Morphisms of Rings

Robert Paré

Abstract Natural questions related to the double category of rings with homomorphisms and bimodules lead to a reevaluation of what a morphism of rings is. We introduce matrix-valued homomorphisms and then drop preservation of identities, giving what are sometimes called amplification homomorphisms. We show how these give extensions of the double category of rings and give some arguments justifying their study.

Introduction

In 1967 when I started my PhD under Jim Lambek's direction, he had already shifted his main interest from ring theory to category theory. I never had a course in ring theory from him but I learned in his category theory course, and in more detail, from his book [5], that rings were not necessarily commutative but had an identity element 1, and that homomorphisms, in addition to preserving sum and product, should also preserve 1.

In the last chapter of that same book, he introduces bimodules and their tensor product and shows that it is associative and unitary up to isomorphism. The tensor product is of course only defined if the rings of scalars match up properly just like composition in categories. It would seem then that we have a sort of category whose objects are rings and whose morphisms are bimodules. Although he doesn't say so there (it wasn't the place), it's clear from his later work that he knew then or shortly after that rings, bimodules and linear maps form a bicategory [1].

Robert Paré
Department of Mathematics and Statistics, Dalhousie University, Halifax, NS, Canada, B3H 4R2, e-mail: pare@mathstat.dal.ca

We are used to thinking of morphisms as structure-preserving functions, so what are we to make of bimodules as morphisms? And what's the relationship with homomorphisms? These are the questions we address here.

In sections 1 and 2 we introduce the relevant double category theory motivated by the example which concerns us, viz. the double category of rings, homomorphisms, bimodules and equivariant maps. We are led naturally to matrix-valued homomorphisms, which gives us a graded category of rings whose degree 1 part is the usual one. Accompanying this is a new double category whose basic properties we expose. This is the content of sections 3 and 4. Then double categorical considerations lead to a further extension of the category (and double category) of rings, to what have been called amplification morphisms. Their basic properties are treated in section 5.

1 Double categories

The category of rings, **Ring**, has rings with 1 as objects and homomorphisms preserving 1 as morphisms. This is a nice category. It is complete, cocomplete, regular, locally finitely presentable, etc.

Given rings R and S, an S-R-bimodule M is a simultaneous left S-module and right R-module whose left and right actions commute;

$$(sm)r = s(mr) \ .$$

If T is another ring and N a T-S-bimodule, the tensor product over S, $N \otimes_S M$ is naturally a T-R-bimodule. We have associativity isomorphisms

$$P \otimes_T (N \otimes_S M) \cong (P \otimes_T N) \otimes_S M$$

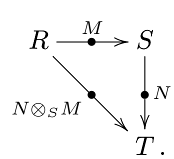
and unit isomorphisms

$$M \otimes_R R \cong M \cong S \otimes_S M$$

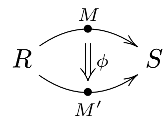
as clearly exposed in Chapter 5 of [5]. To keep track of the various rings involved and what's acting on what and on which side we can write

$$M \colon R \longrightarrow\!\!\!\!\!\!\bullet\;\; S$$

to mean that M is an S-R-bimodule. Then the tensor product looks like a composition



So we get a sort of category whose composition is not really associative nor unitary but only so up to isomorphism. In order to formalize this we must incorporate these isomorphisms into the structure. They are morphisms between morphisms, called 2-cells. Given two S-R-bimodules $M, M' \colon R \xrightarrow{\bullet} S$, a 2-cell



is a linear map of bimodules, i.e. a function such that

$$\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$$

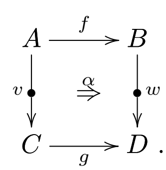
$$\phi(sm) = s\phi(m)$$

$$\phi(mr) = \phi(m)r \ .$$

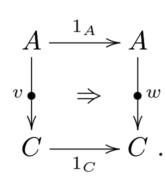
This is the data for a *bicategory*: objects (rings), arrows (bimodules), 2-cells (linear maps). They can be composed in various ways and satisfy a host of equations, all of which would seem obvious to anyone working with bimodules. See the seminal work [1] for details.

Now we have a structure (rings) with two candidates for morphism, homomorphism and bimodule, and we might ask which is the right one. In fact they are both good but for different purposes. So a better question is, how are they related? The answer will come from the theory of double categories.

A *double category* $\mathbb{A}$ has objects (A, B, C, D below) and *two* kinds of morphism, which we call *horizontal* (f, g below) and *vertical* (v, w below). These are related by a further kind of morphism, double cells as in



The horizontal arrows form a category $\mathbf{Hor}\mathbb{A}$ with composition denoted by juxtaposition and identities by 1_A. Cells can also be composed horizontally also forming a category. The vertical arrows compose to give a bicategory $\mathcal{V}ert\mathbb{A}$ whose 2-cells are the *globular cells* of $\mathbb{A}$, i.e. those with identities on the top and bottom



Vertical composition is denoted by $\bullet$ and vertical identities by id_A. Finally double cells can also be composed vertically (also denoted $\bullet$ and id_f). The composition of cells is as associative and unitary as possible, provided the coherence isomorphisms are factored in to make the boundaries match. Rather than give a formal definition, we will be better served by a couple of representative examples. The reader is referred to [4] for a precise definition.

Example 1 The double category which will concern us here is the double category of rings, $\mathbb{R}\mathrm{ing}$. Its objects are rings (with 1) and its horizontal arrows are unitary homomorphisms. Its vertical arrows are bimodules. More precisely a vertical arrow from R to S is an S-R-bimodule. A double cell

$$\begin{array}{ccc} R & \xrightarrow{\;f\;} & R' \\ {\scriptstyle M}\big\downarrow\!\bullet & \overset{\phi}{\Rightarrow} & \bullet\!\big\downarrow{\scriptstyle M'} \\ S & \xrightarrow[\;g\;]{} & S' \end{array}$$

is a map $\phi\colon M \longrightarrow M'$ which is linear in the sense that it preserves addition and is compatible with the actions

$$\phi(sm) = g(s)\phi(m)$$

$$\phi(mr) = \phi(m)f(r)\ .$$

In other words it is an S-R-linear map from M to M' when the codomain is made into an S-R-bimodule by "restriction of scalars".

