Theory and Applications of Categories, Vol. 40, No. 5, 2024, pp. 130–179.

RETROCELLS

For Marta Bunge, constant friend for over half a century

ROBERT PARÉ

ABSTRACT. The notion of retrocell in a double category with companions is introduced and its basic properties established. Explicit descriptions in some of the usual double categories are given. Monads in a double category provide an important example where retrocells arise naturally. Cofunctors appear as a special case. The motivating example of vertically closed double categories is treated in some detail.

Contents

Introduction

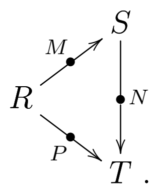
In [17] an in-depth study of the double category $\mathbb{R}$ing of rings, homomorphisms, bimodules and linear maps was made, and several interesting features were uncovered. It became apparent that considering this double category, rather than the category of rings and homomorphisms or the bicategory of bimodules, could provide some important insights into the nature of rings and modules.

An important property of the bicategory of bimodules is that it is biclosed, i.e. the $\otimes$ has right adjoints in each variable so that we have bijections of linear maps

$$\frac{\dfrac{M \longrightarrow N \oslash_T P}{N \otimes_S M \longrightarrow P}}{N \longrightarrow P \oslash_R M}$$

Received by the editors 2023-04-05 and, in final form, 2023-06-14.
Published on 2024-04-02 in the Bunge Festschrift.
2020 Mathematics Subject Classification: 18D05, 18C15, 18D15.
Key words and phrases: Double category, companion, retrocell, cofunctor, closed bicategory.

for bimodules



We use (a slight modification of Lambek's notation for the hom bimodules [14]): $P \oslash_R M$ is the T-S bimodule of R-linear maps $M \longrightarrow P$, and T-linear for $N \obslash_T P$. Both $P \oslash_R M$ and $N \obslash_T P$ are covariant in P but contravariant in the other variables. This is for 2-cells in the bicategory $\mathcal{B}im$ but it does not extend to cells in the double category $\mathbb{R}$ing, which casts a shadow on our contention that $\mathbb{R}$ing works better than $\mathcal{B}im$.

The way out of this dilemma is hinted at in the commuter cells of [11] (there called commutative cells) introduced to deal with the universal property of internal comma objects. That is, to use companions to define new cells, which we call retrocells below, and thus recover functoriality.

After a quick review of companions in Section 1, we introduce retrocells in Section 2 and see that they are the cells of a new double category, and if we apply this construction twice, we get the original double category, up to isomorphism.

Section 3 extends the mates calculus to double categories where we see retrocells appearing as the mates of standard cells. A careful study of dualities in Section 4 completes this.

Retrocells in the standard double categories whose vertical arrows are spans, relations, profunctors or **V**-matrices are analyzed in Section 5. They correspond to various sorts of liftings reminiscent of fibrations.

Section 6 studies retrocells in the context of monads in a double category. It is seen that, while Kleisli objects are certain kinds of universal cells, Eilenberg-Moore objects are universal retrocells. In $\mathbb{S}$pan**A**, monads are category objects in **A** and internal functors are cells preserving identities and multiplication. Retrocells, on the other hand, give cofunctors.

In Section 7 we extend Shulman's closed equipments to general double categories, and establish the functoriality of internal homs, covariant in one variable and retrovariant in the other, formulated in terms of "twisted cospans".

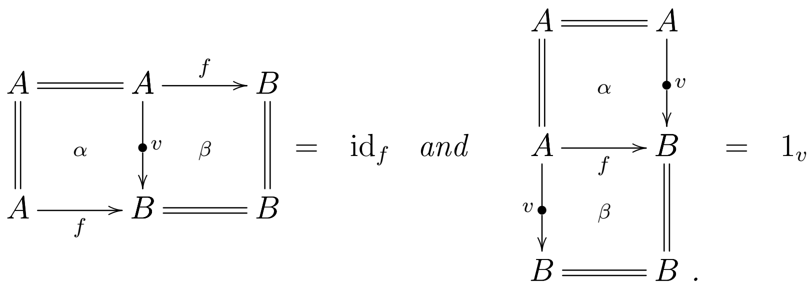
We end in Section 8 by re-examining commuter cells in the light of retrocells and see that this leads to an interesting triple category, though we do not pursue the triple category aspect.

The results of this paper were presented in preliminary form at CT2019 in Edinburgh and in the MIT Categories Seminar in October 2020. We thank Bryce Clarke, Matt Di Meglio and David Spivak for expressing their interest in retrocells. We also thank the anonymous referee for a careful reading and numerous suggestions resulting in a much better presentation.

1. Companions

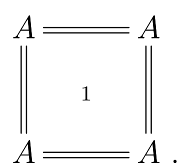
The whole paper will be concerned with double categories that have companions, so we recall the definition, principal properties we will use, and establish some notation (see [10] for more details).

1.1. DEFINITION. *Let* $f: A \longrightarrow B$ *be a horizontal arrow in a double category* $\mathbb{A}$. *A* companion *for* f *is a vertical arrow* $v: A \longrightarrow\!\!\!\!\bullet\; B$ *together with two* binding cells α *and* β *as below, such that*

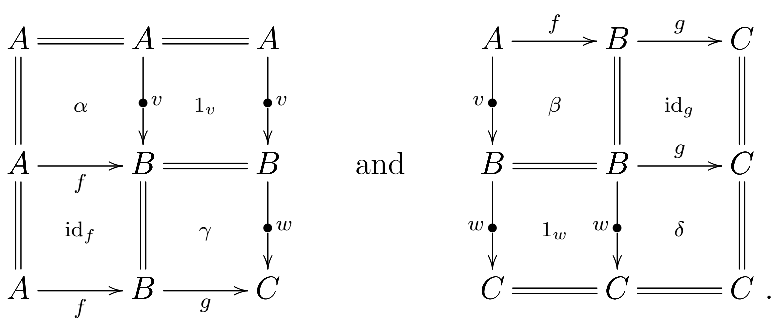


We can always assume the vertical identities are strict and usually denote them by long equal signs in diagrams, as we just did. Of course horizontal identities are always strict, and we use a similar diagrammatic notation.

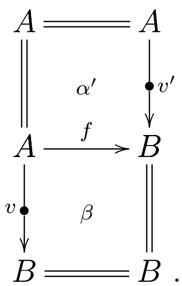
The vertical identity on A, id_A, is a companion to the horizontal identity 1_A, with both binding cells the common value $1_{\mathrm{id}_A} = \mathrm{id}_{1_A}$,



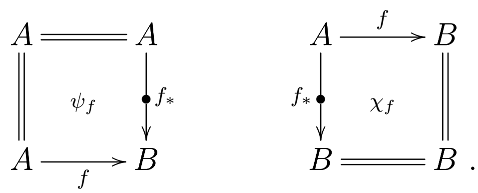
If $f: A \longrightarrow B$ and $g: B \longrightarrow C$ have respective companions (v, α, β) and (w, γ, δ) then gf has $w \bullet v$ as companion with binding cells



Two companions (v, α, β) and (v', α', β') for the same f are isomorphic by the globular isomorphism

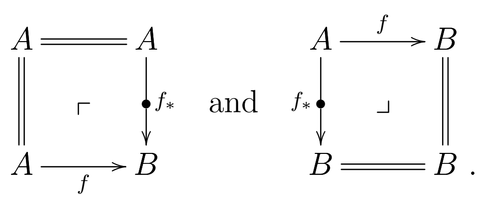


We usually choose a representative companion from each isomorphism class and call it (f_*, ψ_f, χ_f)



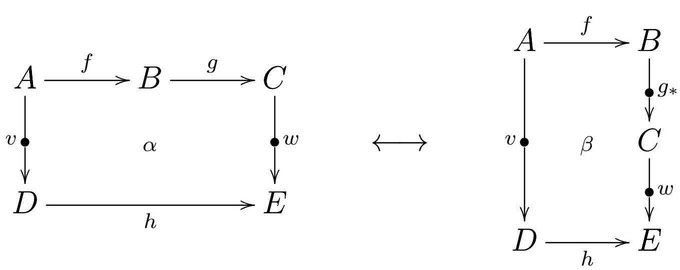
The choice is arbitrary but it simplifies things if we choose the companion of 1_A to be $(\mathrm{id}_A, 1_{\mathrm{id}_A}, 1_{\mathrm{id}_A})$. In all of our examples there is a canonical choice and for that $(1_A)_* = \mathrm{id}_A$.

To lighten the notation, we often write the binding cells ψ_f and χ_f as corner brackets in diagrams:

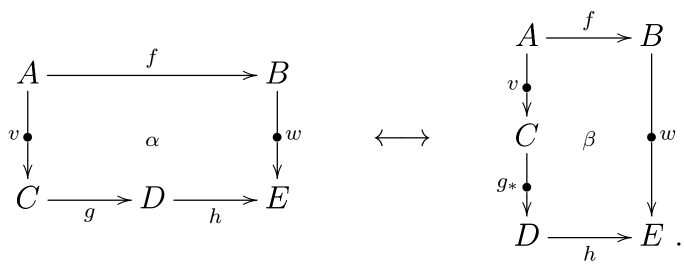


We also use $=$ and $\shortparallel$ for horizontal and vertical identity cells.

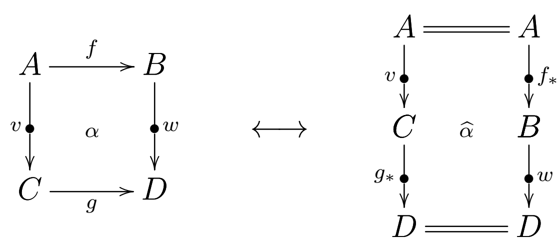
There is a useful technique, called *sliding*, where we slide a horizontal arrow around a corner into a vertical one. Specifically, there are bijections natural in every way that makes sense,



and also



If we combine the two we get a bijection



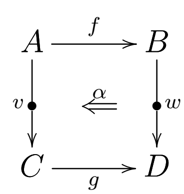
which is, in a sense, the conceptual basis for retrocells. That, and the idea that f and f_* are really the same morphism in different roles.

We refer the reader to [9] for all unexplained double category matters.

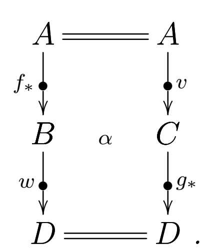
2. Retrocells

Let $\mathbb{A}$ be a double category in which every horizontal arrow has a companion and choose a companion for each (with id_A as the companion of 1_A).

2.1. Definition. *A* retrocell α *in* $\mathbb{A}$*, denoted*

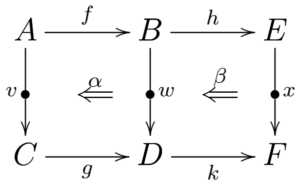


is a (standard) double cell of $\mathbb{A}$ *of the form*

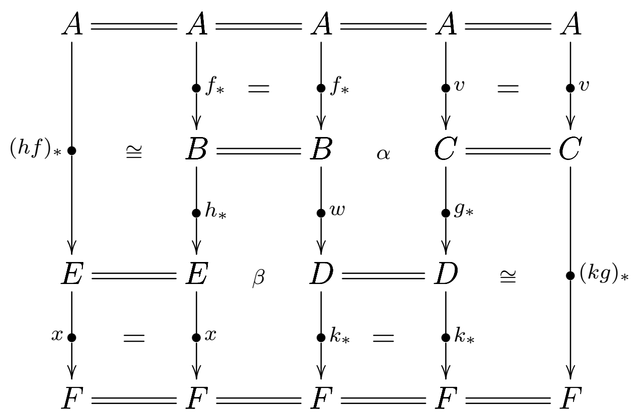


2.2. Theorem. *The objects, horizontal and vertical arrows of* $\mathbb{A}$ *together with retrocells, form a double category* $\mathbb{A}^{ret}$*.*

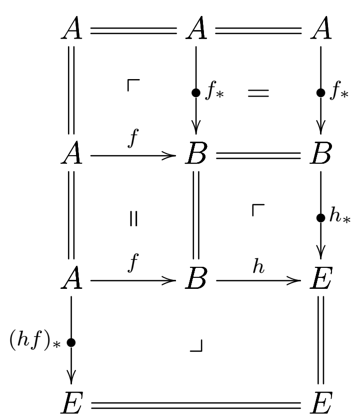
Proof. The horizontal composite $\beta\alpha$ of retrocells



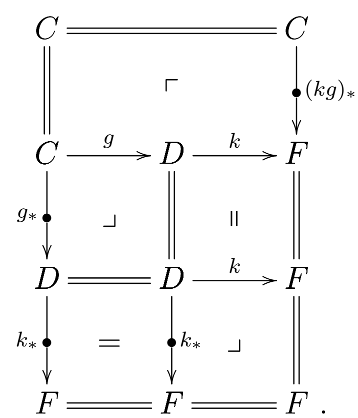
is given by



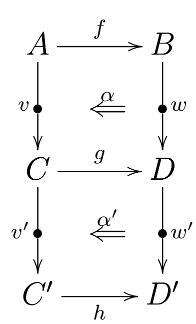
where the $\cong$ represent the canonical isomorphisms $(hf)_* \cong h_* \bullet f_*$ and $(kg)_* \cong k_* \bullet g_*$,



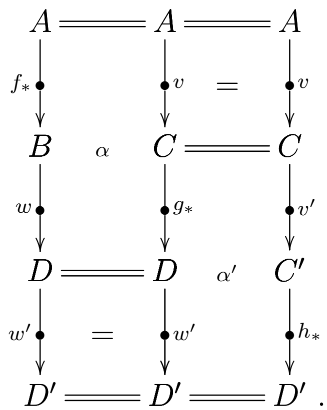
and



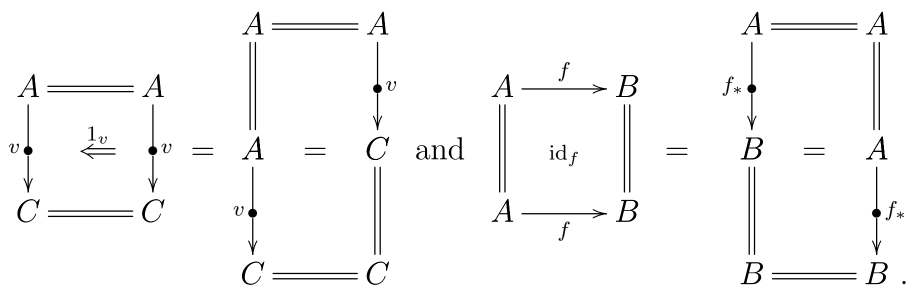
The vertical composite $\alpha' \bullet \alpha$ of retrocells



is



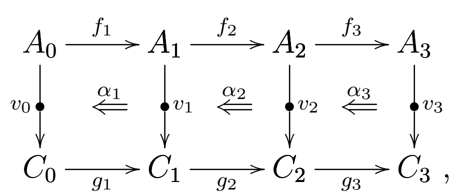
Horizontal and vertical identities are



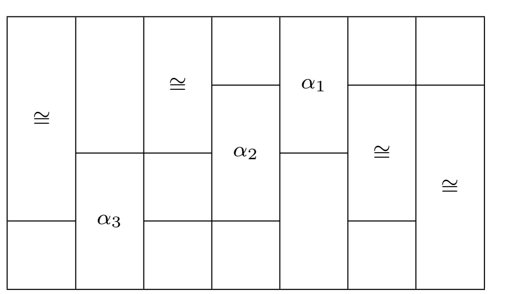
There are a number of things to check (horizontal and vertical unit laws and associativities as well as interchange), all of which are straightforward calculations and will be left to the reader. It is merely a question of writing out the diagrams and following the steps indicated schematically below.

