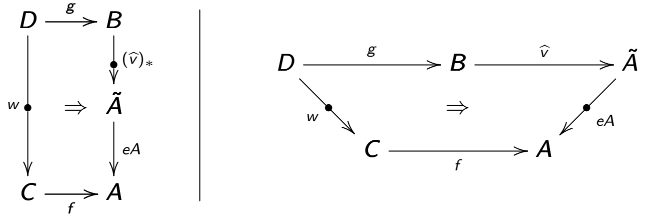


More precisely, for $v : B \longrightarrow\!\!\!\bullet\!\!\!\longrightarrow A$ there exists a $\widehat{v} : B \longrightarrow \tilde{A}$ and an isomorphism



This can be expressed without mention of $(\ )_*$ because we have a bijection

Bijections (cont.)



Yonedafication now yields the single-object definition

Kleisli

- Given a monad (T, η, μ) on **A** we get a double category $\mathbb{K}\mathrm{l}(T)$
  - Objects are those of **A**
  - Horizontal arrows are morphisms of **A**
  - Vertical arrows are Kleisli morphisms i.e.

$$\begin{array}{c} A \\ {\scriptstyle v}\big\downarrow\!\bullet \\ B \end{array} \quad \text{is} \quad A \xrightarrow{\widehat{v}} TB \quad \text{in} \quad \mathbf{A}$$

  - Cells

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle v}\big\downarrow\!\bullet & \Rightarrow & \bullet\!\big\downarrow{\scriptstyle v'} \\ B & \xrightarrow[g]{} & B' \end{array} \quad \text{a unique one if} \quad \begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle \widehat{v}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{v'}} \\ TB & \xrightarrow[Tg]{} & TB' \end{array} \quad \text{commutes}$$

Kleisli (cont.)

- $\mathbb{Kl}(T)$ is companionable
  For $f : B \longrightarrow A$,

$$\begin{array}{c} B \\ {\scriptstyle f_*}\big\downarrow\!\!\bullet \\ A \end{array} \quad \text{is given by} \quad \begin{array}{c} B \\ \big\downarrow {\scriptstyle f} \\ A \\ \big\downarrow {\scriptstyle \eta A} \\ TA \end{array} \quad \text{i.e.} \quad f_* = (\widehat{\eta A \cdot f})$$

- $\mathbb{Kl}(T)$ is classifiable

$$\begin{array}{c} B \\ {\scriptstyle v}\big\downarrow\!\!\bullet \\ A \end{array} \quad \Bigg| \quad B \xrightarrow{\widehat{v}} TA$$

  - $eA : TA \longrightarrow\!\!\!\!\bullet\; A$ is $\widehat{\mathrm{id}_{TA}}$
  - $hA : A \longrightarrow TA$ is ηA

- Double functors $\mathbb{K}\mathrm{l}(T) \longrightarrow \mathbb{K}\mathrm{l}(S)$ correspond to monad morphisms (F, ϕ)

$$\mathbf{A} \xrightarrow{F} \mathbf{B}$$

$$\phi : FT \longrightarrow SF$$

such that . . .

- Horizontal transformations correspond to the 2-cells in Street's 1972 JPAA paper, *Formal theory of monads*
- Vertical transformations correspond to the 2-cells in Lack & Street's 2002 paper, *Formal theory of monads II*

Restriction functors

- A *restriction functor* $F : \mathbf{A} \longrightarrow \mathbf{B}$ is a functor that preserves the restriction operator, $F(\bar{f}) = \overline{F(f)}$

Proposition

A restriction functor F gives a double functor $\mathbb{D}c(F) : \mathbb{D}c(\mathbf{A}) \longrightarrow \mathbb{D}c(\mathbf{B})$

Question: Is every double functor $F : \mathbb{D}c(\mathbf{A}) \longrightarrow \mathbb{D}c(\mathbf{B})$ of this form? F is determined by a unique functor $\mathbf{A} \longrightarrow \mathbf{B}$ which preserves the order and totality. Does this mean it preserves restriction? Probably not. Does $\mathbb{D}c$ at least reflect isos?