Horizontal composition of morphisms and cells is just function composition. Vertical composition is given by

$$\begin{array}{ccc} R & \xrightarrow{\;f\;} & R' \\ {\scriptstyle M}\big\downarrow\!\bullet & \overset{\phi}{\Rightarrow} & \bullet\!\big\downarrow{\scriptstyle M'} \\ S & \xrightarrow{\;g\;} & S' \\ {\scriptstyle N}\big\downarrow\!\bullet & \overset{\psi}{\Rightarrow} & \bullet\!\big\downarrow{\scriptstyle N'} \\ T & \xrightarrow[\;h\;]{} & T' \end{array} \quad = \quad \begin{array}{ccc} R & \xrightarrow{\quad f \quad} & R' \\ {\scriptstyle N\otimes_S M}\big\downarrow\!\bullet & \overset{\psi\otimes_g\phi}{\Longrightarrow} & \bullet\!\big\downarrow{\scriptstyle N'\otimes_{S'} M'} \\ T & \xrightarrow[\quad h \quad]{} & T' \end{array}$$

where $(\psi \otimes_g \phi)(n \otimes_S m) = \psi(n) \otimes_{S'} \phi(m)$. It is easily checked that this is well defined and is associative in the appropriate sense, giving us the double category $\mathbb{R}\mathrm{ing}$.

Example 2 A more basic example, and one to keep in mind, is the following. The double category $\mathbb{R}\mathrm{el}$ has sets as objects and functions as horizontal arrows, so $\mathbf{Hor}\mathbb{R}\mathrm{el} = \mathbf{Set}$. A vertical arrow $R\colon X \longrightarrow\!\!\!\!\!\!\bullet\;\; Y$ is a relation between X and Y and there is a unique cell

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle R}\bullet\!\downarrow & \Rightarrow & \downarrow\!\bullet{\scriptstyle R'} \\ Y & \xrightarrow[g]{} & Y' \end{array}$$

if (and only if) we have

$$\forall_{x,y}(x \sim_R y \Rightarrow f(x) \sim_{R'} g(y)) \ .$$

$\mathbb{R}$el is a strict double category in the sense that vertical composition is strictly associative and unitary. Strict double categories were introduced by Ehresmann [2] and are (ignoring size considerations) the same as category objects in **Cat**, the category of categories.

$\mathbb{R}$ing on the other hand is a weak double category in that vertical composition is only associative and unitary up to coherent isomorphism as is clear from the above discussion. We consider the weak double categories to be the more important notion, certainly for this work, and call them simply double categories without modifiers. They were introduced in [4], where more examples can be found.

2 Companions and conjoints

Functions are often defined to be relations that are single-valued and everywhere defined. Category theorists would tend to take functions as primitive and define relations in the category of sets, as subobjects of a product. Then every function has an associated relation, its graph

$$Gr(f) = \{(x, y) | f(x) = y\} \ .$$

In any case there is a close relationship between functions and certain relations and this can be formulated in purely double category terms.

Definition 3 Let $\mathbb{A}$ be a double category, $f\colon A \longrightarrow B$ a horizontal arrow, and $v\colon A \longrightarrow\!\!\bullet\!\!\longrightarrow B$ a vertical one in $\mathbb{A}$. We say that v is a *companion* of f if we are given cells, the *binding cells*

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle \mathrm{id}_A}\bullet\!\downarrow & \alpha & \downarrow\!\bullet{\scriptstyle v} \\ A & \xrightarrow[f]{} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\bullet\!\downarrow & \beta & \downarrow\!\bullet{\scriptstyle \mathrm{id}_B} \\ B & \xrightarrow[1_B]{} & B \end{array}$$

such that

$$
\begin{array}{ccccc}
A & \xrightarrow{1_A} & A & \xrightarrow{f} & B \\
{\scriptstyle \mathrm{id}_A}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle v} & \beta & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[f]{} & B & \xrightarrow[1_B]{} & B
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \mathrm{id}_A}\big\downarrow\!\bullet & \mathrm{id}_f & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[f]{} & B'
\end{array}
$$

i.e. $\beta\alpha = \mathrm{id}_f$, and

$$
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
{\scriptstyle \mathrm{id}_A}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle v} \\
A & \xrightarrow{f} & B \\
{\scriptstyle v}\big\downarrow\!\bullet & \beta & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}_B} \\
B & \xrightarrow[1_B]{} & B
\end{array}
\quad \cdot\!=\!\cdot \quad
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
{\scriptstyle v}\big\downarrow\!\bullet & 1_v & \bullet\!\big\downarrow{\scriptstyle v} \\
B & \xrightarrow[1_B]{} & B
\end{array}
$$

i.e. $\beta \bullet \alpha \cdot\!=\!\cdot\, 1_v$. The $\cdot\!=\!\cdot$ sign means equality once the canonical isomorphisms $(v \bullet \mathrm{id}_A \cong v \cong \mathrm{id}_B \bullet v)$ are inserted to make the boundaries agree.

Companions, when they exist, are unique up to isomorphism, and we use the notation f_* to denote a choice of companion for f. Companions compose:

$$(gf)_* \cong g_* \bullet f_*$$

and we also have

$$(1_A)_* \cong \mathrm{id}_A \quad .$$

It's an easy exercise to show that in $\mathbb{R}$el, every function has a companion, its graph (if v is a relation R, then α expresses the fact that $Gr(f) \subseteq R$, and β that $R \subseteq Gr(f)$).