The identity laws are trivial because of our conventions that $(1_A)_* = \mathrm{id}_A$ and vertical identities are as strict as in $\mathbb{A}$.

For retrocells



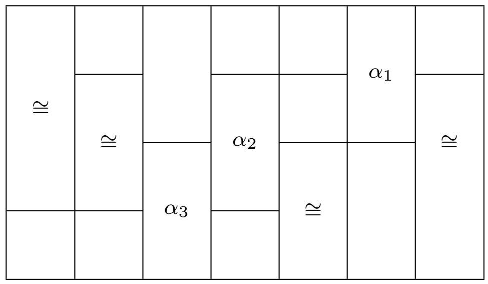
$\alpha_3(\alpha_2\alpha_1)$ is a composite of 17 cells arranged in a 4×7 array represented schematically as



(1)

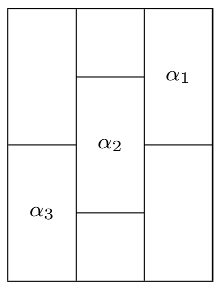
The empty rectangles are horizontal identities and the $\cong$ represent canonical isomorphisms generated by companions.

$(\alpha_3\alpha_2)\alpha_1$ on the other hand is of the form

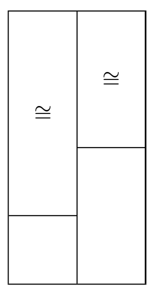


(2)

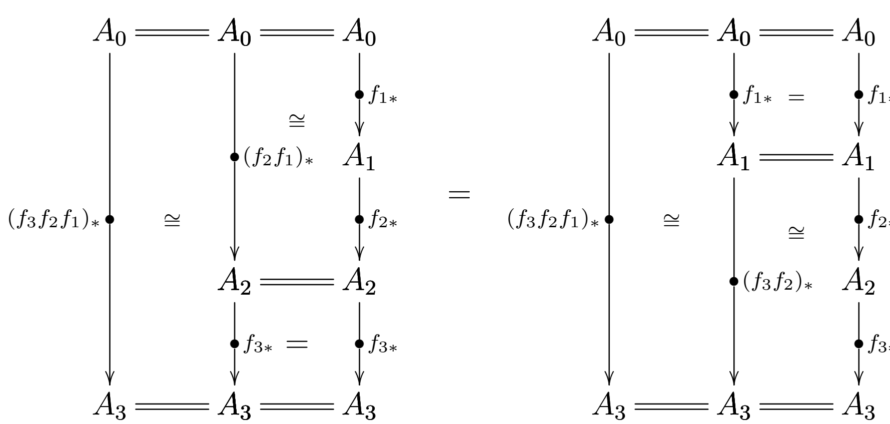
It is now clear what to do. Switch α_3 with $\cong$ in (1) and α_1 with $\cong$ in (2) to get



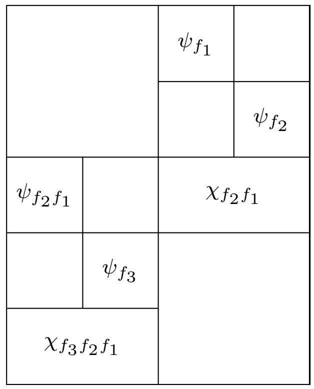
in the middle in both cases. The 4×2 block on the left in (1) becomes



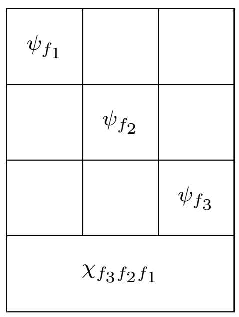
which is not formally the same as 4×2 block in (2), but they are equal by one of the coherence identities for $(\ )_*$. We write it out



There may be something to worry about here because $(f_2 f_1)_* \cong f_{2*} \bullet f_{1*}$ involves $\chi_{f_2 f_1}$ whereas $(f_3 f_2)_* \cong f_{3*} \bullet f_{2*}$ involves $\chi_{f_3 f_2}$ which are unrelated. However both $\chi_{f_2 f_1}$ and $\chi_{f_3 f_2}$ cancel in the composites. The left hand side is



and when we cancel $\chi_{f_2 f_1}$ with $\psi_{f_2 f_1}$ leaving $\mathrm{id}_{f_2 f_1}$, that composite reduces to



as does the right hand side.

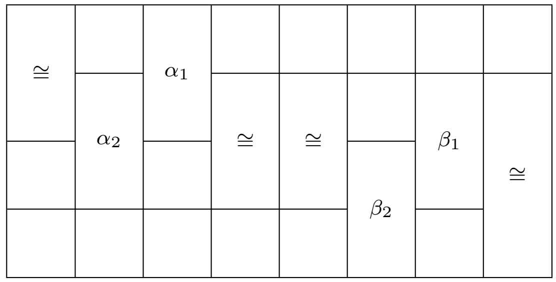
The 4×2 block on the right is the same with the roles of ψ and χ interchanged. This completes the proof of associativity of horizontal composition of retrocells.

The associativity for vertical composition is much simpler as it does not involve ψ's or χ's, only the associativity isomorphisms of $\mathbb{A}$. In particular if $\mathbb{A}$ were strict, then $\mathbb{A}^{ret}$ would be too, and the proof of associativity would be merely a question of writing down the two composites and observing that they are exactly the same.

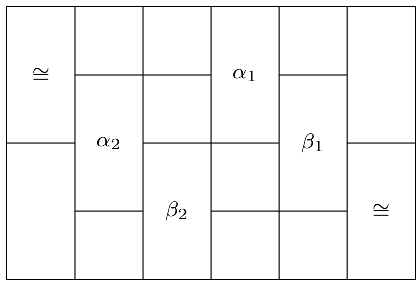
For interchange consider retrocells

$$\begin{array}{ccccc}
A_0 & \xrightarrow{f_1} & A_1 & \xrightarrow{f_2} & A_2 \\
{\scriptstyle v_0}\downarrow\bullet & \overset{\alpha_1}{\Leftarrow} & \bullet\downarrow{\scriptstyle v_1} & \overset{\alpha_2}{\Leftarrow} & \bullet\downarrow{\scriptstyle v_2} \\
B_0 & \xrightarrow{g_1} & B_1 & \xrightarrow{g_2} & B_2 \\
{\scriptstyle w_0}\downarrow\bullet & \overset{\beta_1}{\Leftarrow} & \bullet\downarrow{\scriptstyle w_1} & \overset{\beta_2}{\Leftarrow} & \bullet\downarrow{\scriptstyle w_2} \\
C_0 & \xrightarrow[h_1]{} & C_1 & \xrightarrow[h_1]{} & C_2 \;.
\end{array}$$

Then the pattern for $(\beta_2 \beta_1) \bullet (\alpha_2 \alpha_1)$ is



and for $(\beta_2 \bullet \alpha_2)(\beta_1 \bullet \alpha_1)$ it is

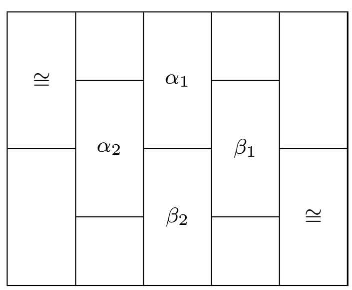


.

The two $\cong$ in the middle of the first one are inverse to each other,

$$g_{2*} \bullet g_{1*} \xrightarrow{\cong} (g_2 g_1)_* \xrightarrow{\cong} g_{2*} \bullet g_{1*} \ ,$$

so each of $(\beta_2\beta_1) \bullet (\alpha_2\alpha_1)$ and $(\beta_2 \bullet \alpha_2)(\beta_1 \bullet \alpha_1)$ is equal to



completing the proof.

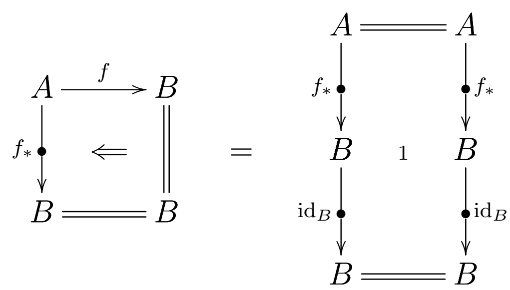
■

2.3. Theorem. *(1) $\mathbb{A}^{ret}$ has a canonical choice of companions.*
(2) There is a canonical isomorphism of double categories with companions

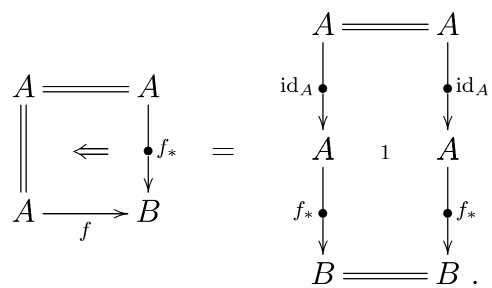
$$\mathbb{A} \xrightarrow{\cong} \mathbb{A}^{ret\ ret}$$

which is the identity on objects and horizontal and vertical arrows.

Proof. The companion of $f\colon A \longrightarrow B$ in $\mathbb{A}^{ret}$ is f_*, f's companion in $\mathbb{A}$ with binding retrocells

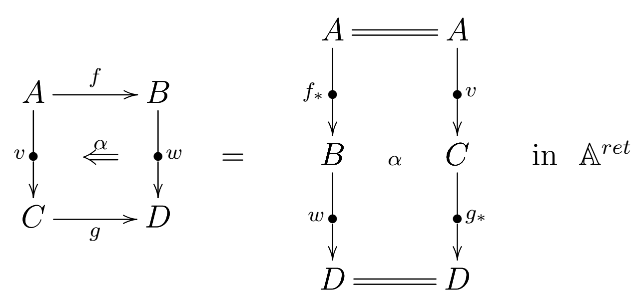


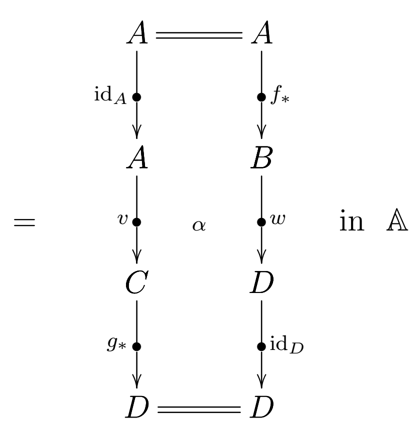
and



The binding equations only involve canonical isos so hold by coherence.

A cell α in $\mathbb{A}^{ret\,ret}$, i.e. a retrocell in $\mathbb{A}^{ret}$ is





and these are in canonical bijection with

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\bullet\!\downarrow & \alpha' & \downarrow\!\bullet{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array}\ .$$

Checking that composition and identities are preserved is a straightforward calculation and is omitted.

■

2.4. Example. If $\mathcal{A}$ is a 2-category, the double category of quintets $\mathbb{Q}\mathcal{A}$ has the same objects as $\mathcal{A}$, the 1-cells of $\mathcal{A}$ as both horizontal and vertical arrows, and cells α

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\bullet\!\downarrow & \alpha & \downarrow\!\bullet{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

i.e. a 2-cell $\alpha\colon kf \longrightarrow gh$. Horizontal and vertical composition are given by pasting. Every horizontal arrow $f\colon A \longrightarrow B$ has a companion, f_*, namely f itself considered as a vertical arrow. A retrocell

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\bullet\!\downarrow & \overset{\alpha}{\Leftarrow} & \downarrow\!\bullet{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

is

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle f_*}\bullet\!\downarrow & & \downarrow\!\bullet{\scriptstyle h} \\ B & \alpha & C \\ {\scriptstyle k}\bullet\!\downarrow & & \downarrow\!\bullet{\scriptstyle g_*} \\ D & = & D \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle kf}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle gh} \\ D & \xrightarrow[1_D]{} & D \end{array}$$

i.e. a coquintet

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}\ .$$

Thus

$$(\mathbb{Q}\mathcal{A})^{ret} = \mathrm{co}\mathbb{Q}\mathcal{A} = \mathbb{Q}(\mathcal{A}^{co})\ .$$

3. Adjoints, companions, mates

The well-known mates calculus says that if we have functors F, G, H, K, U, V as below with $F \dashv U$ and $G \dashv V$, then there is a bijection between natural transformations t and u as below

$$
\begin{array}{ccc}
\mathbf{A} & \xrightarrow{H} & \mathbf{B} \\
{\scriptstyle U}\downarrow & \overset{t}{\Rightarrow} & \downarrow{\scriptstyle V} \\
\mathbf{C} & \xrightarrow[K]{} & \mathbf{D}
\end{array}
\qquad \longleftrightarrow \qquad
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{K} & \mathbf{D} \\
{\scriptstyle F}\downarrow & \overset{u}{\Leftarrow} & \downarrow{\scriptstyle G} \\
\mathbf{A} & \xrightarrow[H]{} & \mathbf{B}
\end{array} .
$$

This is usually stated for bicategories but with the help of retrocells we can extend it to double categories (with companions).

To say that two horizontal arrows are adjoint in a double category $\mathbb{A}$ means they are so in the 2-category of horizontal arrows $\mathcal{H}or\mathbb{A}$. So h left adjoint to f means we are given cells

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{h} & A \\
\| & & \epsilon & & \| \\
A & = & = & = & A
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccccc}
B & = & = & = & B \\
\| & & \eta & & \| \\
B & \xrightarrow[h]{} & A & \xrightarrow[f]{} & B
\end{array}
$$

satisfying the "triangle" identities

$$
\begin{array}{ccccccc}
A & \xrightarrow{f} & B & = & = & = & B \\
\| & = & \| & & \eta & & \| \\
A & \xrightarrow{f} & B & \xrightarrow{h} & A & \xrightarrow{f} & B \\
\| & & \epsilon & & \| & = & \| \\
A & = & = & = & A & \xrightarrow[f]{} & B
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\| & = & \| \\
A & \xrightarrow[f]{} & B
\end{array}
$$

and

$$
\begin{array}{ccccccc}
B & = & = & = & B & \xrightarrow{h} & A \\
\| & & \eta & & \| & = & \| \\
B & \xrightarrow{h} & A & \xrightarrow{f} & B & \xrightarrow{h} & A \\
\| & = & \| & & \epsilon & & \| \\
B & \xrightarrow[h]{} & A & = & = & = & A
\end{array}
\quad = \quad
\begin{array}{ccc}
B & \xrightarrow{h} & A \\
\| & = & \| \\
B & \xrightarrow[h]{} & A
\end{array} .
$$

To say that the vertical arrows are adjoint means that they are so in the vertical bicategory $\mathcal{V}ert\mathbb{A}$. So x is left adjoint to v if we are given cells

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle v}\downarrow & & \| \\ C & \epsilon & \| \\ {\scriptstyle x}\downarrow & & \| \\ A & = & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C & = & C \\ \| & & \downarrow{\scriptstyle x} \\ \| & \eta & A \\ \| & & \downarrow{\scriptstyle v} \\ C & = & C \end{array}$$

also satisfying the triangle identities.

