Hypercategories

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Introduction

- Preliminary report on Cat-indexed categories, a.k.a. hypercategories
- More questions than answers
- Four parts
 - 1. Indexed categories
 - 2. Double categories
 - 3. Families of categories
 - 4. Derivators

I. Indexed Categories

Genesis

- Grew out of the master plan of developing mathematics based on an arbitrary elementary topos **S** (rather than a fixed set theory, *ZFC* e.g.)
- Mathematics is best done using category theory
 - Small category is a category object in S
 - Large category? E.g. Gr(S)
- Idea is that a large category should come equipped with a notion of family of objects parametrized by an object of S

History

- 1973 Lawvere Perugia notes "Theory of categories over a base topos"
- 1974 Bénabou Lectures University of Montreal "Fibrations"
- 1974 Penon Comptes Rendus "Catégories localement internes"
- 1978 Paré Schumacher SLN "Indexed categories"

Idea of families indexed by some structured object

- ullet pprox 1850 Riemann Moduli spaces
- ullet pprox 1960 Mumford et al

Definition

- S a category of parameters has finite limits
- S-indexed category A
 - For every I in **S** a category $\mathcal{A}(I)$ of I-indexed families: $\langle A_i \rangle_{i \in I}$
 - For every $\alpha: J \longrightarrow I$ a functor $\alpha^*: \mathcal{A}(I) \longrightarrow \mathcal{A}(J)$, the reindexing: $\alpha^* \langle A_i \rangle_{i \in J} = \langle A_{\alpha j} \rangle_{j \in J}$
 - Natural isomorphisms

$$\phi_I: 1_{\mathcal{A}(I)} \xrightarrow{\cong} 1_I^*$$

$$\phi_{\alpha,\beta}: \beta^* \alpha^* \xrightarrow{\cong} (\alpha \beta)^*$$

- Satisfying coherence conditions: two unit triangles and one associativity pentagon
- $\mathcal{A}(\): \mathbf{I}^{op} \longrightarrow \underline{CAT}$ is a pseudo-functor
- If ϕ_I , $\phi_{\alpha,\beta}$ are identities, we say \mathcal{A} is rigid

Examples

• **S** = **Set**, **A** arbitrary category

$$\mathcal{A}(I) = \mathbf{A}^I = \prod_I \mathbf{A}$$

- Object is an actual *I*-family $\langle A_i \rangle_{i \in I}$ of **A** objects
- Morphism is $\langle f_i \rangle : \langle A_i \rangle \longrightarrow \langle B_i \rangle$ an *I*-family of morphisms $f_i : A_i \longrightarrow B_i$
- $\alpha^* \langle A_i \rangle_{i \in I} = \langle A_{\alpha j} \rangle_{j \in J}$ reindexing
- S an elementary topos
 - $\mathcal{S}(I) = \mathbf{S}/I$, $\alpha^* : \mathbf{S}/I \longrightarrow \mathbf{S}/J$ is pullback
 - & is S indexed by itself
 - Groups in S are indexed by

$$Gr(S)(I) = Gr(S/I)$$

 α^* preserves groups

Morphisms

- An indexed functor $F: \mathcal{A} \longrightarrow \mathcal{B}$ is a pseudo-natural transformation
 - For every I, F(I) : $\mathcal{A}(I) \longrightarrow \mathcal{B}(I)$ a functor
 - For every α , a natural isomorphism

$$\begin{array}{ccc}
\mathcal{A}(I) \xrightarrow{F(I)} \mathfrak{B}(I) \\
 & & \downarrow^{\alpha^*} & \downarrow^{\alpha^*} \\
\mathcal{A}(J) \xrightarrow{F(J)} \mathfrak{B}(J)
\end{array}$$

- Satisfying "obvious" coherence conditions
- It is *rigid* if all ψ_{α} are identities
- An indexed natural transformation $t: F \longrightarrow G$ is a modification
 - For every I, a natural transformation $t(I): F(I) \longrightarrow G(I)$
 - ullet Compatible with the ψ_{lpha}
- Get a 2-category **S**-<u>IndCat</u>

Externalization

• Let $\mathbb{C} = (C_2 \xrightarrow{p_1 \atop p_2} C_1 \xrightarrow{d_0 \atop e - id} C_0)$ be a category object in **S**

The externalization of $\mathbb C$ is the indexed category $\mathscr Ex(\mathbb C)$