We see that companions give a precise meaning to expressions like "a function is a relation such that..." or more generally "a horizontal arrow is isomorphic to a vertical one".

Proposition 4 *(a) In* $\mathbb{R}$ing*, every homomorphism* $f\colon R \longrightarrow S$ *has a companion, namely* S *considered as an* S*-*R*-bimodule with actions* $\bullet$ *given by*

$$
\begin{aligned}
s' \bullet s &= s's \\
s \bullet r &= sf(r)\,.
\end{aligned}
$$

(b) A bimodule $M\colon R \longrightarrow\!\!\!\!\!\!\bullet\;\; S$ *is a companion if and only if it is free on one generator as a left* S*-module.*

Proof (a) Denote by $S\colon R \longrightarrow\!\!\!\!\!\!\bullet\;\; S$ the bimodule in the statement, i.e. the bimodule gotten from the S-S-bimodule S by "restriction of scalars" along f on the right. Then we have cells

$$
\begin{array}{ccc}
R & \xrightarrow{1_R} & R \\
{\scriptstyle R}\big\downarrow\bullet & \overset{\alpha}{\Rightarrow} & \bullet\big\downarrow{\scriptstyle S} \\
R & \xrightarrow[f]{} & S
\end{array}
\qquad\qquad
\begin{array}{ccc}
R & \xrightarrow{f} & S \\
{\scriptstyle S}\big\downarrow\bullet & \overset{\beta}{\Rightarrow} & \bullet\big\downarrow{\scriptstyle S} \\
S & \xrightarrow[1_g]{} & S
\end{array}
$$

$$\alpha(r) = f(r) \qquad\qquad \beta(s) = s \ .$$

It's an easy calculation to show that α and β are indeed cells and that $\beta\alpha = \mathrm{id}_f$ and $\beta \bullet \alpha \cdot\!=\!\cdot\ 1_S$.

(b) Suppose the bimodule $M\colon R \longrightarrow\!\!\!\!\!\!\bullet\ \ S$ is free as a left S-module with generator $m_0 \in M$. Then for every $r \in R$ there is a unique element $s \in S$ such that

$$m_0 r = s m_0 \ .$$

Call this $s, f(r)$, so that $f(r)$ is uniquely determined by the equation

$$f(r) m_0 = m_0 r \ .$$

It is easy to check that f is a homomorphism $f\colon R \longrightarrow S$. We check that multiplication is preserved, as an example.

$$
\begin{aligned}
f(r_1 r_2) m_0 &= m_0 r_1 r_2 \\
&= f(r_1) m_0 r_2 \\
&= f(r_1) f(r_2) m_0 \,.
\end{aligned}
$$

So $f(r_1 r_2) = f(r_1) f(r_2)$. Also, while we are at it,

$$f(1) m_0 = m_0 1 = 1 m_0$$

so $f(1) = 1$.

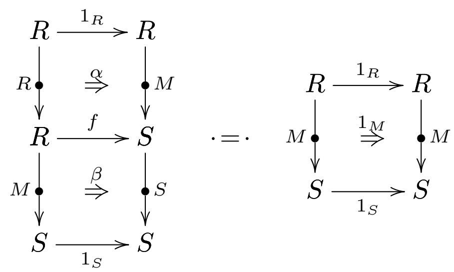
Now define cells

$$
\begin{array}{ccc}
R & \xrightarrow{1_R} & R \\
{\scriptstyle R}\big\downarrow\bullet & \overset{\alpha}{\Rightarrow} & \bullet\big\downarrow{\scriptstyle M} \\
R & \xrightarrow[f]{} & S
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
R & \xrightarrow{f} & S \\
{\scriptstyle M}\big\downarrow\bullet & \overset{\beta}{\Rightarrow} & \bullet\big\downarrow{\scriptstyle S} \\
S & \xrightarrow[1_S]{} & S
\end{array}
$$

by taking $\alpha(r) = m_0 r$, and $\beta(m)$ to be the unique element of S such that $\beta(m) m_0 = m$. The calculations showing that α and β are cells (i.e. linear maps) and that the binding equations

$$
\begin{aligned}
\beta\alpha &= \mathrm{id}_f \\
\beta \bullet \alpha &\cdot\!=\!\cdot\ 1_M
\end{aligned}
$$

hold are easy exercises. We check



in part to illustrate where the $\cdot\!=\!\cdot$ comes in. The left hand diagram has to be modified by placing the canonical isomophism $\rho^{-1}\colon M \longrightarrow M \otimes_R R$ on the left and $\lambda\colon S \otimes_S M \longrightarrow M$ on the right to make the boundaries on both sides of the equation the same. Then for $m \in M$ we have

$$\begin{aligned}\lambda(\beta\otimes\alpha)\rho^{-1}(m) &= \lambda(\beta\otimes\alpha)(m\otimes 1)\\ &= \lambda(\beta m\otimes \alpha 1)\\ &= (\beta m)(\alpha 1)\\ &= (\beta m)(m_0 1)\\ &= m\,.\end{aligned}$$

□

The generator for M is not unique. If m_1 is another one, then there exists an invertible element $a \in S$ such that $m_1 = am_0$. If $f_0, f_1\colon R \longrightarrow S$ are the homomorphisms corresponding to m_0 and m_1 respectively, then we have

$$f_1(r)m_1 = f_1(r)am_0\,.$$

On the other hand, we also have

$$f_1(r)m_1 = m_1 r = am_0 r = af_0(r)m_0$$

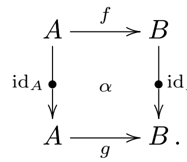
so $f_1(r)a = af_0(r)$ or

$$f_1(r) = af_0(r)a^{-1}\,.$$

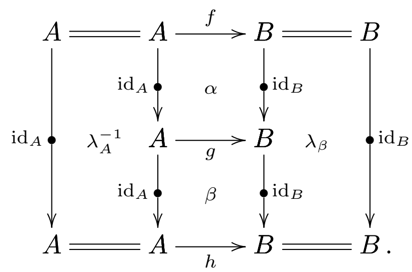
We summarize this in the following proposition.