Suppose we are given horizontal arrows f and h with cells α_1 and β_1 as below. In the presence of companions we can use sliding to transform them. We have bijections

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{h} & A \\ \| & & \alpha_1 & & \| \\ A & = & = & = & A \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \| & \alpha_2 & \downarrow{\scriptstyle h_*} \\ A & = & A \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} A & = & A \\ \| & & \downarrow{\scriptstyle f_*} \\ \| & \alpha_3 & B \\ \| & & \downarrow{\scriptstyle h_*} \\ A & = & A \end{array}$$

$$\begin{array}{ccccc} B & = & = & = & B \\ \| & & \beta_1 & & \| \\ B & \xrightarrow[h]{} & A & \xrightarrow[f]{} & B \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} B & = & B \\ {\scriptstyle h_*}\downarrow & \beta_2 & \| \\ A & \xrightarrow[f]{} & B \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} B & = & B \\ {\scriptstyle h_*}\downarrow & & \| \\ A & \beta_3 & \| \\ {\scriptstyle f_*}\downarrow & & \| \\ B & = & B \end{array} .$$

3.1. Proposition. *h is left adjoint to f with adjunctions α_1 and β_1 if and only if f_* is left adjoint to h_* with adjunctions β_3 and α_3.*

3.2. Theorem. *Consider horizontal morphisms f and g and vertical morphisms v and w as in*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array} .$$

(1) If x is left adjoint to v and y left adjoint to w, then there is a bijection between cells α and retrocells β as in

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\downarrow\bullet & \alpha & \bullet\downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array} \quad\longleftrightarrow\quad \begin{array}{ccc} C & \xrightarrow{g} & D \\ {\scriptstyle x}\downarrow\bullet & \overset{\beta}{\Leftarrow} & \bullet\downarrow{\scriptstyle y} \\ A & \xrightarrow[f]{} & B \end{array} .$$

(2) If h is left adjoint to f and k left adjoint to g, then there is a bijection between cells α and retrocells γ as in

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\downarrow\bullet & \alpha & \bullet\downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array} \quad\longleftrightarrow\quad \begin{array}{ccc} B & \xrightarrow{h} & A \\ {\scriptstyle w}\downarrow\bullet & \overset{\gamma}{\Leftarrow} & \bullet\downarrow{\scriptstyle v} \\ D & \xrightarrow[k]{} & C \end{array} .$$

PROOF. (1) Standard cells α are in bijection with 2-cells $\widehat{\alpha}$ in the bicategory $\mathcal{V}ert\mathbb{A}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\downarrow\bullet & \alpha & \bullet\downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array} \quad\longleftrightarrow\quad \begin{array}{ccc} A & = & A \\ {\scriptstyle v}\downarrow\bullet & & \bullet\downarrow{\scriptstyle \alpha} \\ C & \widehat{\alpha} & B \\ {\scriptstyle g_*}\downarrow\bullet & & \bullet\downarrow{\scriptstyle w} \\ D & = & D \end{array}$$

and retrocells β are defined to be 2-cells in $\mathcal{V}ert\mathbb{A}$

$$\begin{array}{ccc} C & = & C \\ {\scriptstyle g_*}\downarrow\bullet & & \bullet\downarrow{\scriptstyle x} \\ D & \beta & A \\ {\scriptstyle y}\downarrow\bullet & & \bullet\downarrow{\scriptstyle f_*} \\ B & = & B \end{array} .$$

Then our claimed bijection is just the usual bijection from bicategory theory:

$$\frac{\widehat{\alpha}\colon g_* \bullet v \longrightarrow w \bullet f_*}{\beta\colon y \bullet g_* \longrightarrow f_* \bullet x} .$$

(2) From the previous proposition we have f_* is left adjoint to h_* and g_* left adjoint to k_*, and again our bijection follows from the usual bicategory one:

$$\frac{\widehat{\alpha}\colon g_* \bullet v \longrightarrow w_* \bullet f_*}{\gamma\colon v \bullet h_* \longrightarrow k_* \bullet w}$$

■

3.3. Corollary. *(1) If f has a left adjoint h, g a left adjoint k, v a right adjoint x and w a right adjoint y, then we have a bijection of cells*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\Big\downarrow\!\bullet & \alpha & \bullet\!\Big\downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array} \qquad\longleftrightarrow\qquad \begin{array}{ccc} D & \xrightarrow{k} & C \\ {\scriptstyle y}\Big\downarrow\!\bullet & \delta & \bullet\!\Big\downarrow{\scriptstyle x} \\ B & \xrightarrow[h]{} & A \end{array}\ .$$

(2) We get the same bijection if left and right are interchanged in all four adjunctions.

Proof. (1) We have the following bijections

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\Big\downarrow\!\bullet & \alpha & \bullet\!\Big\downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array} \quad\longleftrightarrow\quad \begin{array}{ccc} B & \xrightarrow{h} & A \\ {\scriptstyle w}\Big\downarrow\!\bullet & \overset{\beta}{\Longleftarrow} & \bullet\!\Big\downarrow{\scriptstyle v} \\ D & \xrightarrow[k]{} & C \end{array} \quad\longleftrightarrow\quad \begin{array}{ccc} D & \xrightarrow{k} & C \\ {\scriptstyle y}\Big\downarrow\!\bullet & \delta & \bullet\!\Big\downarrow{\scriptstyle x} \\ B & \xrightarrow[h]{} & A \end{array}$$

the first by direct application of part (2) of Theorem 3.2 and the second by applying part (1) of Theorem 3.2 in $\mathbb{A}^{ret}$ where $x \dashv v$ and $y \dashv w$. Finally δ is a cell in $(\mathbb{A}^{ret})^{ret} \cong \mathbb{A}$.
(2) For this we use (1) first and then (2) in $\mathbb{A}^{ret}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\Big\downarrow\!\bullet & \alpha & \bullet\!\Big\downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array} \quad\longleftrightarrow\quad \begin{array}{ccc} C & \xrightarrow{g} & D \\ {\scriptstyle x}\Big\downarrow\!\bullet & \overset{\gamma}{\Longleftarrow} & \bullet\!\Big\downarrow{\scriptstyle y} \\ A & \xrightarrow[f]{} & B \end{array} \quad\longleftrightarrow\quad \begin{array}{ccc} D & \xrightarrow{k} & C \\ {\scriptstyle y}\Big\downarrow\!\bullet & \delta & \bullet\!\Big\downarrow{\scriptstyle x} \\ B & \xrightarrow[h]{} & A \end{array}\ .$$

■

Note that the statement of the corollary does not refer to retrocells or companions but it does not seem possible to prove it directly without companions. The infamous pinwheel [5] pops up in all attempts to do so.

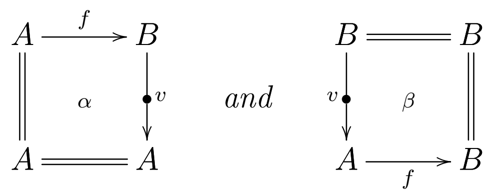
4. Coretrocells

There is a dual situation giving two more bijections in the presence of right adjoints, but the notion of retrocell is not self-dual. In fact there is a dual notion, coretrocell, which also comes up in practice as we will see later.

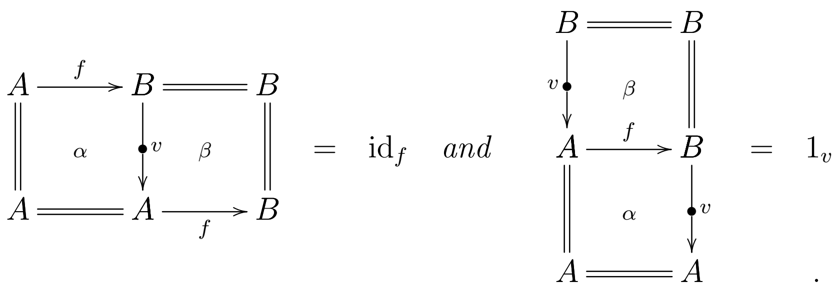
Like for 2-categories there are duals op and co for double categories. $\mathbb{A}^{op}$ has the horizontal direction reversed and $\mathbb{A}^{co}$ the vertical. If $\mathbb{A}$ has companions there is no reason why $\mathbb{A}^{op}$ or $\mathbb{A}^{co}$ should, and even if they did there is no relation between the retrocells there and those of $\mathbb{A}$. Companions in $\mathbb{A}^{op}$ or $\mathbb{A}^{co}$ correspond to conjoints in $\mathbb{A}$ and we will use these to define coretrocells.

For completeness we recall the notion of conjoint. More details can be found in [9].

4.1. Definition. *Let $f\colon A \longrightarrow B$ be a horizontal arrow in $\mathbb{A}$. A* conjoint *for f is a vertical arrow $v\colon B \longrightarrow\!\!\!\bullet\,\, A$ together with two cells (conjunctions)*

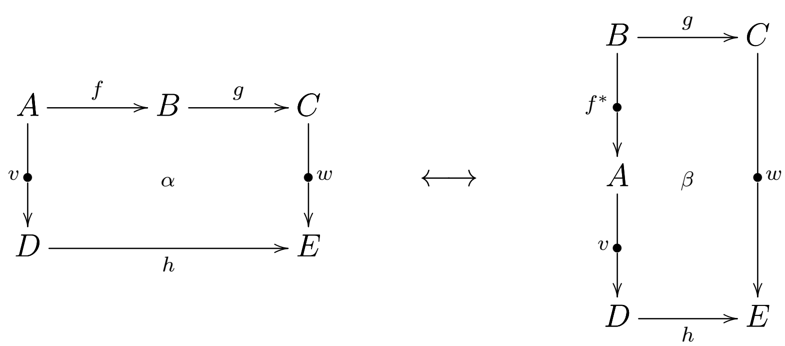


such that

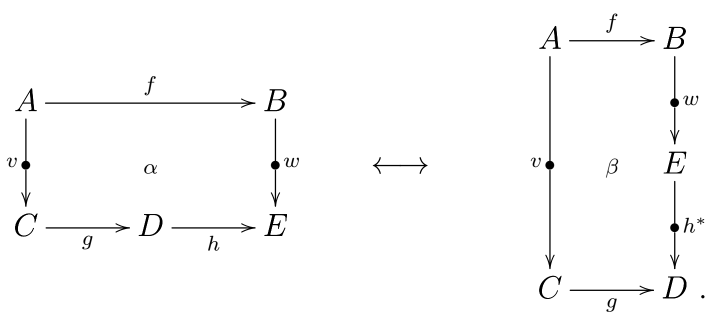


As we said, this is the vertical dual of the notion of companion and therefore has the corresponding properties. They are unique up to globular isomorphism when they exist and we choose representation that we call f^*. We have $(gf)^* \cong f^* \bullet g^*$ and $1_A^* \cong \mathrm{id}_A$. The choice is arbitrary but in practice there is a canonical one and for that 1_A^* is usually id_A, which we will assume.

The dual of sliding is *flipping*: we have bijections, natural in every way that makes sense,

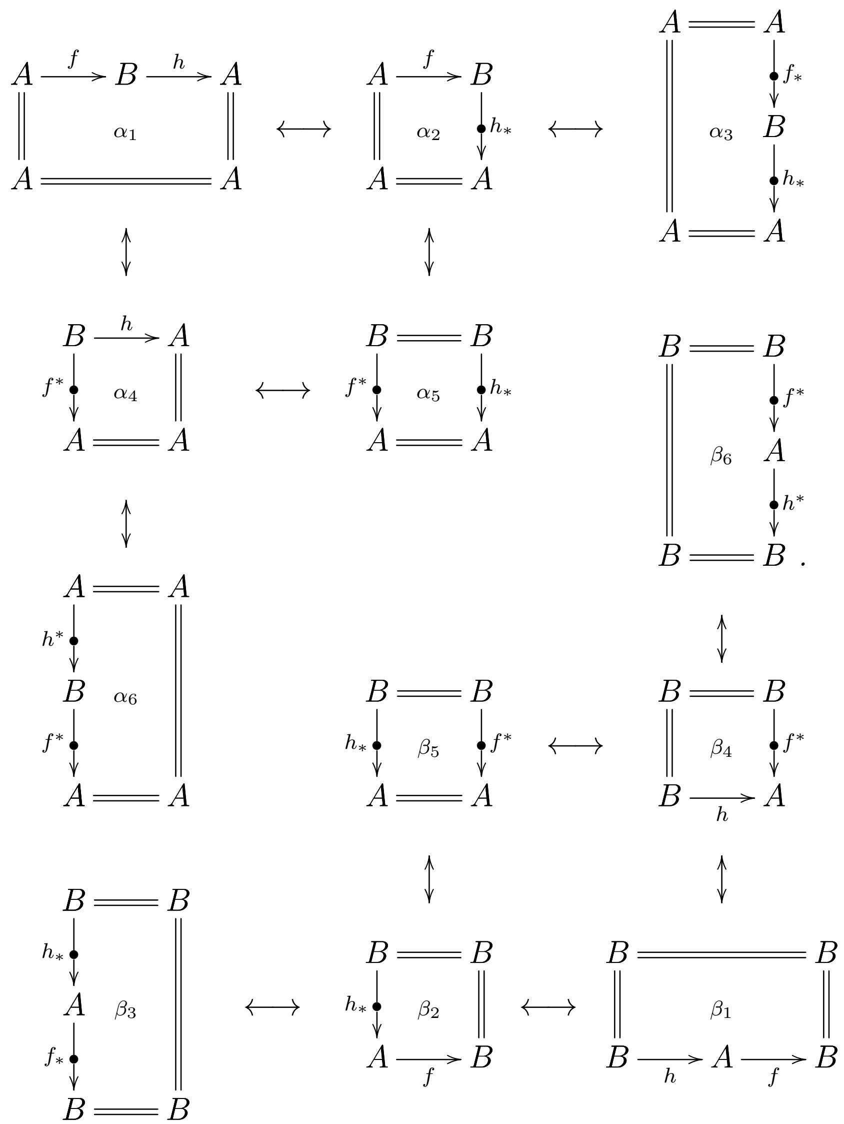


and



We now complete Proposition 3.1.

4.2. PROPOSITION. *Assuming only those companions and conjoints mentioned, we have the following natural bijections*



The following are then equivalent.

(1) *h is left adjoint to f with adjunctions α_1 and β_1*

(2) *h_* is a conjoint for f with conjunctions α_2 and β_2*

(3) *f_* is left adjoint to h_* with adjunctions α_3 and β_3*

(4) *f^* is a companion for h with binding cells α_4 and β_4*

(5) *f^* is isomorphic to h_* with inverse isomorphisms α_5 and β_5*

(6) *f^* is left adjoint to h^* with ajdunctions α_6 and β_6.*

4.3. Definition. *Suppose that in $\mathbb{A}$ every horizontal arrow f has a conjoint f^*, then a* coretrocell

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\big\downarrow\bullet & \Uparrow\alpha & \bullet\big\downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array}$$

is a (standard) cell

$$\begin{array}{ccc} B & = & B \\ {\scriptstyle w}\big\downarrow\bullet & & \bullet\big\downarrow{\scriptstyle f^*} \\ D & \alpha & A \\ {\scriptstyle g^*}\big\downarrow\bullet & & \bullet\big\downarrow{\scriptstyle v} \\ C & = & C \end{array}$$

in $\mathbb{A}$.

Coretrocells are retrocells in $\mathbb{A}^{co}$. So all properties of retrocells dualize to coretrocells. In particular we have a double category $\mathbb{A}^{cor}$ whose cells are coretrocells. Dualities can be confusing so we list them here.

