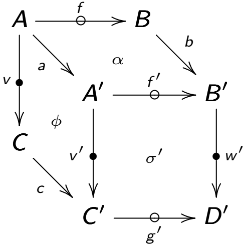
Isotropic Intercategories

Robert Paré
(with Marco Grandis)

Halifax
August 2016

- A kind of lax triple category $\mathcal{A}$

$\mathfrak{a}$:



- Transversal: $\cdot$, 1_A, strictly unitary and associative
- Horizontal: $\circ$, id_A, associative and unitary up to isomorphism
- Vertical: $\bullet$, Id_A, associative and unitary up to isomorphism

Interchange

- $\chi : \dfrac{\sigma_1|\sigma_2}{\sigma_3|\sigma_4} \longrightarrow \dfrac{\sigma_1}{\sigma_3}\bigg|\dfrac{\sigma_2}{\sigma_4}$
- $\delta : \mathsf{Id}_{f_1|f_2} \longrightarrow \mathsf{Id}_{f_1} \,|\, \mathsf{Id}_{f_2}$
- $\mu : \dfrac{\mathsf{id}_{v_1}}{\mathsf{id}_{v_2}} \longrightarrow \mathsf{id}_{\frac{v_1}{v_2}}$
- $\tau : \mathsf{Id}_{\mathsf{id}_A} \longrightarrow \mathsf{id}_{\mathsf{Id}_A}$

- Weak category object in $\mathcal{L}x\mathcal{D}bl$

$$\mathbb{C} \;\; \begin{matrix} \longrightarrow \\ \xrightarrow{\;\circ\;} \\ \longrightarrow \end{matrix} \;\; \mathbb{B} \;\; \begin{matrix} \xrightarrow{\;d_0\;} \\ \xleftarrow{\;\mathrm{id}\;} \\ \xrightarrow[\;d_1\;]{} \end{matrix} \;\; \mathbb{A}$$

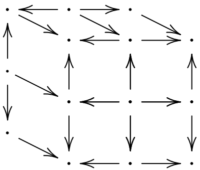
Laxity of $\circ$ - χ, δ
Laxity of id - μ, τ

- Equivalently: a weak category object in $\mathcal{C}x\mathcal{D}bl$

$$\mathbb{X}_2 \;\; \begin{matrix} \longrightarrow \\ \xrightarrow{\;\bullet\;} \\ \longrightarrow \end{matrix} \;\; \mathbb{X}_1 \;\; \begin{matrix} \longrightarrow \\ \xleftarrow{\;\mathrm{Id}\;} \\ \longrightarrow \end{matrix} \;\; \mathbb{X}_0$$

A a category with pullbacks

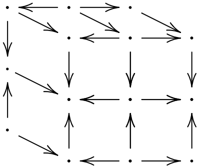
$\mathcal{S}pan^2\mathbf{A}$



χ, δ, μ, τ all isomorphisms

A category with pullbacks and pushouts

$\mathcal{S}pan\mathcal{C}osp$**A**



χ is not an isomorphism
It's the canonical comparison from a pushout of pullbacks to a pullback of pushouts

δ, μ, τ are isomorphisms

Gray categories

- A category $\mathcal{A}$ enriched in $(2\text{-}\mathbf{Cat}, \otimes, \mathbf{1})$

$$\frac{\text{2-functors } \Phi : \mathcal{X} \otimes \mathcal{Y} \longrightarrow \mathcal{Z}}{\text{quasi-functors of two variables } \Psi : \mathcal{X} \times \mathcal{Y} \longrightarrow \mathcal{Z}}$$

$$\Psi(X, -) : \mathcal{Y} \longrightarrow \mathcal{Z}, \quad \Psi(-, Y) : \mathcal{X} \longrightarrow \mathcal{Z} \;\; \text{2-functors}$$

- $\Psi(x, y)$ is not defined, but

$$\begin{array}{ccc} \Psi(X, Y) & \xrightarrow{\Psi(x,Y)} & \Psi(X', Y) \\ {\scriptstyle \Psi(X,y)} \downarrow & \overset{\psi(x,y)}{\Longrightarrow} & \downarrow {\scriptstyle \Psi(X',y)} \\ \Psi(X, Y') & \xrightarrow[\Psi(x,Y')]{} & \Psi(X', Y') \end{array}$$