Proposition 5 *If a bimodule $M\colon R \longrightarrow\!\!\!\!\bullet\,\, S$ is a rank one free left S-module, then $M \cong f_*$ for some homomorphism $f\colon R \longrightarrow S$. f is unique up to conjugation by a unit of S.*

Conjugation by an element of S is actually an isomorphism in a 2-category of rings. Every double category $\mathbb{A}$ (strict or not) has a horizontal 2-category, $\mathcal{H}or\,\mathbb{A}$. The objects are those of $\mathbb{A}$, the 1-cells are the horizontal arrows of $\mathbb{A}$, and the 2-cells are the *special cells* of $\mathbb{A}$, i.e. cells of the form

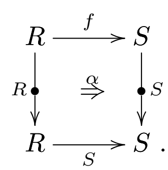


Vertical composition of 2-cells, α and β, in $\mathcal{H}or\mathbb{A}$ uses the canonical isomorphisms $\lambda = \rho\colon \mathrm{id} \bullet \mathrm{id} \longrightarrow \mathrm{id}$



A moderate amount of straightforward calculation shows that $\mathcal{H}or\,\mathbb{A}$ is indeed a 2-category.

When applied to the double category $\mathbb{R}\mathrm{ing}$ we get a 2-category whose objects are rings, whose arrows are homomorphisms and whose 2-cells are linear maps of the form

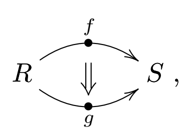


Such an α is determined by its value at 1. We have

$$\begin{aligned} \alpha(r) &= \alpha(r \cdot 1) = g(r)\alpha(1) \\ &= \alpha(1 \cdot r) = \alpha(1)f(r)\,. \end{aligned}$$

This gives the following.

Definition 6 The 2-category of rings, $\mathcal{R}ing$, has rings as objects, homomorphisms as 1-cells and as 2-cells



elements $s \in S$ such that for all r

$$sf(r) = g(r)s\ .$$

This is not that surprising. If we think of a ring as a one-object additive category, then homomorphisms are additive functors, and a 2-cell as above is just a natural transformation. Nevertheless it can be useful to keep in mind.

There is a dual notion to companion, that of conjoint, which we spell out.

Definition 7 Let $f\colon A \longrightarrow B$ be a horizontal arrow in a double category $\mathbb{A}$ and $v\colon B \longrightarrow\!\!\!\!\!\!\bullet\;\; A$ a vertical one. We say that v is *conjoint* to f if we are given cells (*conjunctions*)

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \mathrm{id}_A}\big\downarrow & \overset{\psi}{\Rightarrow} & \big\downarrow{\scriptstyle v} \\
A & \xrightarrow[1_A]{} & A
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
{\scriptstyle v}\big\downarrow & \overset{\chi}{\Rightarrow} & \big\downarrow{\scriptstyle \mathrm{id}_A} \\
B & \xrightarrow[f]{} & A
\end{array}
$$

such that

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{1_B} & B \\
{\scriptstyle \mathrm{id}_A}\big\downarrow & \overset{\psi}{\Rightarrow} & \big\downarrow{\scriptstyle v} & \overset{\chi}{\Rightarrow} & \big\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[1_A]{} & A & \xrightarrow[f]{} & B
\end{array}
=
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \mathrm{id}_A}\big\downarrow & \overset{\mathrm{id}_f}{\Rightarrow} & \big\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[f]{} & B
\end{array}\ ,
$$

i.e. $\chi\psi = \mathrm{id}_f$, and

$$
\begin{array}{ccc}
B & \xrightarrow{1_B} & B \\
{\scriptstyle v}\big\downarrow & \overset{\chi}{\Rightarrow} & \big\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[f]{} & B \\
{\scriptstyle \mathrm{id}_A}\big\downarrow & \overset{\psi}{\Rightarrow} & \big\downarrow{\scriptstyle v} \\
A & \xrightarrow[1_A]{} & A
\end{array}
\quad \cdot\!=\!\cdot \quad
\begin{array}{ccc}
B & \xrightarrow{1_B} & B \\
{\scriptstyle v}\big\downarrow & \overset{1_v}{\Rightarrow} & \big\downarrow{\scriptstyle v} \\
A & \xrightarrow[1_A]{} & A
\end{array}\ ,
$$

i.e. $\psi \bullet \chi \cdot\!=\!\cdot\ 1_v$.

This definition looks very much like that of adjoint, and that is how we think of it: v is right adjoint to f, even though they are different types of arrows. That the notion is the vertical dual to that of companion is clear.

The double category $\mathbb{R}\mathrm{ing}$ is isomorphic to its vertical dual

$$\mathbb{R}\mathrm{ing}^{co} \cong \mathbb{R}\mathrm{ing}\,.$$

The isomorphism takes a ring R to its opposite ring R^{op}, i.e., with multiplication switched. A homomorphism $f\colon R \longrightarrow S$ gives a homomorphism

$f^{op}\colon R^{op} \longrightarrow S^{op}$ (in the same direction) whereas an S-R-bimodule $M\colon R \longrightarrow\!\!\bullet\!\!\to S$ gives an R^{op}-S^{op}-bimodule $M^{op}\colon S^{op} \longrightarrow\!\!\bullet\!\!\to R^{op}$ (in the opposite direction). So the results on companions are readily dualizable. We denote by f^* a conjoint for f. In $\mathbb{R}$ing, every homomorphism $f\colon R \longrightarrow S$ has a conjoint f^*, namely $S\colon S \longrightarrow\!\!\bullet\!\!\to R$ with left action by R given by "restriction"

$$r \bullet s = f(r)s\,.$$

3 Matrix-valued homomorphisms

We saw in the previous section how homomorphisms $f\colon R \longrightarrow S$ correspond to bimodules $M\colon R \longrightarrow\!\!\bullet\!\!\to S$ which are free on one generator as left S-modules. What happens if M is free on p generators? We might expect these to correspond to some kind of homomorphic relation from R to S associating to each $r \in R$, not a unique element of S but rather p of them. This is not exactly what happens, but we do get something interesting.