4.4. Proposition. *(1) If $\mathbb{A}$ has conjoints then $\mathbb{A}^{op}$ and $\mathbb{A}^{co}$ have companions and*
(a) $(\mathbb{A}^{cor})^{op} = (\mathbb{A}^{op})^{ret}$
(b) $(\mathbb{A}^{cor})^{co} = (\mathbb{A}^{co})^{ret}$

(2) If $\mathbb{A}$ has companions then $\mathbb{A}^{op}$ and $\mathbb{A}^{co}$ have conjoints and
(a) $(\mathbb{A}^{ret})^{op} = (\mathbb{A}^{op})^{cor}$
(b) $(\mathbb{A}^{ret})^{co} = (\mathbb{A}^{co})^{cor}$

(3) Under the above conditions
(a) $(\mathbb{A}^{ret})^{coop} = (\mathbb{A}^{coop})^{ret}$
(b) $(\mathbb{A}^{cor})^{coop} = (\mathbb{A}^{coop})^{cor}$.

Passing between $\mathbb{A}$ and $\mathbb{A}^{co}$ switches left adjoints to right (both horizontal and vertical), switches companions and conjoints, and retrocells with coretrocells. Thus we get the dual theorem for mates.

4.5. Theorem. *Assume $\mathbb{A}$ has conjoints.*

(1) If x is right adjoint to v and y right adjoint to w, then there is a bijection between cells α and coretrocells β

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\big\downarrow\bullet & \alpha & \bullet\big\downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array} \qquad \longleftrightarrow \qquad \begin{array}{ccc} C & \xrightarrow{g} & D \\ {\scriptstyle x}\big\downarrow\bullet & \Uparrow\beta & \bullet\big\downarrow{\scriptstyle y} \\ A & \xrightarrow[f]{} & B \end{array}\ .$$

(2) If h is right adjoint to f and k right adjoint to g, then there is a bijection between cells α and coretrocells γ

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\big\downarrow & \alpha & \big\downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array} \qquad \longleftrightarrow \qquad \begin{array}{ccc} B & \xrightarrow{g} & A \\ {\scriptstyle w}\big\downarrow & \Uparrow\gamma & \big\downarrow{\scriptstyle v} \\ D & \xrightarrow[f]{} & C \end{array} .$$

Whereas we think of companions as vertical arrows isomorphic to horizontal ones, it makes sense to think of a cell α as above as a cell

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ C & \widehat{\alpha} & B \\ {\scriptstyle g_*}\big\downarrow & & \big\downarrow{\scriptstyle w} \\ D & = & D \end{array}$$

(which it corresponds to bijectively) and reversing its direction would give a natural notion of a cell in the opposite direction, thus giving retrocells. Coretrocells, on the other hand, are less intuitive. We think of conjoints as vertical arrows adjoint to horizontal ones, and although there is a bijection between cells α and cells

$$\begin{array}{ccc} B & = & B \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle w} \\ A & \alpha^\vee & D \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle g^*} \\ C & = & C \end{array} ,$$

this is more in the nature of a proposition than a tautology. Nevertheless, formally the two bijections are dual, so have the same status. Reversing the direction of the $\alpha^\vee$ gives us coretrocells, and they do come up in practice as we will see in the next sections.

5. Retrocells for spans and such

If $\mathbf{A}$ is a category with pullbacks, we get a double category $\mathbb{S}\mathrm{pan}\mathbf{A}$ whose horizontal part is $\mathbf{A}$, whose vertical arrows are spans and whose cells are span morphisms, modified to account for the horizontal arrows

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle S}\bullet\downarrow & \alpha & \downarrow\bullet{\scriptstyle T} \\ C & \xrightarrow[g]{} & D \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \sigma_0}\uparrow & & \uparrow{\scriptstyle \tau_0} \\ S & \xrightarrow{\alpha} & T \\ {\scriptstyle \sigma_1}\downarrow & & \downarrow{\scriptstyle \tau_1} \\ C & \xrightarrow[g]{} & D\ . \end{array}$$

$\mathbb{S}\mathrm{pan}\mathbf{A}$ has companions f_* and conjoints f^*:

$$f_* \;=\; \begin{array}{c} A \\ \uparrow{\scriptstyle 1_A} \\ A \\ \downarrow{\scriptstyle f} \\ B \end{array} \qquad \text{and} \qquad f^* \;=\; \begin{array}{c} B \\ \uparrow{\scriptstyle f} \\ A \\ \downarrow{\scriptstyle 1_A} \\ A\ . \end{array}$$

A retrocell β is

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle S}\bullet\downarrow & \overset{\beta}{\Leftarrow} & \downarrow\bullet{\scriptstyle T} \\ C & \xrightarrow[g]{} & D \end{array} \quad = \quad \begin{array}{ccc} A & = & A \\ {\scriptstyle p_2}\uparrow & & \uparrow{\scriptstyle \sigma_0} \\ T \times_B A & \xrightarrow{\beta} & S \\ {\scriptstyle \tau_1 p_1}\downarrow & & \downarrow{\scriptstyle g\sigma_1} \\ D & = & D \end{array}$$

where

$$\begin{array}{ccc} T \times_B A & \xrightarrow{p_2} & A \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow[\tau_0]{} & B \end{array}$$

is a pullback.

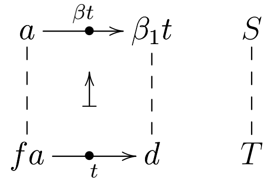
A coretrocell γ is

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle S}\bullet\downarrow & \Uparrow{\scriptstyle \gamma} & \downarrow\bullet{\scriptstyle T} \\ C & \xrightarrow[g]{} & D \end{array} \quad = \quad \begin{array}{ccc} B & = & B \\ {\scriptstyle \tau_0 p_2}\uparrow & & \uparrow{\scriptstyle f\sigma_0} \\ C \times_D T & \xrightarrow{\gamma} & S \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \sigma_1} \\ C & = & C\ . \end{array}$$

When $\mathbf{A} = \mathbf{Set}$ we can represent an element $s \in S$ with $\sigma_0 s = a$ and $\sigma_1 s = c$ by an arrow $a \xrightarrow{s} c$. Then a morphism of spans α is a function

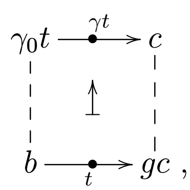
$$(a \xrightarrow{s} c) \longmapsto (fa \xrightarrow{\alpha(s)} gc).$$

For a retrocell β, an element of $T \times_B A$ is a pair $(b \xrightarrow{t} d, a)$ such that $fa = b$ so we can represent it as $fa \xrightarrow{t} d$. Then β is a function $(fa \xrightarrow{t} d) \longmapsto (a \xrightarrow{\beta t} \beta_1 t)$ with $g\beta_1 t = d$. If we picture S as lying over T (thinking of (co)fibrations) then β is a lifting: for every t we are given a βt



So it is like an opfibration but without any of the category structure around (in particular we cannot say that "βt is over t").

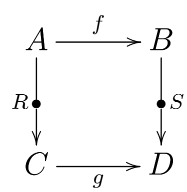
For a coretrocell γ, an element of $C \times_D T$ is a pair $(c, b \xrightarrow{t} d)$ with $gc = d$ which we can write as $b \xrightarrow{t} gc$. γ then assigns to such a t an S element $\gamma_0 t \xrightarrow{\gamma t} c$ with $f\gamma_0 t = b$, i.e. a lifting from T to S



much like a fibration, though without the category structure.

This example shows well the difference between retrocells and coretrocells and their comparison with actual cells.

The story for relations is much the same. If $\mathbf{A}$ is a regular category and



is a boundary in $\mathbb{R}\mathrm{el}\mathbf{A}$, i.e. f and g are morphisms and R and S are relations, then in the internal language of A, there is a (necessarily unique) cell iff

$$a \sim_R c \Rightarrow fa \sim_S gc \;,$$

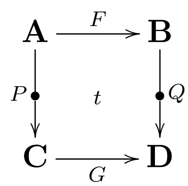
there is a retrocell iff

$$fa \sim_S d \Rightarrow \exists c (a \sim_R c \wedge gc = d)$$

and a coretrocell iff

$$b \sim_S gc \quad \Rightarrow \quad \exists a(a \sim_R c \wedge fa = b).$$

Profunctors are the relations of the $\mathcal{C}at$ world. There is a double category which we call $\mathbb{C}$at whose objects are small categories, horizontal arrows functors, vertical arrows profunctors, and cells the appropriate natural transformations. In a typical cell



t is a natural transformation $P(-,=) \longrightarrow Q(F-, G=)$. $\mathbb{C}$at has companions and conjoints:

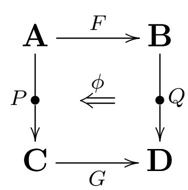
$$F_*(A,B) = \mathbf{B}(FA, B)$$
$$F^*(B,A) = \mathbf{B}(B, FA).$$

We denote an element $p \in P(A,C)$ by an arrow $p: A \longrightarrow\!\!\!\!\bullet\ C$. So the action of t is

$$t: (p: A \longrightarrow\!\!\!\!\bullet\ C) \longmapsto (tp: FA \longrightarrow\!\!\!\!\bullet\ GC)$$

natural in A and C, of course.

A retrocell



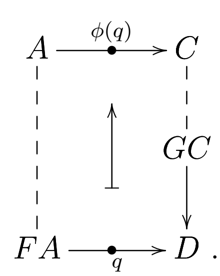
is a natural transformation $\phi: Q \otimes_{\mathbf{B}} F_* \longrightarrow G_* \otimes_{\mathbf{C}} P$. An element of $Q \otimes_{\mathbf{B}} F_*(A,D)$ is an element of $Q(FA,D)$, $g: FA \longrightarrow\!\!\!\!\bullet\ D$. An element of $G_* \otimes_{\mathbf{C}} P(A,D)$ is an equivalence class

$$[p: A \longrightarrow\!\!\!\!\bullet\ C, d: GC \longrightarrow D]_C.$$

So a retrocell assigns to each element of Q, $q: FA \longrightarrow D$, an equivalence class

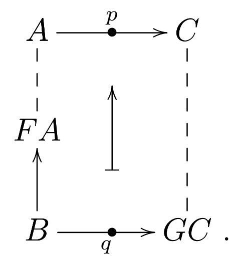
$$[\phi(q): A \longrightarrow\!\!\!\!\bullet\ C, \overline{\phi}(q): GC \longrightarrow D].$$

We can think of it as a lifting, like for spans

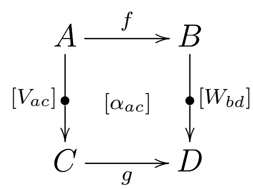


The lifting C does not lie over D, there is merely a comparison $GC \longrightarrow D$. Furthermore the lifting is not unique, but two liftings are connected by a zigzag of **C** morphisms. We have not spelled out the details because we do not know of any occurrences of these retrocells in print.

Coretrocells of profunctors are similar (dual). We get a "lifting"



A final variation on the span theme is **V**-matrices. Let **V** be a monoidal category with coproducts preserved by $\otimes$ in each variable separately. There is associated a double category which we call **V**-$\mathbb{S}$et. Its objects are sets and horizontal arrows functions. A vertical arrow $A \longrightarrow\!\!\!\!\!\bullet\;\, C$ is an $A \times C$ matrix of objects of **V**, $[V_{ac}]$. A cell is a matrix of morphisms



$$\alpha_{ac} \colon V_{ac} \longrightarrow W_{fa,gc}.$$

Vertical composition is matrix multiplication

$$[X_{ce}] \otimes [V_{ac}] = [\sum_{c \in C} X_{ce} \otimes V_{ac}].$$

Every horizontal arrow has a companion

$$f_* = [\Delta_{fa,b}]$$

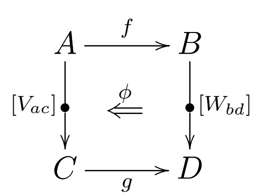
and a conjoint

$$f^* = [\Delta_{b,fa}]$$

where Δ is the "Kronecker delta"

$$\Delta_{b,b'} = \begin{cases} I & \text{if} \quad b = b' \\ 0 & \text{if} \quad b \neq b'. \end{cases}$$

A retrocell



is an $A \times D$ matrix $[\phi_{ad}]$

$$\phi_{ad}\colon W_{fa,d} \longrightarrow \sum_{gc=d} V_{ac}\ .$$

A coretrocell

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle [V_{ac}]}\downarrow & \Uparrow\psi & \downarrow{\scriptstyle [W_{bd}]} \\ C & \xrightarrow[g]{} & D \end{array}$$

is a $B \times C$ matrix $[\psi_{bc}]$

$$\psi_{bc}\colon W_{b,gc} \longrightarrow \sum_{fa=b} V_{ac}\ .$$

For example, if $\mathbf{V} = \mathbf{Ab}$, and we again represent elements of V_{ac} by arrows $a \xrightarrow{v} c$ (resp. of W_{bd} by $b \xrightarrow{w} d$), then ϕ associates to each $fa \xrightarrow{w} d$ a finite number of elements $a \xrightarrow{v_i} c_i$ with $gc_i = d$

$$\begin{array}{ccc} a & \xrightarrow{v_i} & c_i \\ | & \uparrow & | \\ fa & \xrightarrow[w]{} & d \end{array} \quad (i = 1, ..., n)\ .$$

Of course the dual situation holds for coretrocells ψ.

So we see that (co)retrocells in each case give liftings but of a type adapted to the situation. For spans they are uniquely specified, for relations they exist but are not specified, for profunctors only up to a connectedness condition and for matrices of Abelian groups we get a finite number of them.

6. Monads

A monad in $\mathcal{C}at$ is a quadruple $(\mathbf{A}, T, \eta, \mu)$ where $\mathbf{A}$ is a category, $T\colon \mathbf{A} \longrightarrow \mathbf{A}$ an endofunctor, $\eta\colon 1_{\mathbf{A}} \longrightarrow T$ and $\mu\colon T^2 \longrightarrow T$ natural transformations satisfying the well-known unit and associativity laws. In [19] Street introduced morphisms of monads

$$(F, \phi)\colon (\mathbf{A}, T, \eta, \mu) \longrightarrow (\mathbf{B}, S, \kappa, \nu)$$

as functors $F\colon \mathbf{A} \longrightarrow \mathbf{B}$ together with a natural transformation

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{F} & \mathbf{B} \\ {\scriptstyle T}\downarrow & \overset{\phi}{\swarrow} & \downarrow{\scriptstyle S} \\ \mathbf{A} & \xrightarrow[F]{} & \mathbf{B} \end{array}$$

respecting units and multiplications in the obvious way. He called these monad functors, now called lax monad morphisms (see [15]). This was done, not just in $\mathcal{C}at$, but in a general 2-category. Using duality, he also considered what he called monad opfunctors, i.e. oplax morphisms of monads, with the ϕ in the opposite direction.

