

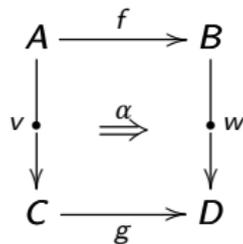
# The double category of Abelian groups

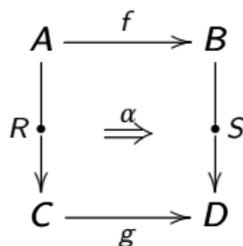
Robert Paré

Octoberfest

October 29, 2022

## Double categories





$A, B, C, D$  sets

$f, g$  functions

$R, S$  relations

$R \subseteq A \times C, S \subseteq B \times D$

$\alpha$  is implication:  $a \sim_R c \Rightarrow fa \sim_S gc$

A thin double category

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow S & \xRightarrow{\alpha} & \downarrow T \\
 C & \xrightarrow{g} & D
 \end{array}$$

$A, B, C, D$  sets

$f, g$  functions

$S, T$  spans

$\alpha$  morphism of spans:

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \uparrow & & \uparrow \\
 S & \xrightarrow{\alpha} & T \\
 \downarrow & & \downarrow \\
 C & \xrightarrow{g} & D
 \end{array}$$

Vertical composition uses pullback

Spans are “constructive relations”: instead of  $a \sim c$  we have  $a \xrightarrow{\bullet \xrightarrow{S}} c$

$$\begin{array}{ccc}
 R & \xrightarrow{f} & R' \\
 \downarrow M \bullet & \xRightarrow{\phi} & \downarrow M' \\
 S & \xrightarrow{g} & S'
 \end{array}$$

$R, S, R', S'$  rings

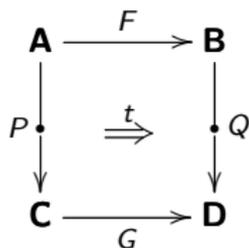
$f, g$  homomorphisms

$M, M'$  bimodules

$\phi$  linear: preserves  $+$  and  $\phi(smr) = g(s)\phi(m)f(r)$

Vertical composition is tensor

$$N \bullet M = N \otimes_S M$$



$A, B, C, D$  small categories

$F, G$  functors

$P, Q$  profunctors:  $P: \mathbf{A}^{op} \times \mathbf{C} \rightarrow \mathbf{Set}$ ,  $Q: \mathbf{B}^{op} \times \mathbf{D} \rightarrow \mathbf{Set}$

$t$  natural transformation:  $t: P(-, =) \rightarrow Q(F-, G=)$

Vertical composition is given by a coend, “matrix multiplication”

Are there useful vertical morphisms of Abelian groups?

What are we looking for?

- Come up in practice (homology, cohomology, representation theory)
- Have good double category properties
- Relate well to other double categories,  $\mathbf{Set}$ ,  $\mathbf{Ring}$ ,  $R\text{-Mod}$  (?), Graded Abelian groups, ...
- $\text{Hom}$ ,  $\otimes$
- etc. ???

Because  $\mathbf{Ab}$  is a nice category

1.  $\text{Span}(\mathbf{Ab})$
2.  $\mathbb{R}\text{el}(\mathbf{Ab})$  (Puppe, Mac Lane, Grandis, Zanasi, ...)
3.  $\text{coRel}(\mathbf{Ab})$
4.  $\text{coSpan}(\mathbf{Ab})$  (Fong, Baez, ...)

## Candidates II

Because **Ab** relates well to other categories

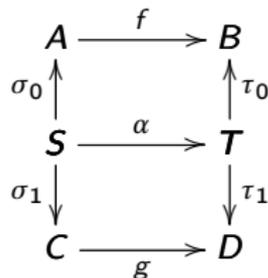
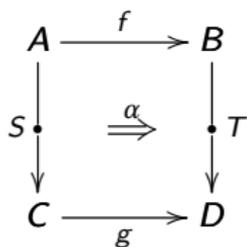
5. Affine superspans (P. Octoberfest 2015)
6. Affine relations (Zanasi)
7. Affine corelations
8. Affine structured cospans (Fong)
9.  $\mathbb{K}l(\mathbb{G})$   $\mathbb{G}$  = free Abelian group comonad in **Ab**
10.  $\mathbb{K}l(\mathbb{T})$   $\mathbb{T}$  = tensor algebra monad in **Ab**

## Candidates III

Because an Abelian group is itself a category

11. An Abelian group is a monoid (so a category), and we can take profunctors  
Get sets with left and right actions (Egger)
12. An Abelian group is a monoid in **Group**  
We get groups with left and right actions
13. An Abelian group is a monoid in **Ab**  
We get Abelian group with left and right actions  
⚠ Actions are with respect to  $\oplus$ ;  $A \oplus X \rightarrow A$   
These are equivalent to cospans (Egger)
14. An Abelian group is a group category (i.e. a group in **Cat**)  
We can consider a category with left and right actions (Daniel Teixeira)
15. An Abelian group is a discrete monoidal closed category – monoidal profunctors

# Span(Ab)



For  $s \in S$  with  $\sigma_0(s) = a$  and  $\sigma_1(s) = c$  write  $s: a \dashrightarrow c$