As a warm-up, let's assume M is free on two generators m_1, m_2 as a left S-module. Nothing is said about the right action (as before). Then for each $r \in R$ we get unique $s_{11}, s_{12}, s_{21}, s_{22} \in S$ such that

$$\begin{aligned} m_1 r &= s_{11} m_1 + s_{12} m_2 \\ m_2 r &= s_{21} m_1 + s_{22} m_2\,. \end{aligned}$$

Let's denote s_{ij} by $f_{ij}(r)$. So to each r we associate not 2 but 4 elements of S! Of course the "same" is true if M is free on p generators $m_1, \ldots, m_p$:

$$m_i r = \sum_{j=1}^{p} f_{ij}(r) m_j\,.$$

Consider

$$m_i(rr') = \sum_{j=1}^{p} f_{ij}(rr') m_j$$

and

$$\begin{aligned} (m_i r) r' &= \textstyle\sum_{j=1}^{p} f_{ij}(r) m_j r' \\ &= \textstyle\sum_{j=1}^{p} f_{ij}(r) \left(\sum_{k=1}^{p} f_{jk}(r') m_k\right) \\ &= \textstyle\sum_{k=1}^{p} \left(\sum_{j=1}^{p} f_{ij}(r) f_{jk}(r')\right) m_k\,. \end{aligned}$$

So $f_{ik}(rr') = \sum_{j=1}^{p} f_{ij}(r) f_{jk}(r')$, i.e. we get a homomorphism

$$f\colon R \longrightarrow Mat_p(S)$$

into the ring of $p \times p$ matrices in S. This leads to the following:

Theorem 8 *(a) Any matrix-valued homomorphism $f\colon R \longrightarrow Mat_p(S)$ induces an S-R-bimodule structure on $S^{(p)}$.*

(b) Any S-R-bimodule $M\colon R \longrightarrow\!\!\!\!\bullet\!\!\!\!\longrightarrow S$ which is free on p generators as a left S-module is isomorphic (as an S-R-bimodule) to $S^{(p)}$ with R-action induced by a homomorphism $f\colon R \longrightarrow Mat_p(S)$ as in (a).

(c) The homomorphism f in (b) is unique up to conjugation by an invertible $p \times p$ matrix A in $Mat_p(S)$.

Proof (a) Let $S^{(p)}$ denote the set of row vectors in S of length p, i.e. $1 \times p$ matrices. Then for any element $\mathbf{s} = [s_1, \dots, s_p] \in S^{(p)}$ let

$$s' \bullet \mathbf{s} = [s's_1, \dots, s's_p]$$

and

$$\mathbf{s} \bullet r = \mathbf{s} f(r) \qquad \text{(matrix multiplication).}$$

The bimodule conditions are easily verified.

(b) We saw just above how to construct a homomorphism $f\colon R \longrightarrow Mat_p(S)$ from a bimodule $M\colon R \longrightarrow\!\!\!\!\bullet\!\!\!\!\longrightarrow S$ with an S-basis $m_1, \dots, m_p$. It is uniquely determined by

$$\mathbf{m} \bullet r = f(r)\mathbf{m}$$

with $\mathbf{m}$ = column vector of m_i's. Because M is free on $m_1, \dots, m_p$ as an S-module we already have an S-isomorphism

$$\phi\colon S^{(p)} \longrightarrow M$$

$$\phi(\mathbf{s}) = \mathbf{s}\mathbf{m}.$$

Now make $S^{(p)}$ into an S-R-bimodule as in (a), i.e. $\mathbf{s} \bullet r = \mathbf{s} f(r)$. Then

$$\begin{aligned} \phi(\mathbf{s} \bullet r) &= \phi(\mathbf{s} f(r)) \\ &= \mathbf{s} f(r)\mathbf{m} \\ &= \mathbf{s}\mathbf{m} \bullet r \\ &= \phi(\mathbf{s}) \bullet r\,, \end{aligned}$$

so ϕ is an S-R-isomorphism.

(c) This is the usual change of bases calculation. It's just a question of taking care to get everything on the right side. If $\mathbf{m}'$ is another S-basis for M we get an invertible S-matrix A such that

$$\mathbf{m}' = A\,\mathbf{m}$$

so if f' is the homomorphism we get from $\mathbf{m}'$, we have

$$\begin{aligned} \mathbf{m}' \bullet r &= f'(r)\mathbf{m}' \\ A\,\mathbf{m} \bullet r &= f'(r)A\,\mathbf{m} \\ \mathbf{m} \bullet r &= A^{-1}f'(r)A\,\mathbf{m} \end{aligned}$$

so $f(r) = A^{-1}f'(r)A$. □

We'd like to think of a homomorphism $R \longrightarrow Mat(S)$ as a kind of homomorphic relation from R to S. So let's look at some special cases.