The lax morphisms work well with Eilenberg-Moore algebras, giving a functor

$$\mathbf{EM}(F,\phi)\colon \mathbf{EM}(\mathbb{T}) \longrightarrow \mathbf{EM}(\mathbb{S})$$

$$(TA \xrightarrow{a} A) \longmapsto (SFA \xrightarrow{\phi A} FTA \xrightarrow{Fa} FA)$$

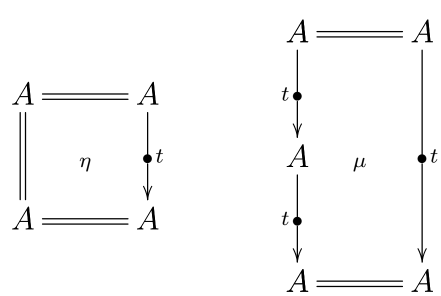
whereas the oplax ones give functors on the Kleisli categories

$$\mathbf{Kl}(F,\psi)\colon \mathbf{Kl}(\mathbb{T}) \longrightarrow \mathbf{Kl}(\mathbb{S})$$

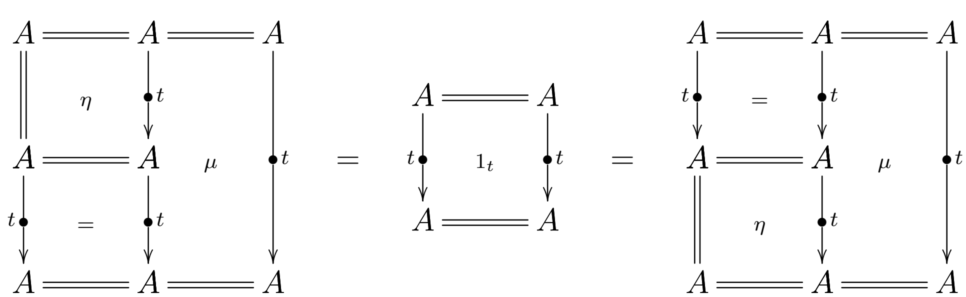
$$(A \xrightarrow{f} TB) \longmapsto (FA \xrightarrow{Ff} FTB \xrightarrow{\psi B} SFB).$$

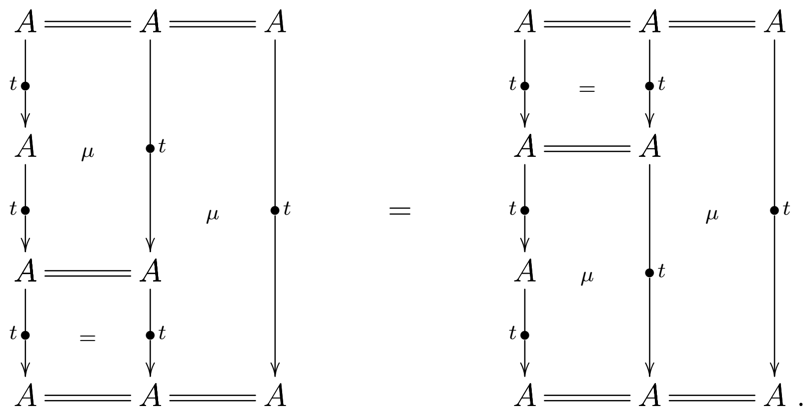
The story for monads in a double category is this (see [8, 7], though note that there horizontal and vertical are reversed). In general we just get one kind of morphism, the oplax ones. If we have companions then we also get the lax ones, and if we also have conjoints we have another kind. The 2-category case considered by Street corresponds to the double category of coquintets which has companions but not conjoints.

Let $\mathbb{A}$ be a double category. A vertical *monad* in $\mathbb{A}$, $t = (A, t, \eta, \mu)$ consists of an object A, a vertical endomorphism t and two cells η and μ as below

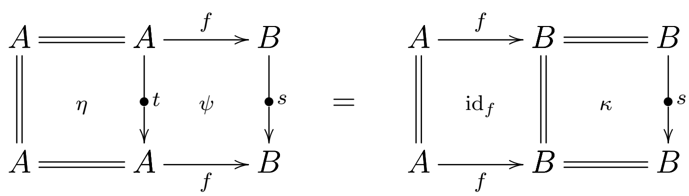


satisfying

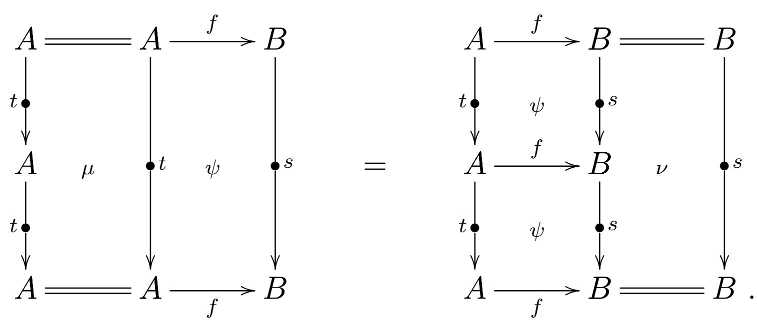




A (horizontal) *morphism of monads* $(f, \psi)\colon (A, t, \eta, \mu) \longrightarrow (B, s, \kappa, \nu)$ consists of a horizontal arrow f and a cell ψ as below, such that



and

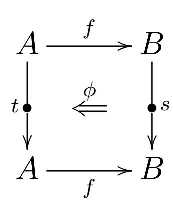


These are the oplax morphisms referred to above.

There are also vertical morphisms of monads, "bimodules", whose composition requires certain well-behaved coequalizers. They are interesting (see e.g. [18]), of course, but will not concern us here.

If $\mathbb{A}$ has companions we can also define retromorphisms of monads. (See [3, 16].)

6.1. Definition. *A* retromorphism of monads $(f, \phi)\colon (A, t, \eta, \mu) \longrightarrow (B, s, \kappa, \nu)$ *consists of a horizontal arrow* f *and a retrocell* ϕ



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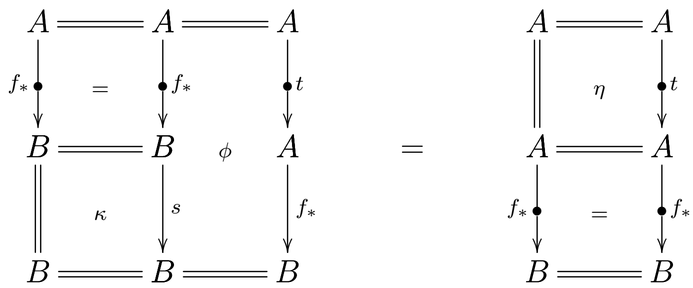
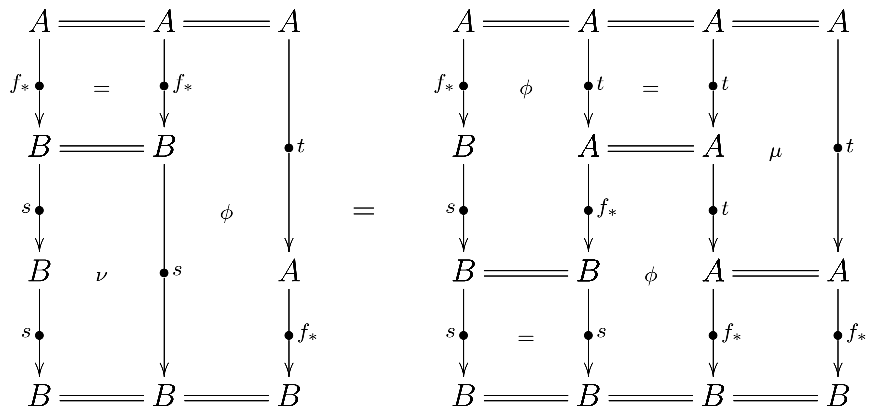




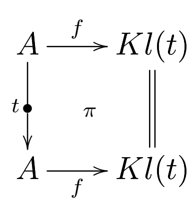
6.2. Proposition. *The identity retrocell is a retromorphism $(A, t, \eta, \mu) \longrightarrow (A, t, \eta, \mu)$. The composite of two retromorphisms of monads is again one.*

Proof. Easy calculation. ∎

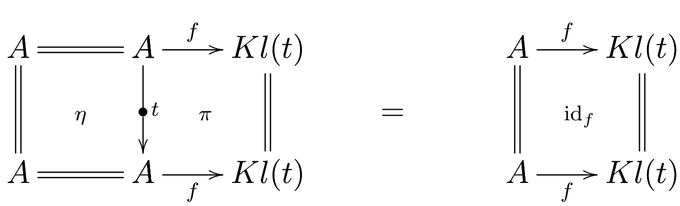
For a monad $t = (A, t, \eta, \mu)$, Kleisli is a colimit construction, a universal morphism of the form

$$(A, t, \eta, \mu) \longrightarrow (X, \mathrm{id}_X, 1, 1).$$

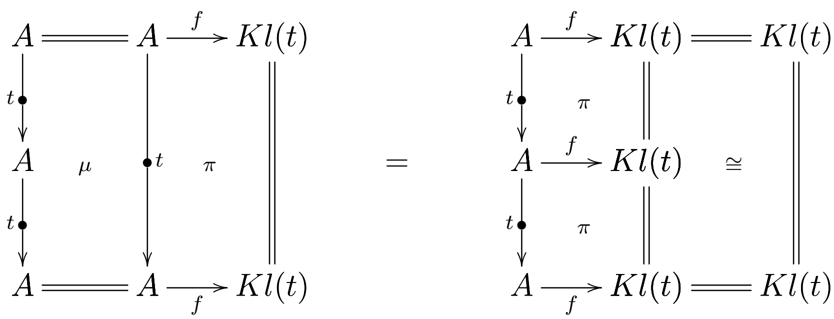
6.3. Definition. *The* Kleisli object *of a vertical monad in a double category, if it exists, is an object $Kl(t)$, a horizontal arrow f and a cell*



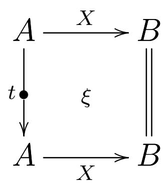
such that
(1)



(2)

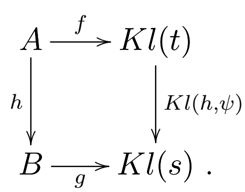


and universal with those properties. That is, for any



such that (1) $\xi\eta = \mathrm{id}$ *and (2)* $\xi\mu = \xi \cdot \xi$*, there exists a unique* $h\colon Kl(t) \longrightarrow B$ *such that (1)* $hf = x$ *and (2)* $h\pi = \xi$*.*

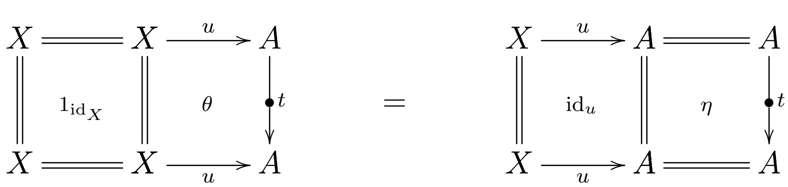
Just by universality, if we have a morphism of monads $(h, \psi)\colon (A, t, \eta, \mu) \longrightarrow (B, s, \kappa, \nu)$ and the Kleisli objects $Kl(t)$ and $Kl(s)$ exist, we get a horizontal arrow $Kl(h, \psi)$ such that



This does not work for Eilenberg-Moore objects. Asking for a universal morphism of the form

$$(X, \mathrm{id}_X, 1, 1) \longrightarrow (A, t, \eta, \mu)$$

is not the right thing as can be seen from the usual $\mathcal{C}at$ example, but also in general. For such a morphism (u, θ), the unit law says



i.e. θ must be ηu and this is a morphism. Thus monad morphisms (u, θ) are in bijection with horizontal arrows $X \longrightarrow A$. The universal such is 1_A, i.e. we get

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \| & \eta & \big\downarrow{\bullet}\, t \\ A & \xrightarrow[1_A]{} & A \end{array}$$

not the Eilenberg-Moore object.

6.4. Definition. *The* Eilenberg-Moore object *of a vertical monad* (A, t, η, μ) *is the universal retromorphism of monads*

$$(X, \mathrm{id}_X, 1, 1) \xrightarrow{(u,\theta)} (A, t, \eta, \mu)$$

$$\begin{array}{ccc} X & \xrightarrow{u} & A \\ \| & \overset{\theta}{\Leftarrow} & \big\downarrow{\bullet}\, t \\ X & \xrightarrow[u]{} & A\ . \end{array}$$

6.5. Proposition. *Let $\mathcal{A}$ be a 2-category and (A, t, η, μ) a monad in $\mathcal{A}$. Then (A, t, η, μ) is also a monad in the double category of coquintets* $\mathrm{co}\mathbb{Q}\mathcal{A}$*, and a retromorphism*

$$(u, \theta) \colon (X, \mathrm{id}_X, 1, 1) \longrightarrow (A, t, \eta, \mu)$$

is a 1-cell $u \colon X \longrightarrow A$ and a 2-cell $\theta \colon tu \longrightarrow u$ in $\mathcal{A}$ satisfying the unit and associativity laws for a t-algebra. The universal such is the Eilenberg-Moore object for t.

Proof. This is merely a question of interpreting the definition of retromorphism in $\mathrm{co}\mathbb{Q}\mathcal{A}$. ∎

We now see immediately how a retrocell $(f, \phi) \colon (A, t, \eta, \mu) \longrightarrow (B, s, \kappa, \nu)$ produces, by universality, a horizontal arrow

$$\begin{array}{ccc} EM(t) & \xrightarrow{EM(f,\phi)} & EM(s) \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle u'} \\ A & \xrightarrow[f]{} & B\ . \end{array}$$

6.6. Example. Let $\mathcal{A}$ be a 2-category and $\mathbb{Q}\mathcal{A}$ the double category of quintets in $\mathcal{A}$. Recall that a cell in $\mathbb{Q}\mathcal{A}$ is a quintet in $\mathcal{A}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\big\downarrow & \overset{\alpha}{\swarrow} & \big\downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D\ . \end{array}$$

Every horizontal arrow f has a companion, namely f itself but viewed as a vertical arrow. A (vertical) monad in $\mathbb{Q}\mathcal{A}$ is a comonad in $\mathcal{A}$. A morphism of monads in $\mathbb{Q}\mathcal{A}$ is then a lax morphism of comonads, and a retromorphism of monads in $\mathbb{Q}\mathcal{A}$ is an oplax morphism of comonads in $\mathcal{A}$.

To make the connection with Street's monad functors and opfunctors, we must take coquintets (the α in the opposite direction) co$\mathbb{Q}\mathcal{A}$. Now a monad in co$\mathbb{Q}\mathcal{A}$ is a monad in $\mathcal{A}$, a monad morphism in co$\mathbb{Q}\mathcal{A}$ is an oplax morphism of monads, i.e. a monad opfunctor in $\mathcal{A}$, whereas a retromorphism of monads is now a lax morphism of monads, i.e. a monad functor.

It is unfortunate that the most natural morphisms from a double category point of view are not the established ones in the literature. At the time of [19], people were more interested in the Eilenberg-Moore algebras for a monad as a generalization of Lawvere theories and their algebras, so it was natural to choose the monad morphisms that worked well with those, namely lax morphisms, as monad functors. Now, with the advent of categorical computer science, Kleisli categories have come into their own, and it is not so clear what the leading concept is, and double category theory suggests that it may well be the oplax morphisms.

6.7. Example. Let $\mathbf{C}$ be a category with (a choice of) pullbacks. As is well-known a monad in $\mathbb{S}\mathrm{pan}\mathbf{C}$ is a category object in $\mathbf{C}$. A morphism of monads in $\mathbb{S}\mathrm{pan}\mathbf{C}$ is an internal functor.