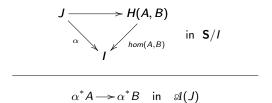
- $\mathscr{E}x(\mathbb{C})(I)$ Objects $I \longrightarrow C_0$
 - Morphisms $I \longrightarrow C_1$
- $\alpha: J \longrightarrow I$, $\alpha^*: \mathcal{E}x(\mathbb{C})(I) \longrightarrow \mathcal{E}x(\mathbb{C})(J)$

$$(I \longrightarrow C_0) \longmapsto (J \xrightarrow{\alpha} I \longrightarrow C_0)$$

- $\mathscr{C}x(\mathbb{C})$ is a rigid **S**-indexed category
- Also have $\mathscr{E}x(F)$, $\mathscr{E}x(t)$ for internal functors and natural transformations
- $\underline{\underline{Cat}}(S) \xrightarrow{\mathscr{C}_{x}} \underline{\underline{Rigid}} \underline{\underline{S}} \underline{IndCat}$ is 2-full and faithful

Smallness

- $\mathscr A$ is *small* if it is isomorphic to $\mathscr Ex(\mathbb C)$ for some category object $\mathbb C$ in **S**
- \mathcal{A} has *small homs* if for every I and A, $B \in \mathcal{A}(I)$ there exist $hom(A, B) : H(A, B) \longrightarrow I$ and a natural bijection



• When S = Set, $S/I \simeq Set^I$, $\mathcal{A}(I) = A^I$

$$\mathsf{hom}(A,B) = \langle \mathbf{A}(A_i,B_i) \rangle_{i \in I}$$

• If \mathcal{A} has small homs, $\mathcal{A}(I)$ enriched in S/I (with cartesian product)

$\sum \prod$

- A has indexed sums if
 - for every $\alpha: J \longrightarrow I$, $\alpha^*: \mathcal{A}(I) \longrightarrow \mathcal{A}(J)$ has a left adjoint

$$\sum_{\alpha}: \mathcal{A}(J) \longrightarrow \mathcal{A}(I)$$

• (Beck condition) for every pullback

$$\begin{array}{c|c}
L \xrightarrow{\beta} & K \\
\delta & & \gamma \\
J \xrightarrow{\alpha} & I
\end{array}$$

the canonical morphism

$$\mathcal{A}(L) \xrightarrow{\sum_{\beta}} \mathcal{A}(K)$$

$$\delta^* \Big| \Longrightarrow \Big| \gamma^*$$

$$\mathcal{A}(J) \xrightarrow{\sum_{\beta}} \mathcal{A}(I)$$

is invertible

- Beck condition says that \sum_{α} is "pointwise"
- For **S** = **Set**

$$\sum_{\alpha} (\langle A_j \rangle_{j \in J}) = \langle \sum_{\alpha(j)=i} A_j \rangle_{i \in I}$$

- ullet \prod_{α} is dual: right adjoint to α^* satisfying a similar Beck condition
- \bullet For $\mathcal{A}=\mathcal{S}$, \sum_{α} , \prod_{α} are well-known from topos theory

 \sum exists for any \boldsymbol{S} and \prod exists if and only if \boldsymbol{S} is locally cartesian-closed

Remarks

- Once the basic position of equipping a category with a notion of abstract families is taken, the general theory is easy, especially for one with a basic acquaintance with topos theory
- ullet The more like sets ullet is, the more like ordinary category theory ullet-indexed categories will be. E.g. ullet a topos or spaces of some sort
- The basic ideas are
 - · natural notion of family parametrized by objects of S
 - · a notion of sums and products of these families
 - naturally occurring category objects in S
- The families intuition can be very useful

Categories as Parametrizers

- Cat is a good candidate for parameters
 - Almost a topos: cartesian-closed, sums are disjoint and universal
 - Well, certainly very "space like"
 - Families indexed by categories fibrations, diagrams, . . .
 - Kan extensions are sums or products of such families
 - Category objects in Cat are double categories
- Basic theory is straightforward concentrate on the differences specific to our choice of Cat as category of parameters
 - Not locally cartesian-closed but we have a characterization of those functors for which
 ∏ exists, the Conduché fibrations
 - Quotients are not good Cat is not a regular category