Example 9 (Pairs of homomorphisms)
Let $f, g \colon R \longrightarrow S$ be homomorphisms. Then we get a homomorphism $h \colon R \longrightarrow Mat_2(S)$ given by

$$h(r) = \begin{bmatrix} f(r) & 0 \\ 0 & g(r) \end{bmatrix}$$

In general, we have the subring of diagonal matrices

$$S^{(p)} \subseteq Mat_p(S)$$

so p homomorphisms $f_i \colon R \longrightarrow S$ give a matrix-valued homomorphism $f \colon R \longrightarrow Mat_p(S)$.

Example 10 (Derivations)
Let $f \colon R \longrightarrow S$ be a homomorphism and d an f-derivation, i.e. an additive function $d \colon R \longrightarrow S$ such that

$$d(rr') = d(r)f(r') + f(r)d(r')\,.$$

Then we get a homomorphism $R \longrightarrow Mat_2(S)$

$$r \longmapsto \begin{bmatrix} f(r) & 0 \\ d(r) & f(r) \end{bmatrix}$$

In fact the set of matrices

$$D = \left\{ \begin{bmatrix} s & 0 \\ s' & s \end{bmatrix} \middle| \, s, s' \in S \right\}$$

is a subring of $Mat_2(S)$, and derivations correspond exactly to homomorphisms $R \longrightarrow Mat_2(S)$ that factor through D.

More generally we can consider the subring of lower triangular matrices

$$L = \left\{ \begin{bmatrix} s & 0 \\ s' & s'' \end{bmatrix} \middle| \, s, s', s'' \in S \right\}\,.$$

Then a homomorphism $R \longrightarrow Mat_2(S)$ that factors through L corresponds to a pair of homomorphisms $f, g \colon R \longrightarrow S$ and a derivation d from f to g, i.e. an additive function $d \colon R \longrightarrow S$ such that

$$d(rr') = d(r)f(r') + g(r)d(r')\,.$$

Example 11 We give one more, somewhat mysterious, example to illustrate the variety of morphisms we get just in the 2×2 case. For any ring S we can construct a ring of "complex numbers" over S:

$$\mathbb{C}(S) = \left\{ \left[\begin{array}{rr} s & s' \\ -s' & s \end{array} \right] \middle| s, s' \in S \right\} .$$

This is a subring of $Mat_2(S)$. A homomorphism $R \longrightarrow Mat_2(S)$ that factors through $\mathbb{C}(S)$ corresponds to two additive functions $c, s \colon R \longrightarrow S$ with the properties

$$\begin{aligned} c(rr') &= c(r)c(r') - s(r)s(r') \\ s(rr') &= s(r)c(r') + c(r)s(r') . \end{aligned}$$

Hopefully these examples will have convinced the reader that considering homomorphisms $R \longrightarrow Mat_p(S)$ as a kind of relation from R to S is an interesting idea worth pursuing.

If they really are a kind of morphism from R to S we should be able to compose them. To get an idea of how this might work, the previous theorem says that a homomorphism $f \colon R \longrightarrow Mat_p(S)$ corresponds to a bimodule $S^{(p)} \colon R \longrightarrow\!\!\!\!\!\bullet\;\, S$ and we know how to compose bimodules. So given another homomorphism $g \colon S \longrightarrow Mat_q(T)$ we get $T^{(q)} \colon S \longrightarrow\!\!\!\!\!\bullet\;\, T$ and if we compose these we get

$$T^{(q)} \otimes_S S^{(p)} \cong T^{(q)} \otimes_S (\oplus_p S) \cong \oplus_p (T^{(q)} \otimes_S S) \cong T^{(pq)} .$$

So what we can expect is a graded composition, graded by the multiplicative monoid of positive integers $(\mathbb{N}^+, \cdot)$. If we are a bit more careful with the above isomorphisms we get an explicit description of the graded composition.

For homomorphisms $f \colon R \longrightarrow Mat_p(S)$ and $g \colon S \longrightarrow Mat_q(T)$, let $f_* = S^{(p)} \colon R \longrightarrow\!\!\!\!\!\bullet\;\, S$ and $g_* = T^{(q)} \colon S \longrightarrow\!\!\!\!\!\bullet\;\, T$ be the bimodules induced by f and g as in the above theorem. So we get a bimodule

$$g_* \otimes_S f_* = T^{(q)} \otimes_S S^{(p)} \colon R \longrightarrow\!\!\!\!\!\bullet\;\, T .$$

Let $\mathbf{e}_1, \ldots, \mathbf{e}_p$ be the standard basis for $S^{(p)}$ and $\mathbf{e}'_1, \ldots, \mathbf{e}'_q$ the standard basis for $T^{(q)}$. Then the $\mathbf{e}'_j \otimes \mathbf{e}_i$ for $1 \leq i \leq p$, $1 \leq j \leq q$ form a basis for $T^{(q)} \otimes_S S^{(p)}$. We have

$$\begin{aligned} (\mathbf{e}'_j \otimes \mathbf{e}_i) r &= \mathbf{e}'_j \otimes \left(\textstyle\sum_k f_{ki}(r) \mathbf{e}_k\right) \\ &= \textstyle\sum_k \mathbf{e}'_j \otimes f_{ki}(r) \mathbf{e}_k \\ &= \textstyle\sum_k \mathbf{e}'_j f_{ki}(r) \otimes \mathbf{e}_k \\ &= \textstyle\sum_k \sum_l g_{lj}(f_{ki}(r)) \mathbf{e}'_l \otimes \mathbf{e}_k . \end{aligned}$$