A retromorphism of monads

$$\begin{array}{ccc} A_0 & \xrightarrow{F} & B_0 \\ {\scriptstyle A_1}\big\downarrow\!\bullet & \overset{\phi}{\Leftarrow} & \bullet\!\big\downarrow{\scriptstyle B_1} \\ A_0 & \xrightarrow[F]{} & B_0 \end{array}$$

is first of all a morphism $F\colon A_0 \longrightarrow B_0$ and then a cell

$$\begin{array}{ccc} A_0 & = & A_0 \\ {\scriptstyle p_2}\big\uparrow & & \big\uparrow{\scriptstyle d_0} \\ B_1 \times_{B_0} A_0 & \xrightarrow{\phi} & A_0 \\ {\scriptstyle d_1 p_1}\big\downarrow & & \big\downarrow{\scriptstyle F d_1} \\ B_0 & = & B_0 \end{array}$$

which must satisfy the unit law

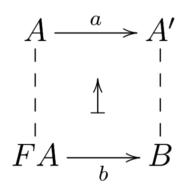
$$\begin{array}{ccc} A_0 & \xrightarrow{\langle \mathrm{id}F, 1_{A_0}\rangle} & B_1 \times_{B_0} A_0 \\ & {\scriptstyle \mathrm{id}}\searrow & \big\downarrow{\scriptstyle \phi} \\ & & A_1 \end{array}$$

and the composition law

$$
\begin{array}{ccccc}
B_1 \times_{B_0} B_1 \times_{B_0} A_0 & \xrightarrow{B_1 \times_{B_0} \phi} & B_1 \times_{B_0} A_1 & \xrightarrow{\cong} & B_1 \times_{B_0} A_0 \times_{A_0} A_1 \\
 & & & & \downarrow{\scriptstyle \phi \times_{A_0} A_1} \\
\downarrow{\scriptstyle \nu \times_{B_0} A_0} & & & & A_1 \times_{A_0} A_1 \\
 & & & & \downarrow{\scriptstyle \mu} \\
B_1 \times_{B_0} A_0 & & \xrightarrow[\phi]{\qquad\qquad} & & A_1 \; .
\end{array}
$$

This is precisely an internal cofunctor [1, 2].

When $\mathbf{C} = \mathbf{Set}$, a cofunctor $F \colon \mathbf{A} \longrightarrow\!\!\!\!\!\!+\;\, \mathbf{B}$ consists of an object function $F \colon \mathrm{Ob}\mathbf{A} \longrightarrow \mathrm{Ob}\mathbf{B}$ and a lifting function $\phi \colon (b \colon FA \longrightarrow B) \longmapsto (a \colon A \longrightarrow A')$ with $FA' = B$



satisfying
(1) (unit law) $\phi(A, 1_{FA}) = 1_A$
(2) (composition law)

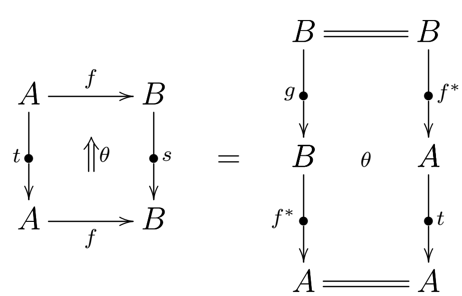
$$\phi(b'b, A) = \phi(b', A')\phi(b, A).$$

So F is like a split opfibration given algebraically but without the functor part.

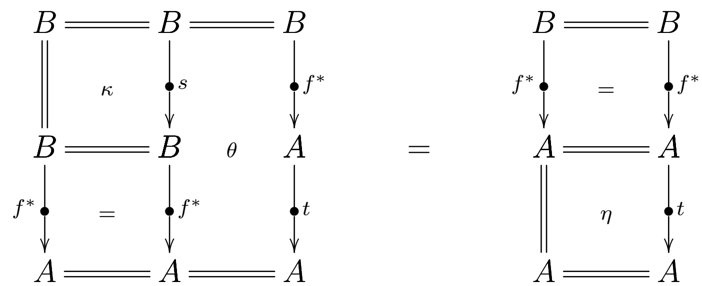
If $\mathbb{A}$ has conjoints, we can define coretromorphisms of monads as retromorphisms in $\mathbb{A}^{op}$ which now has companions, and monads in $\mathbb{A}^{op}$ are the same as monads in $\mathbb{A}$. Explicitly, an opretromorphism

$$(f, \theta) \colon (A, t, \eta, \mu) \longrightarrow (B, s, \kappa, \nu)$$

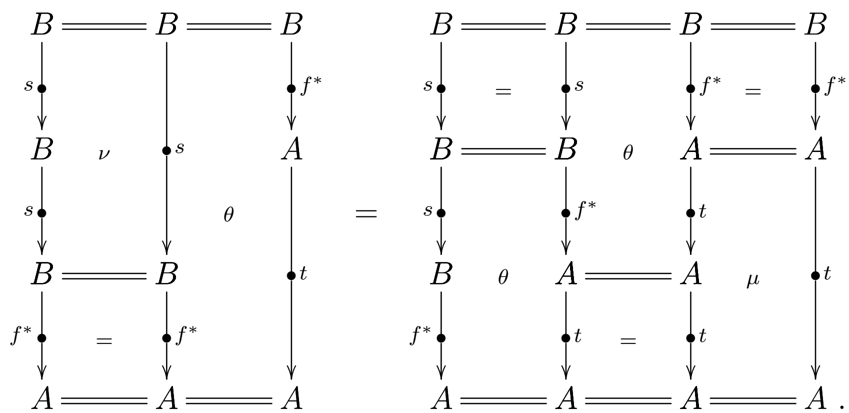
consists of a horizontal morphism $f \colon A \longrightarrow B$ in $\mathbb{A}$ and an opretromorphism θ



such that



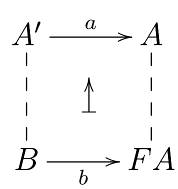
and



Coretromorphisms do not come up in the formal theory of monads because the double category of coquintets of a 2-category seldom has conjoints, but $\mathbb{S}\text{pan}\mathbf{C}$ does, and we get opcofunctors, i.e. cofunctors $\mathbf{A}^{op} \longrightarrow \mathbf{B}^{op}$. These consist of an object function $F\colon \text{Ob}\mathbf{A} \longrightarrow \text{Ob}\mathbf{B}$ and a lifting function

$$\theta\colon (b\colon B \longrightarrow FA) \longmapsto (a\colon A' \longrightarrow A)$$

with $FA' = A$



satisfying
(1) $\theta(A, 1_{FA}) = 1_A$
(2) $\theta(A, bb') = \theta(A, b)\theta(A', b')$.

This again illustrates well the difference between retromorphisms and coretromorphisms and, at the same time, the symmetry of the concepts. They all move objects forward. Functors move arrows forward

$$(a\colon A \longrightarrow A') \longmapsto (Fa\colon FA \longrightarrow FA'),$$

cofunctors move arrows of the form $FA \longrightarrow B$ backward

$$(b\colon FA \longrightarrow B) \longmapsto (\phi b\colon A \longrightarrow A')$$

and opcofunctors move arrows of the form $B \longrightarrow FA$ backward

$$(b\colon B \longrightarrow FA) \longmapsto (\theta b\colon A' \longrightarrow A).$$

All of this can be extended to the enriched setting for a monoidal category $\mathbf{V}$ which has coproducts preserved by the tensor in each variable. Then a monad in $\mathbf{V}$-$\mathbb{S}$et is exactly a small $\mathbf{V}$-category and the retromorphisms are exactly the enriched cofunctors of Clarke and Di Meglio [4], to which we refer the reader for further details.

7. Closed double categories

Many bicategories that come up in practice are closed, i.e. composition $\otimes$ has right adjoints in each variable,

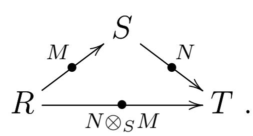
$$Q \otimes (-) \dashv Q \obslash (\ )$$
$$(\ ) \otimes P \dashv (\ ) \oslash P\ .$$

Thus we have bijections

$$\begin{array}{c} P \longrightarrow Q \obslash R \\ \hline Q \otimes P \longrightarrow R \\ \hline Q \longrightarrow R \oslash P \end{array}$$

We adapt (and adopt) Lambek's notation for the internal homs. $\otimes$ is a kind of multiplication and $\obslash$ and $\oslash$ divisions.

7.1. Example. The original example in [14], though not expressed in bicategorical terms, was $\mathcal{B}im$ the bicategory whose objects are rings, 1-cells bimodules and 2-cells linear maps. Composition is $\otimes$



(M is an S-R-bimodule, i.e. left S - right R bimodule, etc.) Given $P\colon R \longrightarrow\!\!\bullet\!\!\rightarrow T$, we have the usual bijections

$$\begin{array}{cc} N \longrightarrow P \oslash_R M & T\text{-}S \text{ linear} \\ \hline N \otimes_S M \longrightarrow P & T\text{-}R \text{ linear} \\ \hline M \longrightarrow N \obslash_T P & S\text{-}R \text{ linear} \end{array}$$

where

$$P \oslash_R M = \mathrm{Hom}_R(M, P)$$
$$N \obslash_T P = \mathrm{Hom}_T(N, P)$$

are the hom bimodules of R-linear (resp. T-linear) maps.

7.2. Example. The bicategory of small categories and profunctors is closed. For profunctors

$$\begin{array}{ccc} & \mathbf{B} & \\ {\scriptstyle P}\nearrow\!\!\bullet & & \bullet\!\!\searrow{\scriptstyle Q} \\ \mathbf{A} & \xrightarrow[R]{\bullet} & \mathbf{C} \end{array}$$

we have

$$(Q \oslash_{\mathbf{C}} R)(A, B) = \{n.t.\ Q(B, -) \longrightarrow R(A, -)\}$$

and

$$(R \oslash_{\mathbf{A}} P)(B, C) = \{n.t.\ P(-, B) \longrightarrow R(-, C)\}\ .$$

7.3. Example. If $\mathbf{A}$ has finite limits, then it is locally cartesian closed if and only if the bicategory of spans in $\mathbf{A}$, $\mathcal{S}pan\mathbf{A}$, is closed (Day [6]).

For spans $A \xleftarrow{p_0} R \xrightarrow{p_1} B$ and $B \xleftarrow{\tau_0} T \xrightarrow{\tau_1} C$, the composite is given by the pullback $T \times_B R$, which we could compute as the pullback P below and then composing with τ_1

$$\begin{array}{ccccc} R & \longleftarrow & P & & \\ \downarrow & PB & \downarrow & & \\ A \times B & \xleftarrow[A\times\tau_0]{} & A \times T & \xrightarrow[A\times\tau_1]{} & A \times C \end{array}$$

i.e. $T \otimes_B (\ )$ is the composite

$$\mathbf{A}/(A \times B) \xrightarrow{(A\times\tau_0)^*} \mathbf{A}/(A \times T) \xrightarrow{\sum_{A\times\tau_1}} \mathbf{A}/(A \times C)\ .$$

$\sum_{A\times\tau_1}$ always has a right adjoint $(A \times \tau_1)^*$ and if $\mathbf{A}$ is locally cartesian closed so will $(A \times \tau_0)^*$, namely $\prod_{A\times\tau_0}$. So, for $A \xleftarrow{\sigma_0} S \xrightarrow{\sigma_1} C$,

$$T \oslash_C S = \prod_{A\times\tau_0} (A \times \tau_1)^* S\ .$$

If we interpret this for $\mathbf{A} = \mathbf{Set}$, in terms of fibers

$$(T \oslash_C S)_{ab} = \prod_c S_{ac}^{T_{bc}}\ .$$

The situation for $\oslash_A$ is similar

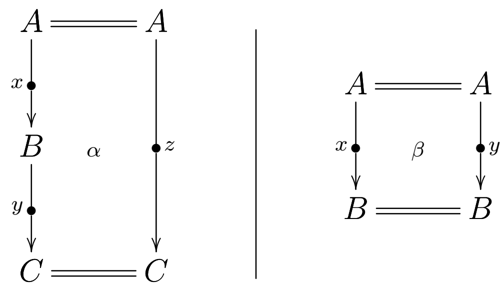
$$(S \oslash_A R)_{bc} = \prod_a S_{ac}^{R_{ab}}\ .$$

These bicategories, and in fact most bicategories that occur in practice, are the vertical bicategories of naturally occurring double categories. So a definition of a (vertically) closed double category would seem in order. And indeed Shulman in [18] did give one. A double category is closed if its vertical bicategory is. This definition was taken up by Koudenburg [13] in his work on pointwise Kan extensions. But both were working with "equipments", double categories with companions and conjoints. Something more is needed for general double categories.

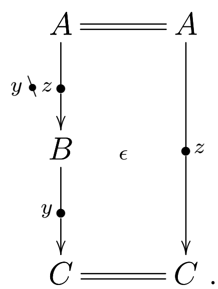
7.4. Definition. *(Shulman)* $\mathbb{A}$ *has* globular left homs *if for every* y, $y \bullet (\ )$ *has a right adjoint* $y \backslash\!\!\bullet (\ )$ *in* $\mathcal{V}ert\mathbb{A}$.

Thus for every z we have a bijection

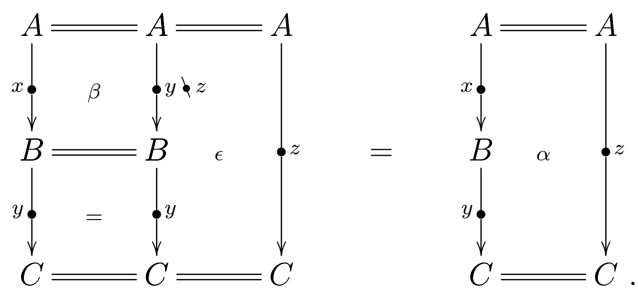
$$\frac{y \bullet x \longrightarrow z}{x \longrightarrow y \backslash\!\!\bullet z} \qquad \text{in } \mathcal{V}ert\mathbb{A}$$



Of course there is the usual naturality condition on x, which is guaranteed by expressing the above bijection as composition with an evaluation cell $\epsilon\colon y \bullet (y \backslash\!\!\bullet z) \longrightarrow z$

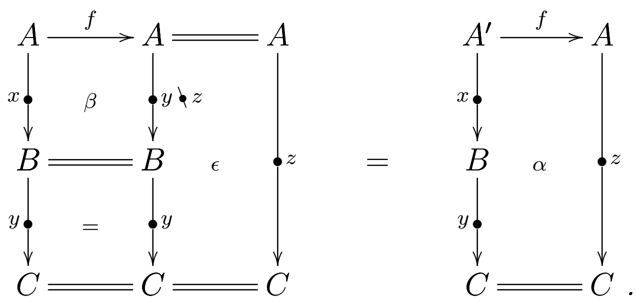


The universal property is then: for every α there is a unique β, as below, such that



This shows clearly that $\backslash\!\!\bullet$ has nothing to do with horizontal arrows, and the interplay between the horizontal and vertical is at the very heart of double categories.

