The double category of Abelian groups

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Double categories

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] & B \arrow[d, "w", "\bullet" marking] \\
C \arrow[r, "g"'] & D
\arrow[from=1-1, to=2-2, phantom, "\overset{\alpha}{\Longrightarrow}"]
\end{tikzcd}$$

$\mathbb{R}$el

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "R"', "\bullet" marking] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & B \arrow[d, "S", "\bullet"' marking] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

A, B, C, D sets

f, g functions

R, S relations

$R \subseteq A \times C,\ S \subseteq B \times D$

α is implication: $\quad a \sim_R c \Rightarrow fa \sim_S gc$

A thin double category

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "S"', "\bullet" marking] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & B \arrow[d, "T", "\bullet" marking] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

A, B, C, D sets

f, g functions

S, T spans

α morphism of spans:

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ S \arrow[u] \arrow[d] \arrow[r, "\alpha"] & T \arrow[u] \arrow[d] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Vertical composition uses pullback

Spans are "constructive relations": instead of $a \sim c$ we have $a \xrightarrow{s} c$

$\mathbb{R}$ing

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "M"', "\bullet" marking] \arrow[dr, phantom, "\overset{\phi}{\Longrightarrow}"] & R' \arrow[d, "M'", "\bullet" marking] \\ S \arrow[r, "g"'] & S' \end{tikzcd}$$

R, S, R', S' rings

f, g homomorphisms

M, M' bimodules

ϕ linear: preserves + and $\phi(smr) = g(s)\phi(m)f(r)$

Vertical composition is tensor

$$N \bullet M = N \otimes_S M$$

$$
\begin{tikzcd}
\mathbf{A} \arrow[r, "F"] \arrow[d, "P"', "\bullet" marking] \arrow[dr, phantom, "\overset{t}{\Rightarrow}"] & \mathbf{B} \arrow[d, "Q", "\bullet"' marking] \\
\mathbf{C} \arrow[r, "G"'] & \mathbf{D}
\end{tikzcd}
$$

$\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ small categories

F, G functors

P, Q profunctors: $\quad P \colon \mathbf{A}^{op} \times \mathbf{C} \longrightarrow \mathbf{Set}, \quad Q \colon \mathbf{B}^{op} \times \mathbf{D} \longrightarrow \mathbf{Set}$

t natural transformation: $\quad t \colon P(-,=) \longrightarrow Q(F-, G=)$

Vertical composition is given by a coend, “matrix multiplication”

$\mathbb{A}\mathrm{b}$

Are there useful vertical morphisms of Abelian groups?

What are we looking for?

- Come up in practice (homology, cohomology, representation theory)
- Have good double category properties
- Relate well to other double categories, $\mathbb{S}\mathrm{et}$, $\mathbb{R}\mathrm{ing}$, R-Mod (?), Graded Abelian groups, ...
- Hom, $\otimes$
- etc. ???

Candidates I

Because **Ab** is a nice category

1. $\mathbb{S}\mathrm{pan}(\mathbf{Ab})$
2. $\mathbb{R}\mathrm{el}(\mathbf{Ab})$ (Puppe, Mac Lane, Grandis, Zanasi, ...)
3. $\mathrm{co}\,\mathbb{R}\mathrm{el}(\mathbf{Ab})$
4. $\mathrm{co}\,\mathbb{S}\mathrm{pan}(\mathbf{Ab})$ (Fong, Baez, ...)

Candidates II

Because **Ab** relates well to other categories

5. Affine superspans (P. Octoberfest 2015)
6. Affine relations (Zanasi)
7. Affine corelations
8. Affine structured cospans (Fong)
9. $\mathbb{K}\mathrm{l}(\mathbb{G})$ $\quad \mathbb{G} =$ free Abelian group comonad in **Ab**
10. $\mathbb{K}\mathrm{l}(\mathbb{T})$ $\quad \mathbb{T} =$ tensor algebra monad in **Ab**

Candidates III

Because an Abelian group is itself a category

11. An Abelian group is a monoid (so a category), and we can take profunctors
    Get sets with left and right actions (Egger)

12. An Abelian group is a monoid in **Group**
    We get groups with left and right actions

13. An Abelian group is a monoid in **Ab**
    We get Abelian group with left and right actions
    ⚠ Actions are with respect to $\oplus$; $A \oplus X \longrightarrow A$
    These are equivalent to cospans (Egger)

14. An Abelian group is a group category (i.e. a group in **Cat**)
    We can consider a category with left and right actions (Daniel Teixeira)

15. An Abelian group is a discrete monoidal closed category – monoidal profunctors

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "S"', "\bullet" marking] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & B \arrow[d, "T", "\bullet"' marking] \\ C \arrow[r, "g"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] & B \\ S \arrow[u, "\sigma_0"] \arrow[r, "\alpha"] \arrow[d, "\sigma_1"'] & T \arrow[u, "\tau_0"'] \arrow[d, "\tau_1"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

For $s \in S$ with $\sigma_0(s) = a$ and $\sigma_1(s) = c$ write $s\colon a \xrightarrow{\bullet} c$

We have

$$s\colon a \xrightarrow{\bullet} c \quad \& \quad s'\colon a' \xrightarrow{\bullet} c' \qquad \rightsquigarrow \qquad s+s'\colon a+a' \xrightarrow{\bullet} c+c'$$

$$0\colon 0 \xrightarrow{\bullet} 0$$

$$-s\colon -a \xrightarrow{\bullet} -c$$

$$\alpha(s)\colon f(a) \xrightarrow{\bullet} g(c)$$

Just because **Ab** has pullbacks

$\mathbb{S}\mathrm{pan}(\mathbf{Ab})$ has:

- Companions f_* and conjoints f^*

$$f_* = (A \xleftarrow{1_A} A \xrightarrow{f} B) \qquad\qquad f^* = (B \xleftarrow{f} A \xrightarrow{1_A} A)$$

$$\frac{a \longrightarrow\!\!\!\!\bullet\!\!\!\!\longrightarrow b}{f(a) = b} \qquad\qquad \frac{b \longrightarrow\!\!\!\!\bullet\!\!\!\!\longrightarrow a}{b = f(a)}$$

- Tabulators

$$\begin{tikzcd} & A \arrow[dd, "\bullet\, S"] \\ \mathrm{Tab}(S) \arrow[ur, "t_0"] \arrow[dr, "t_1"'] \arrow[r, phantom, "\tau"] & {} \\ & B \end{tikzcd} \quad = \quad \begin{tikzcd} S \arrow[r, "\sigma_0"] \arrow[d, equal] & A \\ S \arrow[r, "1"] \arrow[d, equal] & S \arrow[u, "\sigma_0"'] \arrow[d, "\sigma_1"] \\ S \arrow[r, "\sigma_1"'] & B \end{tikzcd}$$

Effective ($S = t_{1*} \bullet t_0^*$) and unitary (tetrahedron)

Still because **Ab** has pullbacks

- Cauchy: Adjoint spans are representable

$$S \dashv R \quad \text{is} \quad f_* \dashv f^*$$

- Dagger structure

$$\begin{tikzcd} A & & B \\ S \arrow[u, "\sigma_0"] \arrow[d, "\sigma_1"'] \arrow[rr, mapsto, "\dagger"] & & S \arrow[u, "\sigma_1"'] \arrow[d, "\sigma_0"] \\ B & & A \end{tikzcd}$$

Because **Ab** has finite limits

$\mathbb{S}\mathrm{pan}(\mathbf{Ab})$ is a Cartesian double category (Aleiferi)

In fact $\oplus\colon \mathbb{S}\mathrm{pan}(\mathbf{Ab}) \times \mathbb{S}\mathrm{pan}(\mathbf{Ab}) \longrightarrow \mathbb{S}\mathrm{pan}(Ab)$ is both a left and a right double adjoint to the diagonal

Because **Ab** has pushouts

$\mathbb{S}\mathrm{pan}(\mathbf{Ab})$ has cotabulators

Unitary

But effective iff σ_0, σ_1 are jointly monic

Because **Ab** is preadditive (an **Ab**-category)

We can add horizontal arrows *and* cells

$$
\begin{tikzcd}
A \arrow[r, "f+f'"] & B \\
S \arrow[u] \arrow[d] \arrow[r, "\alpha+\alpha'"] & T \arrow[u] \arrow[d] \\
C \arrow[r, "g+g'"'] & D
\end{tikzcd}
$$

Adjoint pairs extend to spans

The adjoint pair $\mathbf{Ab} \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{Set}$ extends to an oplax/strict double adjunction

$$\mathbb{S}\mathrm{pan}(\mathbf{Ab}) \underset{F}{\overset{U}{\rightleftarrows}} \mathbb{S}\mathrm{et}$$

which is double monadic

But there doesn't seem to be any good adjunction

$$\mathbb{R}\mathrm{ing} \rightleftarrows \mathbb{S}\mathrm{pan}(\mathbf{Ab})$$

The tensor product functor $\otimes \colon \mathbf{Ab} \times \mathbf{Ab} \longrightarrow \mathbf{Ab}$ extends to an oplax normal double functor

$$\otimes \colon \ \mathbb{S}\mathrm{pan}(\mathbf{Ab}) \times \mathbb{S}\mathrm{pan}(\mathbf{Ab}) \longrightarrow \mathbb{S}\mathrm{pan}(\mathbf{Ab})$$

$$\begin{tikzcd}
A & B & & A \otimes B \\
S \arrow[u] \arrow[d] & T \arrow[u] \arrow[d] \arrow[r, mapsto, "\otimes"] & & S \otimes T \arrow[u] \arrow[d] \\
C & D & & C \otimes D
\end{tikzcd}$$

Associative and unitary but not a strong double functor

We don't get a monoidal double category in the sense of Shulman

Hom

$$
\begin{tikzcd}
A & B & & \mathbf{Ab}(A,B) \\
S \arrow[u] \arrow[d] & T \arrow[u] \arrow[d] \arrow[r, phantom, "\overset{\mathrm{Hom}}{\longmapsto}"] & & \mathrm{Hom}(S,T) \arrow[u] \arrow[d] \\
C & D & & \mathbf{Ab}(C,D)
\end{tikzcd}
$$

$\mathrm{Hom}(S,T)$ is the Abelian group of all cells

$$
\begin{tikzcd}
A \arrow[r] & B \\
S \arrow[u] \arrow[d] \arrow[r] & T \arrow[u] \arrow[d] \\
D \arrow[r] & D
\end{tikzcd}
$$

$\otimes$ – Hom adjointness

Theorem

Hom is a lax normal double functor $\mathbb{S}\mathrm{pan}(\mathbf{Ab})^{op} \times \mathbb{S}\mathrm{pan}(\mathbf{Ab}) \longrightarrow \mathbb{S}\mathrm{pan}(\mathbf{Ab})$.

For each Abelian group we have an oplax/lax double adjunction $(\ ) \otimes A \dashv Hom(A, -)$.