So we apply f to r to get a $p \times p$ matrix and then apply g to each of the entries to get a block $p \times p$ matrix of $q \times q$ matrices. Ordering the basis $\{\mathbf{e}'_j \otimes \mathbf{e}_i\}$ will give us a $(pq) \times (pq)$ matrix. The ordering is arbitrary but a judicious choice will make calculations easier and the block matrix picture suggests just such a choice. We order them lexicographically from the right

$$\langle \mathbf{e}'_1 \otimes \mathbf{e}_1, \mathbf{e}'_2 \otimes \mathbf{e}_1, \ldots, \mathbf{e}'_q \otimes \mathbf{e}_1, \mathbf{e}'_1 \otimes \mathbf{e}_2, \ldots, \mathbf{e}'_q \otimes \mathbf{e}_p \rangle$$

i.e. $\mathbf{e}'_j \otimes \mathbf{e}_i$ is in the $j + q(i-1)$ position. This gives us an isomorphism $Mat_p Mat_q(T) \cong Mat_{pq}(T)$, and with the aid of this we can now compose f with g to get a homomorphism $R \longrightarrow Mat_{pq}(T)$. This composition enlarges the category of rings to an $(\mathbb{N}^+, \cdot)$-graded category. Of course associativity and the unit laws have to be proved, which is a bit messy and a more general categorical approach will clarify things.

First of all for any p, $Mat_p(R)$ is functorial in R, i.e. we have a family of functors $Mat_p \colon \mathbf{Ring} \longrightarrow \mathbf{Ring}$. Then the isomorphisms $Mat_P Mat_q(R) \cong Mat_{pq}(R)$ are natural in R and they satisfy an associativity condition giving us a graded monad.

Graded monads were explicitly defined as such in [3] but certainly go back to Bénabou [1].

Definition 12 Let $(M, \cdot, 1)$ be a monoid. An *M-graded monad* consists of a category $\mathbf{A}$ and for each $m \in M$ an endofunctor $T_m \colon \mathbf{A} \longrightarrow \mathbf{A}$, together with natural transformations

$$\eta \colon 1_{\mathbf{A}} \longrightarrow T_1$$

and

$$\mu_{m,m'} \colon T_m T_{m'} \longrightarrow T_{mm'}$$

satisfying unit laws

$$\begin{array}{ccc} T_m & \xrightarrow{T_m\eta} & T_m T_1 \\ {\scriptstyle \eta T_m}\downarrow & \searrow^{1_{T_m}} & \downarrow{\scriptstyle \mu_{m,1}} \\ T_1 T_m & \xrightarrow[\mu_{1,m}]{} & T_m \end{array}$$

and associativity

$$\begin{array}{ccc} T_m T_{m'} T_{m''} & \xrightarrow{\mu_{m,m'} T_{m''}} & T_{mm'} T_{m''} \\ {\scriptstyle T_m \mu_{m',m''}}\downarrow & & \downarrow{\scriptstyle \mu_{mm',m''}} \\ T_m T_{m'm''} & \xrightarrow[\mu_{m,m'm''}]{} & T_{mm'm''} \,. \end{array}$$

This is nothing but a lax functor

$$T \colon \mathcal{M} \longrightarrow \mathcal{C}at$$

where $\mathcal{M}$ is the locally discrete one-object 2-category with 1-cells given by the elements of M.

Proposition 13 *(1) For every $p \in \mathbb{N}^+$, Mat_p is a functor* $\mathbf{Ring} \longrightarrow \mathbf{Ring}$.
(2) For every $p, q \in \mathbb{N}^+$ we have a natural isomorphism $\mu_{p,q} \colon Mat_p Mat_q \longrightarrow Mat_{pq}$.

(3) The families $\langle Mat_p \rangle_{p \in \mathbb{N}^+}$*,* $\langle \mu_{p,q} \rangle_{p,q \in \mathbb{N}^+}$ *together with the canonical isomorphism* $\eta\colon 1_{\mathbf{Ring}} \longrightarrow Mat_1$ *form an* $(\mathbb{N}^+, \cdot, 1)$*-graded monad* Mat*.*

Proof (1) $Mat_p(f)$ is application of f to a matrix entry-wise. This is obviously functorial.
(2) An element of $Mat_p Mat_q(R)$ is a $p \times p$ matrix of $q \times q$ matrices, and μ_{pq} of such is just the $pq \times pq$ matrix we get by erasing the inside brackets. This is also obviously natural.
(3) It is also clear that the $\langle \mu_{pq} \rangle$ are associative, the only difference between the two candidates being the order in which we erase the brackets inside the "block block" matrix.

The unit $\eta\colon 1_{\mathbf{Ring}} \longrightarrow Mat_1$ consists in putting square brackets around an element to make it a 1×1 matrix, so the unit laws are equally transparent. □

Given a graded monad $\mathbb{T} = (\langle T_m \rangle, \eta, \langle \mu_{m,m'} \rangle)$ we can construct a graded Kleisli category $\mathbf{A}_{\mathbb{T}}$. The objects are those of $\mathbf{A}$ and a morphism of degree m, $(m, f)\colon A \longrightarrow B$ in $\mathbf{A}_{\mathbb{T}}$ is a morphism $f\colon A \longrightarrow T_m B$ in $\mathbf{A}$. Composition

$$A \xrightarrow{(m,f)} B \xrightarrow{(m',g)} C$$

is given by $A \xrightarrow{f} T_m B \xrightarrow{T_m g} T_m T_{m'} C \xrightarrow{\mu_{m,m'}} T_{mm'} C$ and units by $(1, \eta A)\colon A \longrightarrow A$. That $\mathbf{A}_{\mathbb{T}}$ is a graded category is an easy calculation, just like for the usual Kleisli category.

We see now that our matrix-valued homomorphisms are exactly the Kleisli morphisms for the graded monad Mat. This gives a new, larger category of rings, $\mathbf{Ring}_{Mat}$.

Remark 14 Graded monads have recently appeared in the computer science literature (see e.g. [3] and references there). Our Kleisli category is not the same as theirs where their grading is on the objects rather than on the morphisms. The theory of graded monads and its extension to double categories is very interesting but that would take us too far afield so we leave it for future work.