Hypercategories

Definition

A hypercategory is a **Cat**-indexed category

A hyperfunctor is a Cat-indexed functor

A hypernatural transformation is a Cat-indexed natural transformation

"Hypercategory" is the old Eilenberg-Kelly name for 2-category

II. Double Categories

Double Categories

ullet Ehresmann pprox 1960 Double category has objects, two kinds of arrows, horizontal and vertical, and cells



- Horizontal and vertical compositions, giving two category structures, related by interchange
- Category object in Cat

$$\mathbb{A} = \mathbf{A}_2 \xrightarrow{\stackrel{p_1}{\longrightarrow}} \mathbf{A}_1 \xrightarrow{\stackrel{d_0}{\longleftarrow}} \mathbf{A}_0$$

 \mathbf{A}_0 : objects and vertical arrows

A1: horizontal arrows and cells

Double Functors, Horizontal Transformations

• A double functor $F: \mathbb{A} \longrightarrow \mathbb{B}$ is a function taking objects, horizontal (resp. vertical) arrows, and cells of \mathbb{A} to similar elements of \mathbb{B}

preserving everything

- A horizontal transformation $t: F \longrightarrow G$ is
 - for every A in A, a horizontal arrow $tA : FA \longrightarrow GA$ in B
 - for every vertical arrow v in \mathbb{A} , a cell

$$FA \xrightarrow{tA} GA$$

$$Fv \downarrow \qquad \qquad \downarrow \qquad \downarrow Gv$$

$$FA' \xrightarrow{tA'} GA'$$

- horizontally natural
- vertically functorial

Examples

Sets, functions, relations, inclusion

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow & \downarrow \\
R & \downarrow & & \downarrow & \downarrow \\
C & \xrightarrow{g} & D
\end{array}$$

- For any category A, □A squares in A
- For any 2-category $\underline{\underline{A}}$, $\mathbb{Q}\underline{\underline{A}}$ quintets in $\underline{\underline{A}}$

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow h & \swarrow & \downarrow \downarrow \\
C & \xrightarrow{g} & D
\end{array}$$

• For any 2-category \underline{A} , \mathbb{H} or \underline{A}

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\parallel & \psi_{\alpha} & \parallel \\
A & \xrightarrow{g} & B
\end{array}$$

Externalization

- ullet Every double category \mathbb{A} gives a hypercategory $\mathscr{C}x(\mathbb{A})$
 - Objects of $\mathscr{E}x(\mathbb{A})(\mathbf{I})$ are vertical diagrams of shape \mathbf{I} in \mathbb{A}

$$I \longrightarrow A_0$$
 i.e. $\mathbb{V}\mathrm{ert}\ I \longrightarrow \mathbb{A}$

• Morphisms of $\mathscr{E}x(\mathbb{A})(\mathbf{I})$ are

$$I \longrightarrow A_1$$

i.e., horizontal transformations of vertical diagrams

• For $F: \mathbf{J} \longrightarrow \mathbf{I}$, $F^*: \mathscr{C}x(\mathbb{A})(\mathbf{I}) \longrightarrow \mathscr{C}x(\mathbb{A})(\mathbf{J})$ is given by composition

$$(\operatorname{\mathbb{V}\mathrm{ert}} I \overset{\Phi}{\longrightarrow} \mathbb{A}) \vdash^{F^*} \to (\operatorname{\mathbb{V}\mathrm{ert}} J \overset{\operatorname{\mathbb{V}\mathrm{ert}} F}{\longrightarrow} \operatorname{\mathbb{V}\mathrm{ert}} I \overset{\Phi}{\longrightarrow} \mathbb{A})$$

- $\mathscr{E}x(\mathbb{A})(\mathbb{1})$ has
 - objects: the objects of A
 - arrows: the horizontal arrows of A
- $\mathscr{E}x(\mathbb{A})$ is a rigid hypercategory

Examples

•
$$\mathscr{E}x(\Box A)(I) = A^I$$
, $F^* = A^F : A^I \longrightarrow A^J$

 \prod_{E} , \sum_{E} are Kan extensions but Beck condition doesn't hold!