7.5. Definition. $\mathbb{A}$ *has* strong left homs (is left closed) *if for every* y *and* z *as below there is a vertical arrow* $y \backslash\!\!\bullet z$ *and an evaluation cell* ϵ *such that for every* α *there is a* *unique β such that*

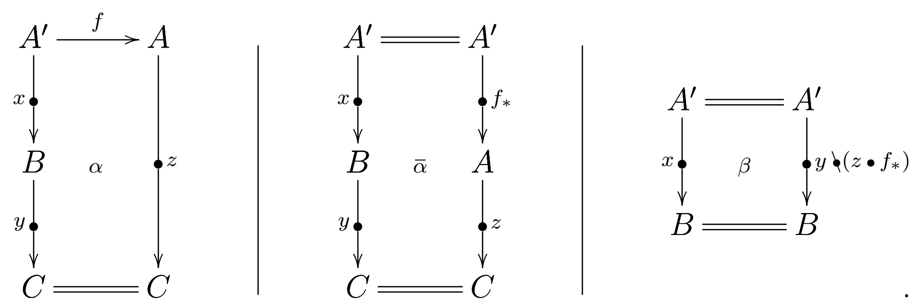


7.6. Proposition. *If $\mathbb{A}$ has companions and has globular left homs, then the strong universal property is equivalent to stability under companions: for every f, the canonical morphism*

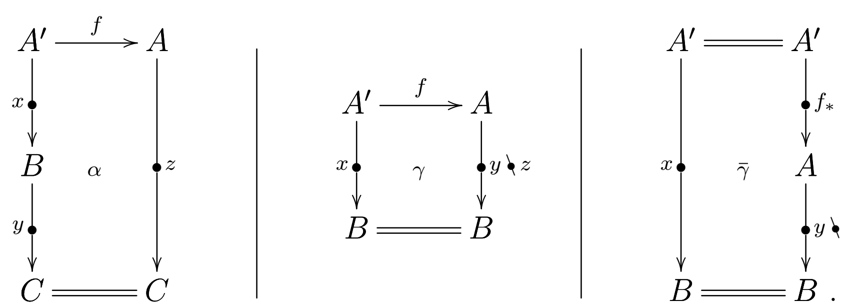
$$(y \backslash\!\!\!\bullet\, z) \bullet f_* \longrightarrow y \backslash\!\!\!\bullet\,(z \bullet f_*)$$

is an isomorphism.

Proof. (Sketch) For every f and x as below we have the following natural bijections of cells



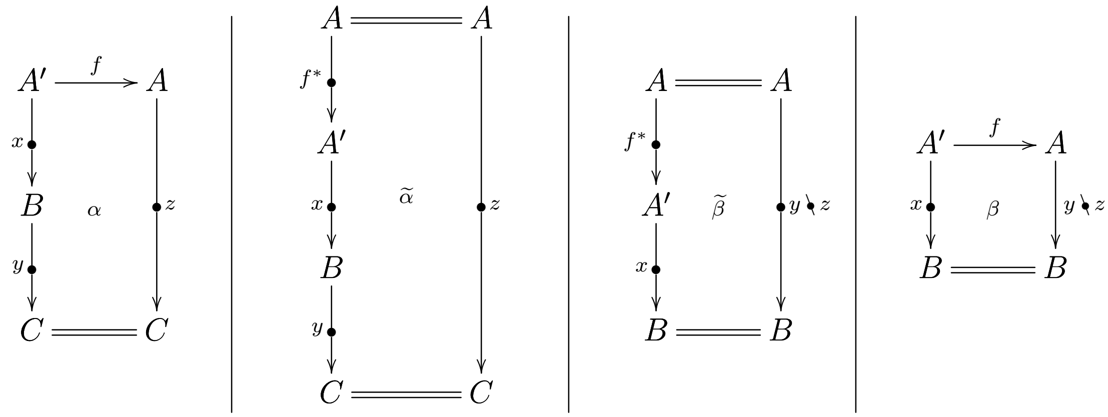
$y \backslash\!\!\!\bullet\, z$ is strong iff we have the following bijections



■

7.7. Proposition. *If $\mathbb{A}$ has conjoints, then the strong universal property is equivalent to the globular one.*

PROOF. (Sketch) For every f and x as below we have the following natural bijections



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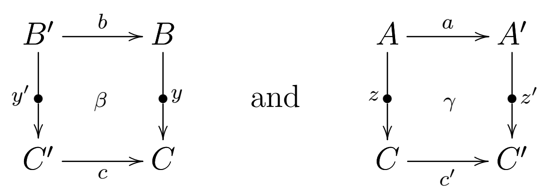
All of the examples above have conjoints so the left homs are automatically strong.

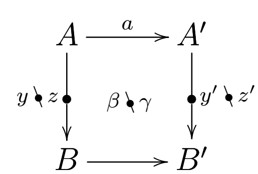
Of course, $y \backslash\!\!\bullet z$ is functorial in y and z, contravariant in y and covariant in z, but only for globular cells β, γ

$$y' \xrightarrow{\beta} y \quad \& \quad z \xrightarrow{\gamma} z' \quad \rightsquigarrow \quad y \backslash\!\!\bullet z \xrightarrow{\beta \backslash\!\!\bullet \gamma} y' \backslash\!\!\bullet z' \ .$$

For general double category cells β, γ

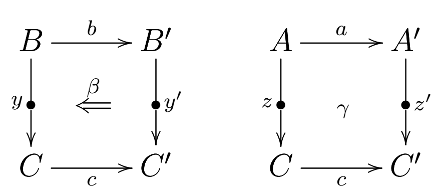


we would hope to get a cell



but b is in the wrong direction, and there are c and c' in opposite directions. If we reverse b and c then β is in the wrong direction. That was the motivation for retrocells.

7.8. PROPOSITION. *Suppose* $\mathbb{A}$ *has companions and is (strongly) left closed. Then a retrocell* β *and a standard cell* γ



induce a canonical cell

$$\begin{array}{ccc} A & \xrightarrow{a} & A' \\ {\scriptstyle y \diagdown z}\big\downarrow & \beta \diagdown \gamma & \big\downarrow{\scriptstyle y' \diagdown z'} \\ B & \xrightarrow[b]{} & B' \end{array} .$$

Proof. (Sketch) A candidate ξ for $\beta \diagdown \gamma$ would satisfy the following bijections

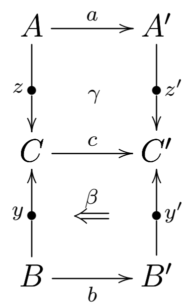
$$\begin{array}{ccc} A & \xrightarrow{a} & A' \\ {\scriptstyle y \diagdown z}\big\downarrow & \xi & \big\downarrow{\scriptstyle y' \diagdown z'} \\ B & \xrightarrow[b]{} & B' \end{array} \quad \Bigg| \quad \begin{array}{ccc} A & \xrightarrow{a} & A' \\ {\scriptstyle y \diagdown z}\big\downarrow & & \\ B & \bar{\xi} & \big\downarrow{\scriptstyle y' \diagdown z'} \\ {\scriptstyle b_*}\big\downarrow & & \\ B' & = & B' \end{array} \quad \Bigg| \quad \begin{array}{ccc} A & \xrightarrow{a} & A' \\ {\scriptstyle y \diagdown z}\big\downarrow & & \\ B & & \\ {\scriptstyle b_*}\big\downarrow & \bar{\bar{\xi}} & \big\downarrow{\scriptstyle z'} \\ B' & & \\ {\scriptstyle y'}\big\downarrow & & \\ C' & = & C' \end{array}$$

and there is indeed a canonical $\bar{\bar{\xi}}$, namely

$$\begin{array}{ccccccc} A & = & A & = & A & \xrightarrow{a} & A' \\ {\scriptstyle y \diagdown z}\big\downarrow & = & \big\downarrow{\scriptstyle y \diagdown z} & & & & \\ B & = & B & \epsilon & \big\downarrow{\scriptstyle z} & \gamma & \big\downarrow{\scriptstyle z'} \\ {\scriptstyle b_*}\big\downarrow & & \big\downarrow{\scriptstyle y} & & & & \\ B' & \beta & C & = & C & \xrightarrow{c} & C' \\ {\scriptstyle y'}\big\downarrow & & \big\downarrow{\scriptstyle c_*} & = & \big\downarrow{\scriptstyle c_*} & & \| \\ C' & = & C' & = & C' & = & C' \end{array} .$$

■

In fact the cell $\beta \diagdown \gamma$ is not only canonical but also functorial, i.e. $(\beta'\beta) \diagdown (\gamma'\gamma) = (\beta' \diagdown \gamma')(\beta \diagdown \gamma)$. To express this properly we must define the categories involved. The codomain of $\diagdown$ is simply $\mathbf{A}_1$, the category whose objects are vertical arrows of $\mathbb{A}$ and whose morphisms are (standard) cells. The domain of $\diagdown$ is the category which, for lack of a better name, we call $\mathbf{TC}(\mathbb{A})$ (twisted cospans). Its objects are cospans of vertical arrows and its cells are pairs (β, γ)

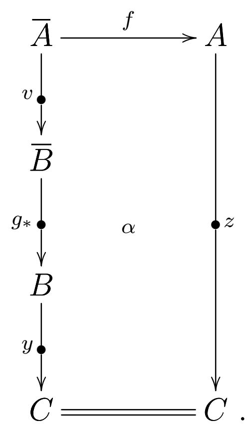


where β is a retrocell and γ a standard cell. Also we must flesh out our sketchy construction of $y \setminus\!\!\!\bullet\, z$. We can express the universal property of $y \setminus\!\!\!\bullet\, z$ as representability of a functor

$$L_{y,z} \colon \mathbf{A}_1^{op} \longrightarrow \mathbf{Set}\ .$$

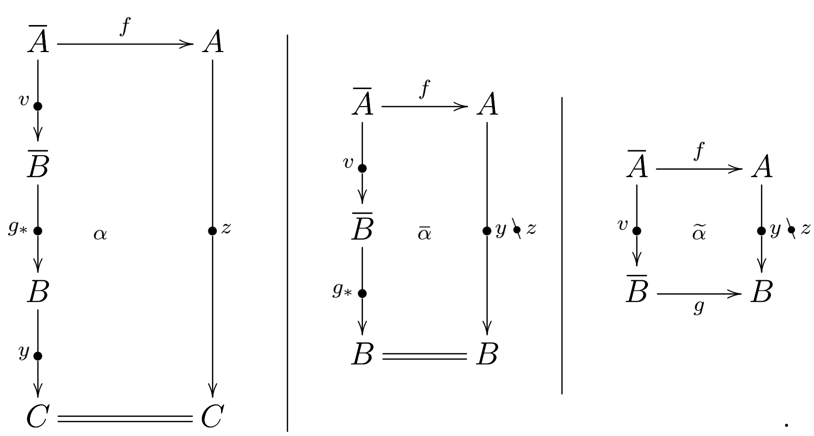
For $v \colon \overline{A} \longrightarrow\!\!\!\!\!\!\bullet\;\; \overline{B}$, $L_{y,z}(v) = \{(f, g, \alpha) | f, g, \alpha \ \text{ as in } (*)\}$

$$f \colon \overline{A} \longrightarrow A, g \colon \overline{B} \longrightarrow B$$



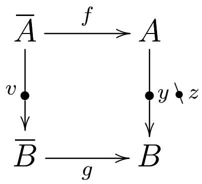
$(*)$

Some straightforward calculation will show that $L_{y,z}$ is indeed a functor. The following bijections show that $y \setminus\!\!\!\bullet\, z$ is a representing object for $L_{y,z}$

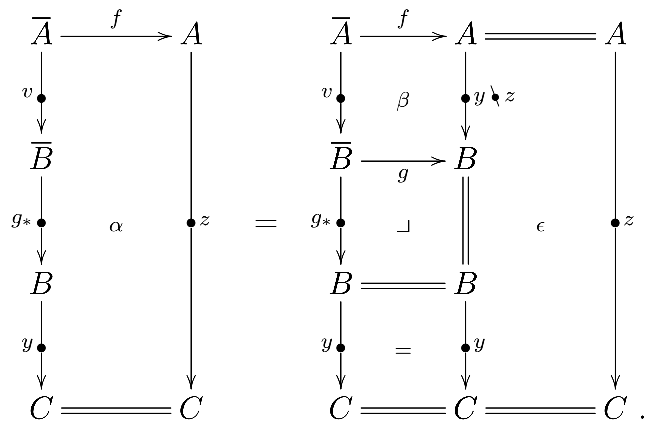


.

This gives the full double category universal property of $\backslash\!\!\!\bullet$: For every boundary

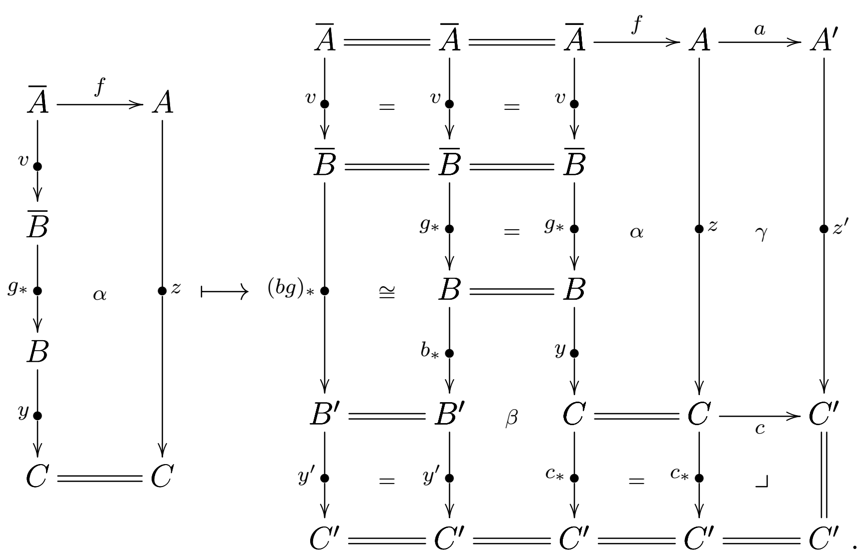


and α as below, there exists a unique fill-in β such that



For (β, γ) in $\mathbf{TC}(\mathbb{A})$ we get a natural transformation

$$\phi_{\beta\gamma} \colon L_{y,z} \longrightarrow L_{y',z'}$$



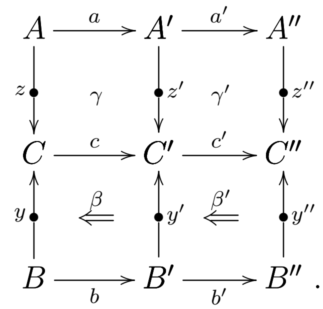
Some calculation is needed to show naturality, which we leave to the reader. This natural transformation is what gives $\beta \backslash\!\!\!\bullet\, \gamma$.

We are now ready for the main theorem of the section.