Theorem

A double functor $\mathbb{D}c Par_{\mathbf{M}}\mathbf{A} \longrightarrow \mathbb{D}c Par_{\mathbf{N}}\mathbf{B}$ comes from a unique functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ which restricts to $\mathbf{M} \longrightarrow \mathbf{N}$ and preserves pullbacks of $m \in M$ by arbitrary $f \in \mathbf{A}$. Thus it does come from a restriction functor

Transformations

Recall that a horizontal transformation $t : F \longrightarrow G$ between double functors $\mathbb{A} \longrightarrow \mathbb{B}$ consists of assignments:

(1) For every A in $\mathbb{A}$ a horizontal morphism $tA : FA \longrightarrow GA$

(2) For every vertical morphism $v : A \longrightarrow\!\!\bullet\!\!\longrightarrow \bar{A}$ a cell

$$\begin{array}{ccc} FA & \xrightarrow{tA} & GA \\ {\scriptstyle Fv}\,\big\downarrow\!\bullet & tv & \bullet\!\big\downarrow\,{\scriptstyle Gv} \\ G\bar{A} & \xrightarrow[t\bar{A}]{} & G\bar{A} \end{array}$$

satisfying

(3) Horizontal naturality (for horizontal arrows and cells)

(4) Vertical functoriality (for identities and composition)

Let $F, G : \mathbf{A} \longrightarrow \mathbf{B}$ be restriction functors. Then a horizontal transformation

$$t : \mathbb{D}c(F) \longrightarrow \mathbb{D}c(G)$$

(1) assigns to each A in $\mathbf{A}$ a total morphism

$$tA : FA \longrightarrow GA$$

(2) such that for every $f : A \longrightarrow\!\!\!\!\!\bullet\!\!\!\!\longrightarrow \bar{A}$ in $\mathbf{A}$ we have

$$\begin{array}{ccc} FA & \xrightarrow{tA} & GA \\ {\scriptstyle Ff}\downarrow\bullet & \leq & \bullet\downarrow{\scriptstyle Gf} \\ F\bar{A} & \xrightarrow[t\bar{A}]{} & G\bar{A} \end{array}$$

(3) and t is natural for horizontal arrows (i.e. for f total, we have equality in (2))

This is what C & L call a lax restriction transformation

Proposition

Let $\mathbf{M} \subseteq \mathbf{A}$ *and* $\mathbf{N} \subseteq \mathbf{B}$ *be stable systems of monics and* $F, G : \mathbf{A} \longrightarrow \mathbf{B}$ *functors that preserve the given monics and their pullbacks*

Then horizontal transformations $\mathbb{D}c(F) \longrightarrow \mathbb{D}c(G)$ *correspond to arbitrary natural transformations* $F \longrightarrow G$

Restriction transformations correspond to cartesian ones

There is a notion of commuter cell in double categories, and requiring the cells in (2) to be commuter cells makes them equalities

Vertical transformations

A vertical transformation $\phi : \mathbb{D}c(F) \longrightarrow \mathbb{D}c(G)$

(1) assigns to each object A of $\mathbf{A}$ an arbitrary morphism of B

$$tA : FA \longrightarrow\!\!\!\!\bullet\!\!\!\!\longrightarrow GA$$

(2) will be automatic

(3) is natural with respect to all morphisms

(4) is vacuous

Question: Is this any good?

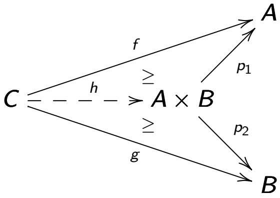
There are other notions of vertical transformation, e.g. the *modules* of

- "Yoneda Theory for Double Categories", Theory and Applications of Categories, Vol. 25, No. 17, 2011, pp. 436-489
  which generalize to double categories the modules of
- Cockett, J.R.B., Koslowski, J., Seely, R.A.G., Wood, R.J., Modules, Theory Appl. Categ. 11 (2003), No. 17, pp. 375-396

Project: Investigate the significance of lax (oplax) double functors and modules for restriction categories

Cartesian restriction categories

A restriction category **A** is *cartesian* if for every pair of objects A, B there is an object $A \times B$ and morphisms $p_1 : A \times B \longrightarrow A$, $p_2 : A \times B \longrightarrow B$ with the following universal property



For every f, g there exists a unique h such that

$$p_1 h = f\bar{g}$$

$$p_2 h = g\bar{f}$$

There is also a terminal object condition

Double products

Recall that $\mathbb{A}$ has binary products if

(1) for every A, B there is an object $A \times B$ and horizontal arrows $p_1 : A \times B \longrightarrow A$, $p_2 : A \times B \longrightarrow B$ which have the usual universal property with respect to horizontal arrows