- For a Gray category, composition will be a quasi-functor

$$\mathcal{A}(A,B) \times \mathcal{A}(B,C) \longrightarrow \mathcal{A}(A,C)$$

- There is no horizontal composition of 2-cells, only whiskering
- If we define $\Psi(x,y) = \Psi(x,Y)\Psi(X',y)$ we get a lax functor

$$\Psi : \mathcal{X} \times \mathcal{Y} \longrightarrow \mathcal{Z}$$

satisfying:

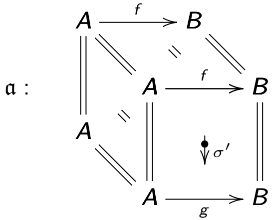
  - $\Psi(x,1)\Psi(x',y') \longrightarrow \Psi(xx',y)$ is an identity
  - $\Psi(x,y)\Psi(1,y') \longrightarrow \Psi(x,yy')$ is an identity
  - $1 \longrightarrow \Psi(1,1)$ is an identity

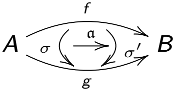
- We can put all of the homs of a Gray category together to get

$$\sum_{A,B,C} \mathcal{A}(A,B) \times \mathcal{A}(B,C) \overset{\circ}{\underset{}{\Rrightarrow}} \sum_{A,B} \mathcal{A}(A,B) \underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\text{id}}}} Ob\ \mathcal{A}$$

a category object in $\mathcal{L}x\mathcal{D}bl$, i.e. an intercategory $\mathcal{A}_I$

- General cube looks like



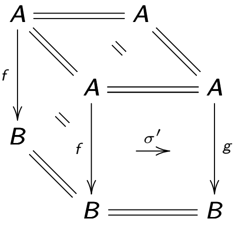


- χ is not an isomorphism, but $\chi\begin{pmatrix} * & * \\ \mathsf{Id} & * \end{pmatrix}$ and $\chi\begin{pmatrix} * & \mathsf{Id} \\ * & * \end{pmatrix}$ are identities

  δ, μ, τ are identities

- If instead we define $\Psi(x, y) = \Psi(X, y)\Psi(x, Y')$ we get a colax functor, which gives a different intercategory $\mathcal{A}_c$

$\mathfrak{a}$:



- Now

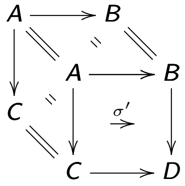
$$\chi \begin{pmatrix} \text{id} & * \\ * & * \end{pmatrix} \text{ and } \chi \begin{pmatrix} * & * \\ * & \text{id} \end{pmatrix} \text{ are identities}$$

“Symmetric” case

- A better way of representing a Gray category as an intercategory $\mathcal{A}_s$

  A general cube

$\mathfrak{a}$:

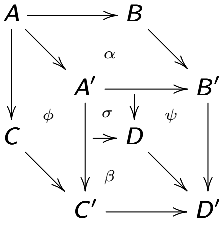


- The basic cells are co-quintets (No choice!)
  - The horizontal composition has to be the lax one
  - The vertical composition is the oplax one
  - The (co-)quintet composition determines these

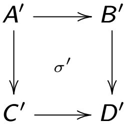
- We have $\chi\begin{pmatrix} \mathrm{id} & * \\ * & * \end{pmatrix}$, $\chi\begin{pmatrix} * & * \\ * & \mathrm{id} \end{pmatrix}$, $\chi\begin{pmatrix} \mathrm{Id} & * \\ * & * \end{pmatrix}$, $\chi\begin{pmatrix} * & * \\ * & \mathrm{Id} \end{pmatrix}$ all identities

Transversal invariance

An intercategory $\mathcal{A}$ is *transversally invariant* if for every open box of cells