- $\mathscr{E}x(\mathbb{Q}\underline{A})(\mathbf{I})$
 - objects are 2-functors $I \longrightarrow \underline{A}$
 - morphisms are lax transformations
- $\mathscr{E}x(\mathbb{H}\mathrm{or}\underline{A})(\mathbf{I})$
 - objects are constant on components of I, i.e. a function $\pi_0 I \longrightarrow \mathrm{Ob} \ \underline{A}$
 - · morphisms are not constant
 - e.g. if I is connected, an object of $\mathscr{E}x(\mathbb{H}\mathrm{or}\underline{\underline{A}})(I)$ is an object of $\underline{\underline{A}}$, and a morphism $A \longrightarrow B$ is a functor $I \longrightarrow \underline{A}(A,B)$

Small homs

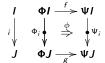
- Every small category, i.e. $\mathscr{E}x(\mathbb{A})$, has small homs
- Let $\Phi, \Psi : \mathbb{V}\mathrm{ert} \ I \longrightarrow \mathbb{A}$ be two I-families $\mathsf{hom}(\Phi, \Psi)$ is given by the pullback

$$H(\Phi, \Psi) \longrightarrow \mathbf{A}_1$$

$$hom_{(\Phi, \Psi)} \downarrow \qquad \qquad \downarrow^{\langle d_0, d_1 \rangle}$$

$$\mathbf{I} \xrightarrow{\langle \Phi, \Psi \rangle} \mathbf{A}_0 \times \mathbf{A}_0$$

- an object of $H(\Phi, \Psi)$ is a pair $(I, \Phi I \xrightarrow{f} \Psi I)$
- a morphism is a pair (i, ϕ)



Remark

 $hom(\Phi, \Psi)$ is a family of categories, an object of Cat/I (general theory). It can be an arbitrary category over I, $X \longrightarrow I$ (take $I + X + X + I \longrightarrow I + X + I \longrightarrow I + I$)

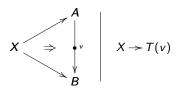
Tabulators

 $\bullet \ \, \mathsf{For} \,\, \mathscr{A} = \mathscr{C}x\mathbb{A}$

$$\prod_{2}: \mathcal{A}(2) \longrightarrow \mathcal{A}(1)$$

gives tabulators for \mathbb{A}

• The adjointness $2^* \longrightarrow \prod_2$ gives the 1-dimensional universal property

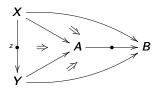


Tetrahedron

Beck for

$$\begin{array}{c|c}
2 \times 2 \stackrel{P_2}{\longrightarrow} 2 \\
P_1 \downarrow & \downarrow 2 \\
2 & \xrightarrow{2} 1
\end{array}$$

gives the tetrahedron condition





2-Functoriality

• $\mathscr{E}x(\mathbb{A})(-): \mathbf{Cat}^{op} \longrightarrow \mathbf{CAT}$ is a functor but not usually a 2-functor!

Given an I-family $\Phi: I \longrightarrow A_0$ we get

$$\Phi F \xrightarrow{\Phi t} \Phi G$$

$$\parallel \qquad \qquad \parallel$$

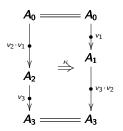
$$F^* \Phi \longrightarrow G^* \Psi$$

But this is a vertical transformation $(\Phi t(J))$ is a vertical arrow) A natural transformation $F^* \longrightarrow G^*$ is a horizontal transformation

- $\mathscr{E}x(\Box \mathbf{A})$ is 2-functorial
- $\mathscr{E}x(\mathbb{Q}\underline{A})$ is not in general

Weak Double Categories

- Most double categories of structures are weak
- A weak double category has the same data as a double category except that vertical composition is only unitary and associative up to coherent special isomorphism, e.g.



• It can be viewed as a weak category object



in CAT

Cat

- There is a weak double category Cat that plays the role of Cat in the double category universe
- It consists of small categories, functors, profunctors, natural transformations



$$t: P(-,-) \longrightarrow Q(F-,G-)$$

- Profunctors are "relations between categories"
- $P: \mathbf{A} \longrightarrow \mathbf{B}$ is $P: \mathbf{A}^{op} \times \mathbf{B} \longrightarrow \mathbf{Set}$

Other Examples

• Set: sets, functions, spans

• Ring: rings, homomorphisms, bimodules

• $Vert\underline{B}$: \underline{B} a bicategory

• V-Cat: suitable V

• V-Set: sets, functions, V-matrices

The Hypercategory of a Weak Double Category

- ℋℊր(A)(I)
 - objects pseudo-functors Vert I → A
 - morphisms are pseudo-natural transformations
 - F* is precomposition with F
- $\mathcal{H}yp(\mathbb{A})$ is a rigid hypercategory
- $\mathcal{H}yp(\mathbb{A})$ has small homs
- The natural notion of morphism is pseudo-functor (or weak functor) $\Phi: \mathbb{A} \longrightarrow \mathbb{B}$
- This gives $\mathscr{H}yp(\mathbb{A}) \xrightarrow{\mathscr{H}yp(\Phi)} \mathscr{H}yp(\mathbb{B})$ a rigid hyperfunctor