7.9. Theorem. *For $\mathbb{A}$ a left closed double category with companions, the internal hom is a functor*

$$\backslash\!\!\bullet : \mathbf{TC}(\mathbb{A}) \longrightarrow \mathbf{A}_1 \ .$$

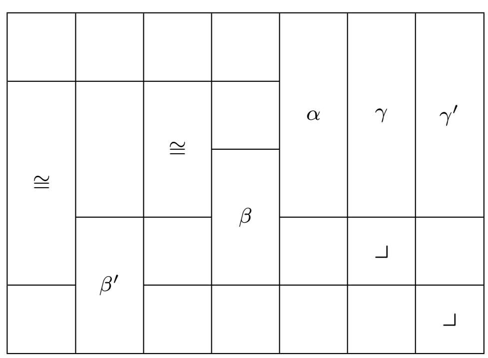
Proof. Let (β, γ) and (β', γ') be composable morphisms in $\mathbf{TC}(\mathbb{A})$



Then

$$L_{y,z} \xrightarrow{\phi_{\beta,\gamma}} L_{y',z'} \xrightarrow{\phi_{\beta',\gamma'}} L_{y'',z''}$$

takes v to the composite of 23 cells (most of which are bookkeeping – identities, canonical isos, ...) arranged in a 5×7 array, with 38 objects, and best represented schematically as

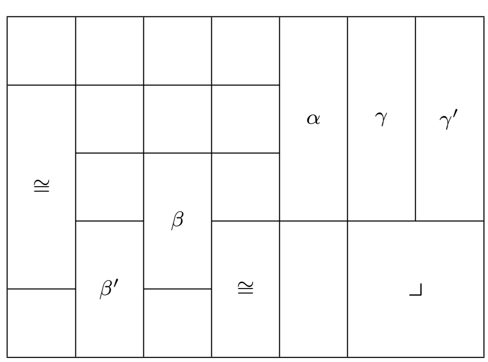


(*)

whereas

$$L_{yz} \xrightarrow{\phi_{\beta'\beta,\gamma'\gamma}} L_{y''z''}$$

takes v to



(**)

The three bottom right cells of (**) compose to the 2×2 block on the bottom right of (*), so the 5×3 part on the right of (*) is equal to the 5×4 part on the right of (**). And the rest are equal too by coherence. It follows that

$$(\beta' \setminus\!\!\!\bullet\, \gamma')(\beta \setminus\!\!\!\bullet\, \gamma) = (\beta'\beta) \setminus\!\!\!\bullet\, (\gamma'\gamma).$$

For identities $1_{y \setminus\!\!\!\bullet\, z} = 1_y \setminus\!\!\!\bullet\, 1_z$.

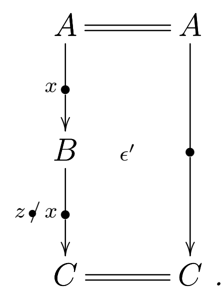
■

Right closure is dual but the duality is op, switching the direction of vertical arrows which switches companions with conjoints and retrocells with coretrocells. We outline the changes.

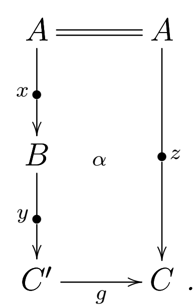
7.10. Definition. *(Shulman)* $\mathbb{A}$ *has* globular right homs *if for every* x, $(\ ) \bullet x$ *has a right adjoint* $(\ ) /\!\!\!\bullet\, x$ *in* $\mathcal{V}ert\mathbb{A}$,

$$\frac{y \bullet x \longrightarrow z}{y \longrightarrow z /\!\!\!\bullet\, x} \quad in \ \ \mathcal{V}ert\mathbb{A}.$$

This bijection is mediated by an evaluation cell



The right homs are strong *if* $z /\!\!\!\bullet\, x$ *has the universal property for cells of the form*

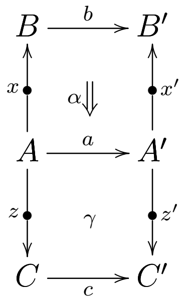


7.11. Proposition. *If* $\mathbb{A}$ *has conjoints and globular right homs, then the strong universal property is equivalent to the canonical morphism*

$$g^* \bullet (z /\!\!\!\bullet\, x) \longrightarrow (g^* \bullet z) /\!\!\!\bullet\, x$$

being an isomorphism. If instead $\mathbb{A}$ *has companions, then strong is equivalent to globular.*

Finally, if $\mathbb{A}$ has conjoints, $z \mathrel{\not\bullet} x$ is functorial in z and x, for standard cells in z and for coretrocells in x. More precisely, $\not\bullet$ is defined on the category $\mathbf{TS}(\mathbb{A})$ whose objects are spans of vertical arrows, (x, z), as below, and whose morphisms are pairs of cells



where α is a coretrocell and γ a standard one.

7.12. Theorem. *If* $\mathbb{A}$ *has conjoints and is right closed, then* $\not\bullet$ *is a functor*

$$\not\bullet : \mathbf{TS}(\mathbb{A}) \longrightarrow \mathbf{A}_1.$$

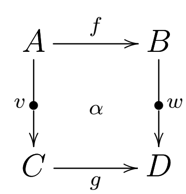
For completeness sake, we end this section with a definition.

7.13. Definition. *A double category* $\mathbb{A}$ *is closed if it is right closed and left closed.*

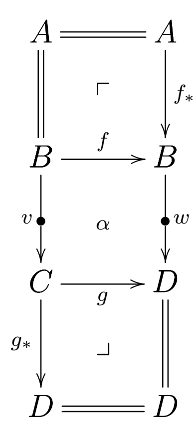
8. A triple category

As mentioned in the introduction, one of the inspirations for retrocells was the commuter cells of [11].

8.1. Definition. *Let* $\mathbb{A}$ *be a double category with companions. A cell*



is a commuter *cell if the associated globular cell* $\widehat{\alpha}$

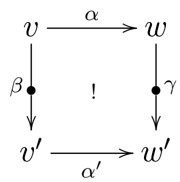


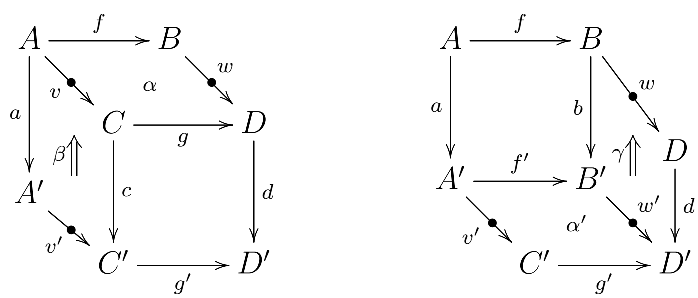
is a horizontal isomorphism.

The intent is that the cell α itself is an isomorphism making the square commute (up to isomorphism).

The inverse of $\widehat{\alpha}$ is a retrocell, so the question is, how do we express that a cell and a retrocell are inverse to each other?

Cells and retrocells form a double category (and ultimately a triple category). For a double category with companions $\mathbb{A}$, we define a new (vertical arrow) double category $\mathbb{V}\mathrm{ar}(\mathbb{A})$ as follows. Its objects are the vertical arrows of $\mathbb{A}$, its horizontal arrows are standard cells of $\mathbb{A}$, and its vertical arrows are retrocells. It is a thin double category with a unique cell

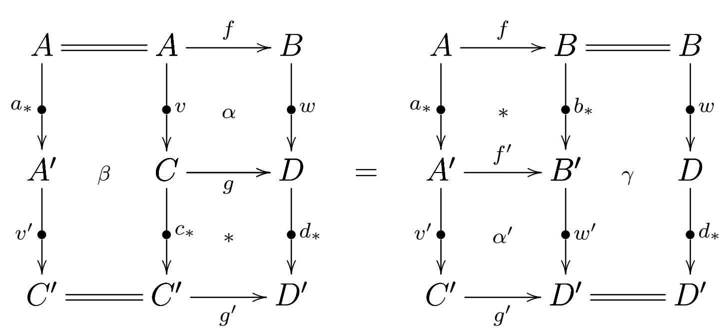




if we have

$$f'a = bf$$
$$g'c = dg\ ,$$

and

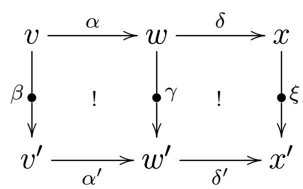


where the starred cells are the canonical ones gotten from the equations $g'c = dg$ and $f'a = bf$ by "sliding".

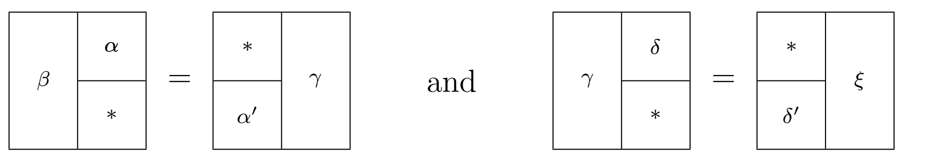
8.2. Proposition. $\mathbb{V}\mathrm{ar}(\mathbb{A})$ *is a strict double category.*

PROOF. We just have to check that cells compose horizontally and vertically. We simply give a sketch of the proof.

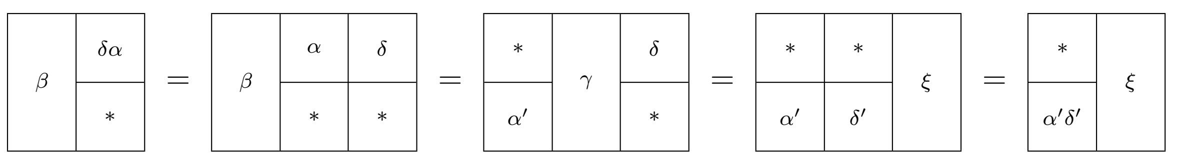
Suppose we have two cells,



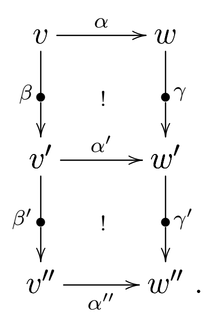
i.e. we have



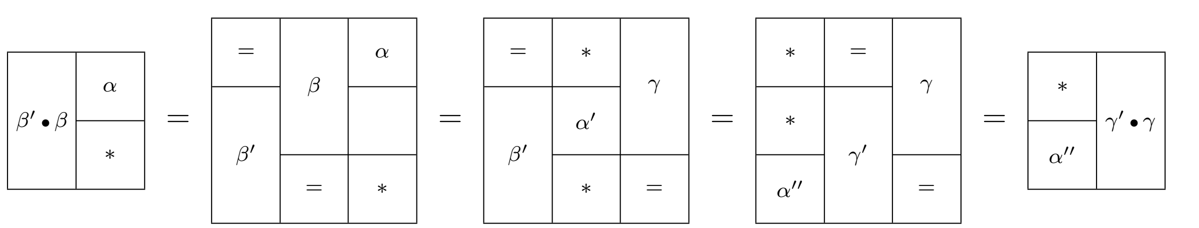
Thus



Consider cells



We did not say, but vertical composition of arrows in $\mathbb{V}\mathrm{ar}(\mathbb{A})$ is given by horizontal composition of retrocells. It could not be otherwise given their boundaries. Then we have the following



So horizontal and vertical composition of cells are again cells.

Identities pose no problem.

∎

8.3. PROPOSITION. *A cell*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\Big\downarrow\!\bullet & \alpha & \bullet\!\Big\downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array}$$

in $\mathbb{A}$ *is a commuter cell iff* $\alpha\colon v \longrightarrow w$ *has a companion in* $\mathbb{V}\mathrm{ar}\mathbb{A}$.

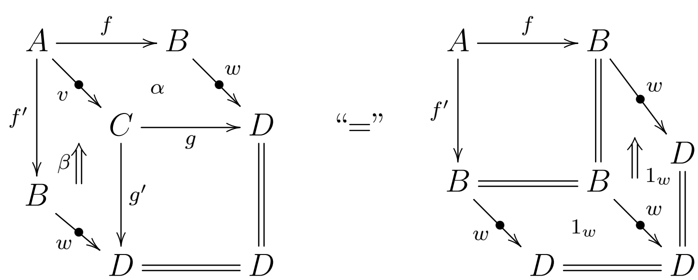
PROOF. A companion $\beta\colon v \multimap w$ for α will have cells

$$\begin{array}{ccc} v & \xrightarrow{\alpha} & w \\ {\scriptstyle \beta}\Big\downarrow\!\bullet & ! & \bullet\!\Big\downarrow{\scriptstyle \mathrm{id}_w} \\ w & = & w \end{array} \quad \text{and} \quad \begin{array}{ccc} v & = & v \\ {\scriptstyle \mathrm{id}_v}\Big\downarrow\!\bullet & ! & \bullet\!\Big\downarrow{\scriptstyle \beta} \\ v & \xrightarrow[\alpha]{} & w \end{array}$$

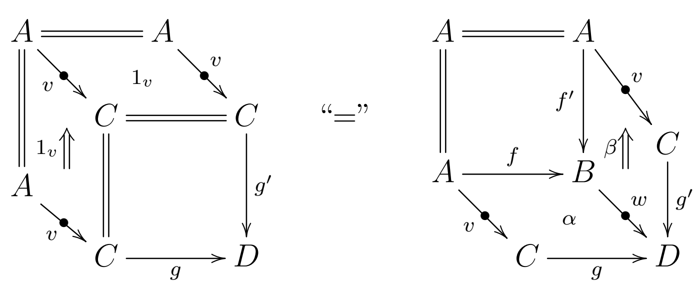
i.e. it is a retrocell

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ {\scriptstyle v}\Big\downarrow\!\bullet & \overset{\beta}{\Longleftarrow} & \bullet\!\Big\downarrow{\scriptstyle w} \\ C & \xrightarrow[g']{} & D \end{array}$$

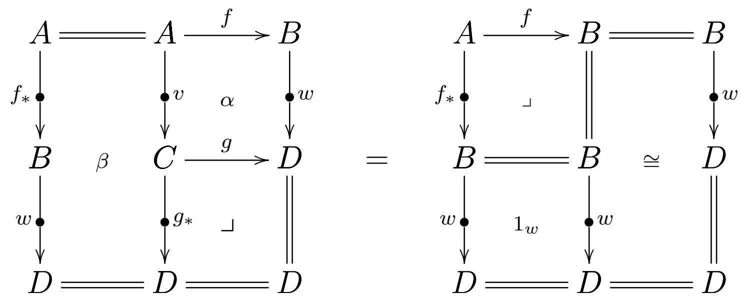
making the following cubes "commute"



and



So, first of all $f = f'$ and $g = g'$. The first "equation" says

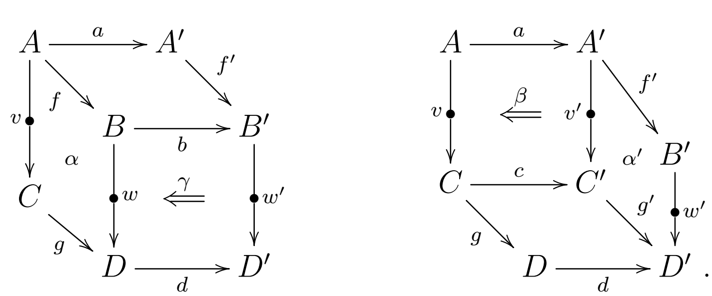


which by sliding is equivalent to $\widehat{\alpha}\beta = 1_{w \bullet f_*}$. Similarly the second equation says $\beta\widehat{\alpha} = 1_{g_* \bullet v}$.

■

We end by acknowledging the "triple category in the room". The cubes we have been discussing are clearly the triple cells of a triple category $\mathfrak{Ret}\mathbb{A}$. We orient the cubes to be in line with our intercategories conventions of [12] where the faces of the cubes are horizontal, vertical (left and right), and basic (front and back) in decreasing order of strictness (or fancyness). The order here will be commutative, cell, and retrocell.

1. Objects are the objects of $\mathbb{A}$, $(A, A', B, ..)$
2. Transversal arrows are the horizontal arrows of $\mathbb{A}$, (f, f', g, g')
3. Horizontal arrows are the horizontal arrows of $\mathbb{A}$, (a, b, c, d)
4. Vertical arrows are the vertical arrows of $\mathbb{A}$, (v, v', w, w')
5. Horizontal cells are commutative squares of horizontal arrows
6. Vertical cells are double cells in $\mathbb{A}$, (α, α')
7. Basic cells are retrocells in $\mathbb{A}$, (β, γ)
8. Triple cells are "commutative" cubes as discussed above



We leave the details to the interested reader.

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