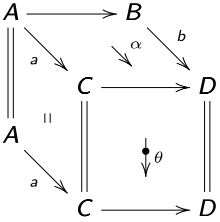
with $\alpha, \beta, \phi, \psi$ transversal isomorphisms, there exist a basic cell

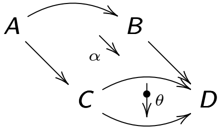


closing the box and a transversally invertible cube $\mathfrak{s} : \sigma \longrightarrow \sigma'$ filling it

I.e. basic cells are transportable along isomorphisms

$\mathcal{A}$ is *(horizontally) cylindrical* if its vertical arrows and cells are identities





Proposition

If $\mathcal{A}$ is horizontally cylindrical, it is transversally invariant if and only if all transversally invertible horizontal cells have basic companions

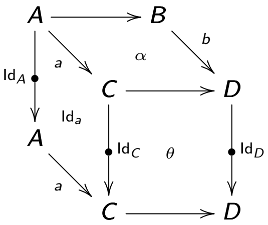
Cylindrification

Theorem

If $\mathcal{A}$ is a transversally invariant intercategory, then there is a cylindrical intercategory $\mathcal{Z}\mathcal{A}$ gotten by taking the vertical arrows to be Id_A and vertical cells to be Id_a and the rest full on this. The inclusion $\Phi : \mathcal{Z}\mathcal{A} \longrightarrow \mathcal{A}$ is strict-pseudo. Furthermore $\mathcal{Z}\mathcal{A}$ is transversally invariant

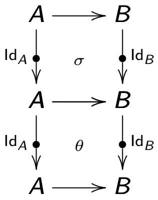
Proof.

A general cube looks like



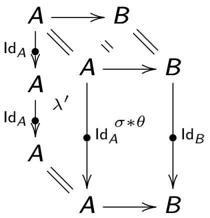
These compose transversally and horizontally as in $\mathcal{A}$
There is no choice for the vertical composite $\mathrm{Id}_A \bullet \mathrm{Id}_A$: it has to be Id_A, so the inclusion Φ only preserves composition up to isomorphism $\lambda' = \rho' : \mathrm{Id}_A \bullet \mathrm{Id}_A \longrightarrow \mathrm{Id}_A$.

For vertical composition of basic cells



we use transversal invariance to choose (arbitrarily) a basic cell $\sigma * \theta$ and an invertible cube $\mathfrak{g}(\sigma, \theta)$

$$\mathfrak{g}(\sigma, \theta) : \sigma \bullet \theta \longrightarrow \sigma * \theta$$

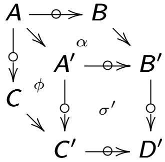


Vertical composition of cubes is by conjugation

$$\mathfrak{a} * \mathfrak{b} = \mathfrak{g}^{-1} \cdot (\mathfrak{a} \bullet \mathfrak{b}) \cdot \mathfrak{g}$$

- Let $\mathcal{A}$ be (horizontally) cylindrical and transversally invariant. We wish to construct a new intercategory $\mathcal{Q}\mathcal{A}$ whose basic cells are quintets
- A general cube in $\mathcal{Q}\mathcal{A}$ will look like

$\mathfrak{a}$:



A basic cell $\begin{array}{ccc} A & \xrightarrow{f} & B \\ x\downarrow & \downarrow\sigma & \downarrow y \\ C & \xrightarrow[g]{} & D \end{array}$ is a cell $\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{y} & D \\ \| & & \sigma & & \| \\ A & \xrightarrow[x]{} & C & \xrightarrow[g]{} & D \end{array}$ in $\mathcal{A}$

- Composition of arrows (transversal, horizontal, vertical) and of horizontal and vertical cells is performed in $\mathcal{A}$
- Horizontal composition of basic cells

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{h} & E \\ {\scriptstyle x}\downarrow & \sigma & \downarrow{\scriptstyle y} & \theta & \downarrow{\scriptstyle z} \\ C & \xrightarrow[g]{} & D & \xrightarrow[k]{} & F \end{array} \qquad ?$$