Proposition

There is an equivalence of categories

$$\frac{\mathbb{A} \xrightarrow{\text{weak}} \mathbb{B}}{\mathcal{H}yp(\mathbb{A}) \xrightarrow{\text{rigid}} \mathcal{H}yp(\mathbb{B})}$$

Essential Smallness

- A category is essentially small if it is equivalent to a small one
- Equivalence for indexed categories (Bunge-Paré, Cahiers 1980)
 - \bullet Strong equivalence $\mathscr{A} \xrightarrow{F} \mathscr{B}$ such that $FG \cong 1, \ GF \cong 1$
 - Weak equivalence
 - F full and faithful (F(I) is full and faithful for every I)
 - F essentially surjective on objects: for every I, $B \in \mathcal{B}(I)$ there is a cover $e: J \longrightarrow I$, $A \in \mathcal{A}(J)$, and an isomorphism $e^*B \cong F(J)(A)$
- There cover meant regular epimorphism, and S was assumed to be regular

Effective Descent Morphisms

- Cat is not regular
- A good notion of cover is an effective descent morphism

Theorem

(Janelidze, Sobral, Tholen) A functor $F: J \longrightarrow I$ is of effective descent type in Cat if and only if it is onto on objects, arrows, and composable pairs of arrows

Theorem

If $\mathbb A$ is a weak double category, then there exist a strict double category $\mathbb B$ and a weak equivalence $\mathscr Ex(\mathbb B) \longrightarrow \mathscr Hyp(\mathbb A)$

III. Families of Categories

The Indexing of Cat

- There are many possible notions of what a family of categories or functors might be
 - Cat/I categories over I with commutative triangles
 - Cat // I categories over I with lax triangles
 - Fib/I fibrations over I with cartesian functors
 - Cond/I Conduché fibrations with commutative triangles
 - Pseudo(I^{op} , Cat)
 - Etc.

The Standard Indexing

- Cat(I) = Cat/I, F^* is pullback along F
- $\mathscr{C}at(1) \cong \mathsf{Cat}$
- • Cat(2) has objects profunctors A → B
 morphisms natural transformations



 \bullet $\mathscr{C}at$ is a flexible hypercategory

Lax Functors

Theorem

(Bénabou) A category over I, $U: A \longrightarrow I$ is equivalent to a normal lax functor $I \longrightarrow Prof$

- In fact Cat/I is equivalent to the category of normal lax functors Vert I → Cat with horizontal transformations
- The correspondence is given by

$$I \mapsto \mathbf{A}_I = U^{-1}(I)$$

$$i \longmapsto P_i$$

$$P_I(A, A') = \{a : A \longrightarrow A' | Ua = i\}$$

- This does not use choice
- If instead we take this as our definition of Cat, then we get an equivalent but rigid hypercategory

Some Properties of $\mathscr{C}at$

- $\mathscr{C}at$ has \sum_F satisfying the Beck condition (true in general for any **S**)
- $\mathscr{C}at()$: Cat^{op} \longrightarrow CAT is not a 2-functor
 - Let $I, I' : \mathbb{I} \longrightarrow I$ be two functors and $i : I \longrightarrow I'$ a natural transformation. Then we have

$$\mathscr{C}at(\mathbf{I}) \xrightarrow{I^*} \mathscr{C}at(1)$$

$$\begin{array}{ccc}
A & & \stackrel{I^*}{\longmapsto} & A_I \\
\downarrow & & \stackrel{I^{**}}{\longmapsto} & A_{I'}
\end{array}$$

but there may not be any functor $\mathbf{A}_{l} \longrightarrow \mathbf{A}_{l'}$ (if $\mathbf{A}_{l} \neq \emptyset = \mathbf{A}_{l'}$)

Powerful Families

• An object A in a cartesian category \mathbf{A} is *powerful* if $A \times (\) : \mathbf{A} \longrightarrow \mathbf{A}$ has a right adjoint $(\)^A$