We have

$$s: a \dashrightarrow c \quad \& \quad s': a' \dashrightarrow c' \quad \rightsquigarrow \quad s+s': a+a' \dashrightarrow c+c'$$

$$0: 0 \dashrightarrow 0$$

$$-s: -a \dashrightarrow -c$$

$$\alpha(s): f(a) \dashrightarrow g(c)$$

## Just because $\mathbf{Ab}$ has pullbacks

$\mathbb{S}\text{pan}(\mathbf{Ab})$  has:

- Companions  $f_*$  and conjoints  $f^*$

$$f_* = (A \xleftarrow{1_A} A \xrightarrow{f} B)$$

$$\frac{a \xrightarrow{\bullet} b}{f(a) = b}$$

$$f^* = (B \xleftarrow{f} A \xrightarrow{1_A} A)$$

$$\frac{b \xrightarrow{\bullet} a}{b = f(a)}$$

- Tabulators

$$\begin{array}{ccc} & & A \\ & \nearrow t_0 & \downarrow S \\ \text{Tab}(S) & \tau & S \\ & \searrow t_1 & \downarrow B \end{array} = \begin{array}{ccc} S & \xrightarrow{\sigma_0} & A \\ \parallel & & \uparrow \sigma_0 \\ S & \xrightarrow{1} & S \\ \parallel & & \downarrow \sigma_1 \\ S & \xrightarrow{\sigma_1} & B \end{array}$$

Effective ( $S = t_{1*} \bullet t_0^*$ ) and unitary (tetrahedron)

## Still because **Ab** has pullbacks

- Cauchy: Adjoint spans are representable

$$S \dashv R \text{ is } f_* \dashv f^*$$

- Dagger structure

$$\begin{array}{ccc} A & & B \\ \sigma_0 \uparrow & & \uparrow \sigma_1 \\ S & \xrightarrow{\dagger} & S \\ \sigma_1 \downarrow & & \downarrow \sigma_0 \\ B & & A \end{array}$$

## Because $\mathbf{Ab}$ has finite limits

$\text{Span}(\mathbf{Ab})$  is a Cartesian double category (Aleiferi)

In fact  $\oplus: \text{Span}(\mathbf{Ab}) \times \text{Span}(\mathbf{Ab}) \longrightarrow \text{Span}(\mathbf{Ab})$  is both a left and a right double adjoint to the diagonal

## Because **Ab** has pushouts

$\text{Span}(\mathbf{Ab})$  has cotabulators

Unitary

But effective iff  $\sigma_0, \sigma_1$  are jointly monic

Because **Ab** is preadditive (an **Ab**-category)

We can add horizontal arrows *and* cells

$$\begin{array}{ccc} A & \xrightarrow{f+f'} & B \\ \uparrow & & \uparrow \\ S & \xrightarrow{\alpha+\alpha'} & T \\ \downarrow & & \downarrow \\ C & \xrightarrow{g+g'} & D \end{array}$$

## Adjoint pairs extend to spans

The adjoint pair  $\mathbf{Ab} \begin{array}{c} \xrightarrow{U} \\ \xleftarrow{F} \end{array} \mathbf{Set}$  extends to an oplax/strict double adjunction

$$\mathbf{Span}(\mathbf{Ab}) \begin{array}{c} \xrightarrow{U} \\ \xleftarrow{F} \end{array} \mathbf{Set}$$

which is double monadic

But there doesn't seem to be any good adjunction

$$\mathbf{Ring} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathbf{Span}(\mathbf{Ab})$$



The tensor product functor  $\otimes: \mathbf{Ab} \times \mathbf{Ab} \rightarrow \mathbf{Ab}$  extends to an oplax normal double functor

$$\otimes: \mathbb{S}\text{pan}(\mathbf{Ab}) \times \mathbb{S}\text{pan}(\mathbf{Ab}) \rightarrow \mathbb{S}\text{pan}(\mathbf{Ab})$$

$$\begin{array}{ccc} \begin{array}{c} A \\ \uparrow \\ S \\ \downarrow \\ C \end{array} & \begin{array}{c} B \\ \uparrow \\ T \\ \downarrow \\ D \end{array} & \xrightarrow{\otimes} \\ & & \begin{array}{c} A \otimes B \\ \uparrow \\ S \otimes T \\ \downarrow \\ C \otimes D \end{array} \end{array}$$

Associative and unitary but not a strong double functor

We don't get a monoidal double category in the sense of Shulman

$$\begin{array}{ccc}
 A & B & \\
 \uparrow & \uparrow & \\
 S & T & \xrightarrow{\text{Hom}} \\
 \downarrow & \downarrow & \\
 C & D & \\
 & & \text{Ab}(A, B) \\
 & & \uparrow \\
 & & \text{Hom}(S, T) \\
 & & \downarrow \\
 & & \text{Ab}(C, D)
 \end{array}$$

$\text{Hom}(S, T)$  is the Abelian group of all cells

$$\begin{array}{ccc}
 A & \longrightarrow & B \\
 \uparrow & & \uparrow \\
 S & \longrightarrow & T \\
 \downarrow & & \downarrow \\
 D & \longrightarrow & D
 \end{array}$$

## ⊗ – Hom adjointness

### Theorem

*Hom is a lax normal double functor  $\text{Span}(\mathbf{Ab})^{\text{op}} \times \text{Span}(\mathbf{Ab}) \longrightarrow \text{Span}(\mathbf{Ab})$ .*

*For each Abelian group we have an oplax/lax double adjunction  $(\ ) \otimes A \dashv \text{Hom}(A, -)$ .*