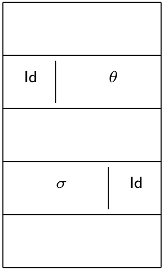
If horizontal composition were strict it would be:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{h} & E & \xrightarrow{z} & F \\ \| & \mathrm{Id}_f & \| & & \theta & & \| \\ A & \xrightarrow[f]{} & B & \xrightarrow[y]{} & D & \xrightarrow[k]{} & F \\ \| & & \sigma & & \| & \mathrm{Id}_k & \| \\ A & \xrightarrow[x]{} & C & \xrightarrow[g]{} & D & \xrightarrow[k]{} & F \end{array}$$

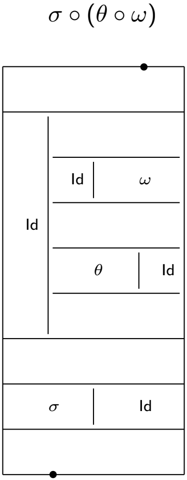
$$
\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{h} & \dot{E} & \xrightarrow{z} & F \\
\| & & & (\kappa^{-1})_* & & & \| \\
A & \xrightarrow[f]{} & \dot{B} & \xrightarrow[h]{} & E & \xrightarrow[z]{} & F \\
\| & \mathrm{Id}_f & \| & & \theta & & \| \\
A & \xrightarrow[f]{} & \dot{B} & \xrightarrow[y]{} & D & \xrightarrow[k]{} & F \\
\| & & & \kappa_* & & & \| \\
A & \xrightarrow[f]{} & B & \xrightarrow[y]{} & \dot{D} & \xrightarrow[k]{} & F \\
\| & & \sigma & & \| & \mathrm{Id}_k & \| \\
A & \xrightarrow[x]{} & C & \xrightarrow[g]{} & \dot{D} & \xrightarrow[k]{} & F \\
\| & & & (\kappa^{-1})_* & & & \| \\
A & \xrightarrow[x]{} & \dot{C} & \xrightarrow[g]{} & D & \xrightarrow[k]{} & F
\end{array}
$$

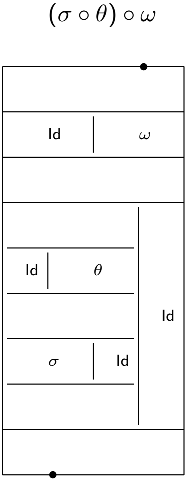
or

$\sigma \circ \theta$



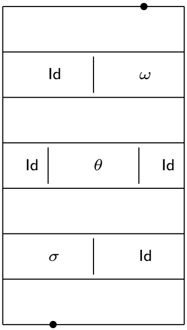
Associativity





Need conditions on $\mathcal{A}$ to get an isomorphism here

The plan is to show that both are canonically isomorphic to



For this we need certain conditions:

Conditions

(0) $\mathcal{A}$ is transversally invariant

(1) $\delta : \mathrm{Id}_{f|g} \longrightarrow \mathrm{Id}_f \mid \mathrm{Id}_g$ is an isomorphism, i.e. $\circ : \mathbb{C} \longrightarrow \mathbb{B}$ is normal

(2) $\tau : \mathrm{Id}_{\mathrm{id}_A} \longrightarrow \mathrm{id}_{\mathrm{Id}_A}$ is an isomorphism, i.e. $\mathrm{id} : \mathbb{A} \longrightarrow \mathbb{B}$ is normal

(3) $\chi \begin{pmatrix} \mathrm{Id} & * \\ \mathrm{Id} & * \end{pmatrix}$ and $\chi \begin{pmatrix} * & \mathrm{Id} \\ * & \mathrm{Id} \end{pmatrix}$ are isomorphisms (whiskers)

(4) $\chi \begin{pmatrix} \mathrm{Id} & * \\ * & \mathrm{Id} \end{pmatrix}$ is an isomorphism (Gray)

Theorem

If a (horizontally) cylindrical intercategory $\mathcal{A}$ satisfies conditions (0)-(4) then $\mathcal{QA}$ is a transversally invariant intercategory

Definition

$\mathcal{A}$ is *isotropic* if it is equivalent to $\mathcal{QZA}$