Theorem

(Giraud, Conduché) $U: \mathbf{A} \longrightarrow \mathbf{I}$ is powerful in $\mathbf{Cat/I}$ if and only if it satisfies the following condition: for every $f: A \longrightarrow A'$, every factorization of Uf, $UA \stackrel{\times}{\longrightarrow} I \stackrel{y}{\longrightarrow} UA'$ lifts to a factorization of f, $A \stackrel{g}{\longrightarrow} \bar{A} \stackrel{h}{\longrightarrow} A'$, Ug = x, Uh = y, and any two liftings are connected by a zigzag path of diagrams

$$\begin{array}{ccc}
A \xrightarrow{g} \bar{A} \xrightarrow{h} A' \\
\parallel & \downarrow^{a} & \parallel \\
A \xrightarrow{g'} \bar{A'} \xrightarrow{h'} A'
\end{array}$$

with Ua = 1

Powerful Categories

Proposition

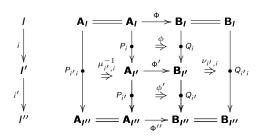
- (1) Powerful morphisms are stable under pullback
- (2) $F^* : \mathbf{Cat/I} \longrightarrow \mathbf{Cat/J}$ has a right adjoint \prod_F if and only if F is powerful
- (3) \prod_{F} satisfies the Beck condition
 - Get a subhypercategory $\mathcal{PC}at$ of $\mathcal{C}at$ with $\mathcal{PC}at(I)$ the full subcategory of \mathbf{Cat}/\mathbf{I} determined by the powerful families
 - \bullet \mathscr{PCat} has small homs
 - \bullet $A \longrightarrow I$ is powerful if and only if the corresponding lax functor $\mathbb{V}\mathrm{ert}\ I \longrightarrow \mathbb{C}\mathrm{at}$ is in fact pseudo
 - $\mathcal{PC}at \cong \mathcal{H}yp(\mathbb{C}at)$

Conduché at Work

- Useful to see in detail how the Conduché condition comes in
 - Let $U: \mathbf{A} \longrightarrow \mathbf{I}$ and $V: \mathbf{B} \longrightarrow \mathbf{I}$ be such that V^U exists in \mathbf{Cat}/\mathbf{I} . Say $V^U: \mathbf{C} \longrightarrow \mathbf{I}$
 - An object of C has to be (I, Φ : A_I →> B_I)
 A morphism of C has to be

$$\begin{array}{ccc}
I & \mathbf{A}_{I} & \xrightarrow{\Phi} \mathbf{B}_{I} \\
\downarrow i & P_{I} & & \Rightarrow & \downarrow G \\
I' & \mathbf{A}_{I'} & \xrightarrow{\Phi'} \mathbf{B}_{I'}
\end{array}$$

- Composition



Contrafamilies

Definition

An I-indexed *contrafamily* of categories is a normal oplax functor $\mathbb{V}\mathrm{ert}I \longrightarrow \mathbb{C}\mathrm{at}$ A morphism of contrafamilies is a horizontal transformation

- If Φ is a contrafamily and Ψ a family, the pointwise exponential $\Psi^{\Phi}(I) = \Psi(I)^{\Phi(I)}$ is a family
- ullet Precomposition gives reindexing functors for a hypercat ${\mathscr Con}{\mathscr Cat}$
- Note that $\operatorname{\mathscr{C}on}\operatorname{\mathscr{C}at}(1)\cong\operatorname{Cat}$

Theorem

ConCat(I) is cartesian closed

Measuring

- Let Φ be a contrafamily and Ψ , Θ families. A Φ -measuring from Ψ to Θ is a morphism of families $\Psi \longrightarrow \Theta^{\Phi}$
 - •

$$M(I): \Phi I \times \Psi I \longrightarrow \Theta I$$

.

$$\begin{array}{c|c}
\Phi I \times \Psi I & \xrightarrow{MI} & \Theta I \\
\Phi(i) \times \Psi(i) \downarrow & \xrightarrow{Mi} & \downarrow \Theta i \\
\Phi I' \times \Psi I' & \xrightarrow{MI'} & \Theta I'
\end{array}$$

• Compatible with the laxity and colaxity cells

Theorem

Given two families $\Psi,\,\Theta,$ there is a universal measuring from Ψ to Θ

$$M(\Psi,\Theta) \times \Psi \longrightarrow \Theta$$

Corollary

 $\mathscr{C}at(\mathbf{I})$ is enriched in $\mathscr{C}on\mathscr{C}at(\mathbf{I})$ and is cotensored

IV. Derivators

Derivators

- Heller Homotopy theories (1988)
- Grothendieck Les Dérivateurs (1990)
- Franke System of triangulated diagram categories (1996?)