(2) for every pair of vertical arrows $v : A \longrightarrow\!\!\!\bullet\; C$ and $w : B \longrightarrow\!\!\!\bullet\; D$ there is a vertical arrow $v \times w : A \times B \longrightarrow\!\!\!\bullet\; C \times D$ and cells

$$\begin{array}{ccc} A \times B & \xrightarrow{p_1} & A \\ {\scriptstyle v \times w}\big\downarrow\bullet & \pi_1 & \bullet\big\downarrow{\scriptstyle v} \\ C \times D & \xrightarrow[q_2]{} & C \end{array} \qquad \begin{array}{ccc} A \times B & \xrightarrow{p_2} & B \\ {\scriptstyle v \times w}\big\downarrow\bullet & \pi_2 & \bullet\big\downarrow{\scriptstyle w} \\ C \times D & \xrightarrow[q_2]{} & D \end{array}$$

with the usual universal property with respect to cells

Proposition

A *is a cartesian restriction category if and only if* $\mathbb{D}c(\mathbf{A})$ *has finite double products*

Proof*

(1) Suppose **A** is a cartesian restriction category. The universal property of product is the usual one when restricted to total maps
Given vertical arrows $v : A \longrightarrow\!\!\!\!\!\bullet\!\!\!\!> C$, $w : B \longrightarrow\!\!\!\!\!\bullet\!\!\!\!> D$ we get a unique $v \times w$

$$\begin{array}{ccccc} A & \xleftarrow{p_1} & A \times B & \xrightarrow{p_2} & B \\ {\scriptstyle v}\downarrow\bullet & \geq & \bullet\downarrow{\scriptstyle v\times w} & \leq & \bullet\downarrow{\scriptstyle w} \\ C & \xleftarrow[q_1]{} & C \times D & \xrightarrow[q_2]{} & D \end{array}$$

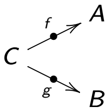
and

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ {\scriptstyle z}\downarrow\bullet & \leq & \bullet\downarrow{\scriptstyle v} \\ Y & \xrightarrow[g]{} & C \end{array} \quad \& \quad \begin{array}{ccc} X & \xrightarrow{h} & B \\ {\scriptstyle z}\downarrow\bullet & \leq & \bullet\downarrow{\scriptstyle w} \\ Y & \xrightarrow[k]{} & D \end{array} \quad \Leftrightarrow \quad \begin{array}{ccc} X & \xrightarrow{(f,h)} & A \times B \\ {\scriptstyle z}\downarrow\bullet & \leq & \bullet\downarrow{\scriptstyle v\times w} \\ Y & \xrightarrow[(g,k)]{} & C \times D \end{array}$$

so $\mathbb{D}c(\mathbf{A})$ has binary double products

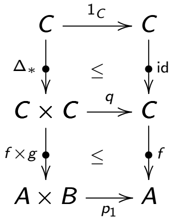
(2) Suppose $\mathbb{D}c(\mathbf{A})$ has finite double products

Given



we have $h = \; C \xrightarrow{\Delta_*} C \times C \xrightarrow{f \times g} A \times B$

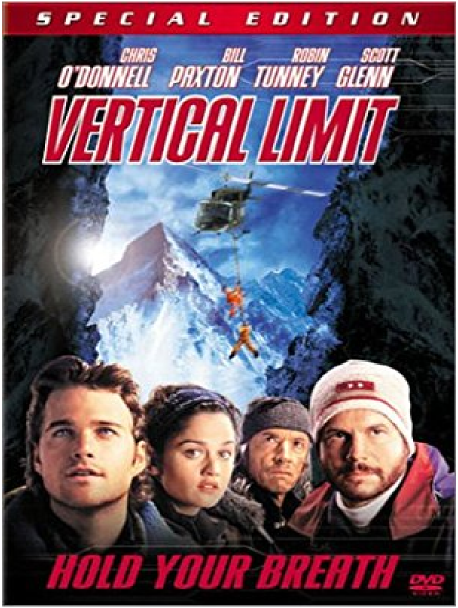
and cells



so $p_1 h = f\overline{(f \times g \bullet \Delta_*)} = f\bar{g}$

*Warning: Some details may not have been checked

Homework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