Definition

A derivator is

- (Der 0) A 2-functor (strict) A : Cat^{op} → CAT
- (Der 1) $\mathcal{A}(\sum \mathbf{I}_{\alpha}) \cong \prod \mathcal{A}(\mathbf{I}_{\alpha})$
- (Der 2) $\mathcal{A}(I) \longrightarrow \prod_{I} \mathcal{A}(1)$ conservative
- (Der 3) F^* has a left adjoint \sum_F and a right adjoint \prod_F
- (Der 4) Beck-Chevalley for comma categories

• (Der 5) The canonical $\mathcal{A}(2 \times \mathbf{I}) \longrightarrow \mathcal{A}(\mathbf{I})^2$ is essentially surjective on objects and full

Derivators as Hypercategories

• What do (Der 0)-(Der 5) mean in terms of hypercategories? In particular for $\mathscr{C}x(\mathbb{A})$?

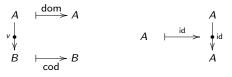
- (Der 0): $\mathcal{A}(-)$: Cat^{op} \longrightarrow CAT is a 2-functor
 - $\bullet \text{ For } \mathbb{1} \xrightarrow{\overset{0}{\underbrace{<-!}}} 2 \text{ we have } 0 \dashv ! \dashv 1 \text{ so } \mathscr{A}(2) \xrightarrow{\overset{0^*}{\underbrace{<-!}^*}} \mathscr{A}(\mathbb{1}), \ 1^* \dashv !^* \dashv 0^*$

For $\mathcal{A} = \mathscr{E}x(\mathbb{A})$

 $\mathscr{E}x(\mathbb{A})(2)$ is the category whose objects are vertical arrows and whose morphisms are cells

 $\mathscr{E}x(\mathbb{A})(\mathbb{1})$ is the category whose objects are those of \mathbb{A} and whose morphisms are horizontal arrows

0* is the domain functor, 1* is the codomain functor and !* is the functor "identity"



 $cod \dashv id \dashv dom$

$cod \dashv id \dashv dom$

• $(dom)(id) \cong 1$ so id is full and faithful

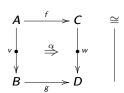
$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow^{id_A} & & \downarrow^{id_B} & & \Rightarrow & f = g \& \alpha = id_f \\
A & \xrightarrow{g} & B
\end{array}$$

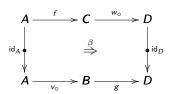
cod ⊢ id

• id ⊢ dom

Companions

- This means that every vertical arrow v has a horizontal companion v_{\circ}
- Furthermore ()_o is functorial
- Finally





• I.e., for any boundary (f, g; v, w) there is at most one cell and there is one if and only if $w_{\circ}f = sv_{\circ}$

Lax Kernel

• Let A, B be two categories with the same objects and $\Phi: A \longrightarrow B$ a functor, identity on objects. We can construct a double category \mathbb{K}_{Φ} with objects those of A (and/or B), vertical arrows the morphisms of A and horizontal arrows the morphisms of A. There is a unique cell

$$\begin{array}{ccc}
A & \xrightarrow{b} & C \\
\downarrow^{a} & \xrightarrow{\alpha} & \downarrow^{a'} \\
B & \xrightarrow{b'} & D
\end{array}$$

iff
$$(\Phi a')b = b'(\Phi a)$$

• \mathbb{K}_{Φ} is a kind of lax kernel of Φ

$$(\Phi,\Phi,\Phi) \xrightarrow{\Longrightarrow} (\Phi,\Phi) \xrightarrow{\Longrightarrow} A \xrightarrow{\Phi} B$$

Proposition

 $\mathscr{E}x(\mathbb{A})$ satisfies (Der 0) if and only if \mathbb{A} is of the form \mathbb{K}_{Φ} for some Φ

- (Der 1) $\mathcal{A}(\sum \mathbf{I}_{\alpha}) \cong \prod \mathcal{A}(\mathbf{I}_{\alpha})$
- This says that a family indexed by a sum of categories is an ordinary family of families indexed by each component
 - $\mathscr{E}x(\mathbb{A})$ always satisfies this
 - So does ℋyp(A)
 - Also Cat, PCat, ConCat

- (Der 2) $\mathcal{A}(I) \longrightarrow \prod_{I} \mathcal{A}(1)$ conservative
- When $\mathcal{A} = \mathscr{E}x(\mathbb{A})$, this says that if f and g in



are invertible, then so is α

- E.g.: For a bicategory $\underline{\underline{B}}$, $\mathbb{V}\mathrm{ert}\underline{\underline{B}}$ satisfies this if and only if $\underline{\underline{B}}$ is groupoid enriched
- For $\mathscr{E}x(\mathbb{A})$ this follows from (Der 0)
- For $\mathscr{E}x(\mathbb{A})$ and $\mathscr{H}yp(\mathbb{A})$ we do have
 - a weaker condition

$$\mathcal{A}(\mathbf{I}) \longrightarrow \prod_{Arr\mathbf{I}} \mathcal{A}(2)$$

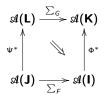
is conservative

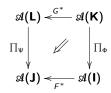
ullet (Der 3) F^* has a left adjoint \sum_F and a right adjoint \prod_F

These are the usual adjoints from indexed category theory, though they are usually required to satisfy the Beck condition for pullbacks

- For $\mathcal{A}=\mathscr{E}x(\mathbb{A})$ or $\mathscr{H}yp(\mathbb{A})$ and $I:I\longrightarrow \mathbb{I},\ \prod_I:\mathcal{A}(I)\longrightarrow \mathcal{A}(\mathbb{I})$ gives the horizontal limit of a vertical diagram, like tabulators e.g.
- For general $F: I \longrightarrow J$, $\prod_F : \mathcal{A}(I) \longrightarrow \mathcal{A}(J)$ is a kind of horizontal Kan extension

• (Der 4) The Beck condition for comma objects requires ${\mathcal A}(\)$ to be a 2-functor in order to get the comparisons





- (Der 5) $H_{\mathbf{I}}: \mathcal{A}(2 \times \mathbf{I}) \longrightarrow \mathcal{A}(\mathbf{I})^2$ essentially surjective on objects and full
- Also depends on A() being a 2-functor
- In fact H_{I} is the embodiment of 2-functoriality

• Given
$$\mathbf{I} \xrightarrow{F} \mathbf{J}$$
 we get $2 \times \mathbf{I} \xrightarrow{T} \mathbf{J}$

so
$$\mathcal{A}(\mathbf{J}) \xrightarrow{T^*} \mathcal{A}(2 \times \mathbf{I})$$

If we compose with $H_{\mathbf{I}}$ we get $\mathcal{A}(\mathbf{J}) \xrightarrow{H_{\mathbf{I}}T^*} \mathcal{A}(\mathbf{I})^2$

Thus
$$t^*: F^* \longrightarrow G^*$$

H as a Hyperfunctor

- We can define new hypercategories from old as follows
 - $\mathcal{A}[2]$ is given by $\mathcal{A}[2](\mathbf{I}) = \mathcal{A}(2 \times \mathbf{I})$

This is the hypercategory of internal (or vertical) arrows of ${\mathscr A}$

• \mathcal{A}^2 is given by $\mathcal{A}^2(\mathbf{I}) = \mathcal{A}(\mathbf{I})^2$

This is the hypercategory of external (or horizontal) arrows of ${\mathcal A}$

• $H: \mathcal{A}[2] \longrightarrow \mathcal{A}^2$ is a hyperfunctor

It is an assignment of horizontal arrows to vertical ones

- \bullet For a double category $\mathbb A$
 - $\mathscr{E}x(\mathbb{A})[2]$ corresponds to $\mathbb{A}^{\mathbb{V}\mathrm{ert}2}$
 - $\mathscr{E}x(\mathbb{A})^2$ corresponds to $\mathbb{A}^{\mathbb{H}\mathrm{or}2}$
 - $H: \mathbb{A}^{\mathbb{V}\mathrm{ert}2} \longrightarrow \mathbb{A}^{\mathbb{H}\mathrm{or}2}$

H for $\mathscr{E}x(\mathbb{K}_{\Phi})$

- H for $\mathscr{E}x(\mathbb{K}_{\Phi})$ is always full and faithful
- For it to be essentially surjective on objects as in (Der 5) means that there is a functor S and an isomorphism



To be continued ...