4 The graded double category of rings

We can extend the double category of rings by adding in the new graded morphisms. The double category $\mathbb{R}\mathrm{ing}_{Mat}$ has objects all rings but horizontal arrows are the graded ones, $(p, f)\colon R \longrightarrow S$, i.e. $f\colon R \longrightarrow Mat_p(S)$. The vertical arrows are still bimodules $M\colon R \longrightarrow\!\!\bullet\!\!\longrightarrow S$. A double cell

$$\begin{array}{ccc} R & \xrightarrow{(p,f)} & R' \\ {\scriptstyle M}\big\downarrow\bullet & \overset{\phi}{\Rightarrow} & \bullet\big\downarrow{\scriptstyle M'} \\ S & \xrightarrow[(q,g)]{} & S' \end{array}$$

is a linear map (a cell in $\mathbb{R}$ing)

$$\begin{array}{ccc} R & \xrightarrow{f} & Mat_p(R') \\ {\scriptstyle M}\big\downarrow\bullet & \overset{\phi}{\Rightarrow} & \bullet\big\downarrow{\scriptstyle Mat_{q,p}(M')} \\ S & \xrightarrow[g]{} & Mat_q(S') \end{array}$$

where $Mat_{q,p}(M')$ is the bimodule of $q \times p$ matrices with entries in M', with the $Mat_q(S')$ action given by matrix multiplication on the left, and similarly for $Mat_p(R')$.

Theorem 15 *(1)* $\mathbb{R}\text{ing}_{Mat}$ *is a double category.*
(2) $\mathbb{R}\text{ing}_{Mat}$ *is vertically self dual, i.e.* $\mathbb{R}\text{ing}^{co}_{Mat} \cong \mathbb{R}\text{ing}_{Mat}$.
(3) Every horizontal arrow has a companion.

Proof (1) This is a straightforward but long and uninformative calculation, best done in the context of graded monads, so is omitted here.
(2) This is not completely obvious because $Mat_p(S)^{op}$ is not the same as $Mat_p(S^{op})$. As sets they are the same but the multiplications are different. If we evaluate A times B in each of these, the b's come before the a's but in the first case it's column times row and the reverse in the second. But they are isomorphic, the isomorphisms given by transpose

$$t_S \colon Mat_p(S)^{op} \longrightarrow Mat_p(S^{op})$$

$$A \longmapsto A^T\,.$$

If we denote the opposite product by $*$, then

$$t_S(A * B) = t_S(BA) = (BA)^T$$

whose $(i,j)^{th}$ entry is

$$\sum_k b_{jk} a_{ki}\,.$$

On the other hand

$$t_S(A) t_S(B) = A^T B^T$$

whose $(i,j)^{th}$ entry is

$$\sum_k a_{ki} * b_{jk} = \sum_k b_{jk} a_{ki}\,.$$

The vertical involution

$$\Theta\colon \mathbb{R}\mathrm{ing}^{co}_{Mat} \longrightarrow \mathbb{R}\mathrm{ing}_{Mat}$$

is defined on objects by taking the opposite ring

$$\Theta(R) = R^{op}$$

and on vertical arrows, $M\colon R \longrightarrow\!\!\!\!\!\!\bullet\;\; S$, it is as before:

$$\Theta(M)\colon S^{op} \longrightarrow\!\!\!\!\!\!\bullet\;\; R^{op}$$

is M considered as a left R^{op} right S^{op} bimodule. For a horizontal arrow $(p, f)\colon R \longrightarrow S$, $\Theta(p, f)\colon R^{op} \longrightarrow S^{op}$ is given by

$$R^{op} \xrightarrow{f^{op}} Mat_p(S)^{op} \xrightarrow{t_S} Mat_p(S^{op})\,.$$

For a cell

$$\begin{array}{ccc} R & \xrightarrow{(p,f)} & R' \\ {\scriptstyle M}\big\downarrow\bullet & \overset{\phi}{\Rightarrow} & \bullet\big\downarrow{\scriptstyle M'} \\ S & \xrightarrow[(q,g)]{} & S' \end{array}$$

$$\begin{array}{ccc} S^{op} & \xrightarrow{\Theta(q,g)} & S'^{op} \\ {\scriptstyle M}\big\downarrow\bullet & \overset{\Theta(\phi)}{\Longrightarrow} & \bullet\big\downarrow{\scriptstyle M'} \\ R^{op} & \xrightarrow[\Theta(p,f)]{} & R'^{op} \end{array}$$

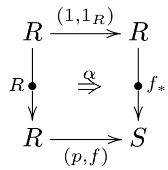
is given by

$$\begin{array}{ccccc} S^{op} & \xrightarrow{g} & Mat_q(S')^{op} & \xrightarrow{t_{S'}} & Mat_q(S'^{op}) \\ {\scriptstyle M}\big\downarrow\bullet & \overset{\phi}{\Longrightarrow} & \bullet\big\downarrow{\scriptstyle Mat_{q,p}(M')} & \overset{t_{M'}}{\Longrightarrow} & \bullet\big\downarrow{\scriptstyle Mat_{p,q}(M')} \\ R^{op} & \xrightarrow[f]{} & Mat_p(R')^{op} & \xrightarrow[t_{R'}]{} & Mat_p(R'^{op})\,. \end{array}$$

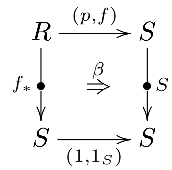
Here $t_{M'}$ is also taking transpose.
(3) Given a horizontal arrow $(p, f)\colon R \longrightarrow S$, i.e. a homomorphism $f\colon R \longrightarrow Mat_p(S)$, its companion is the bimodule

$$f_* = S^{(p)}\colon R \longrightarrow\!\!\!\!\!\!\bullet\;\; S$$

introduced in the previous section. The binding cells



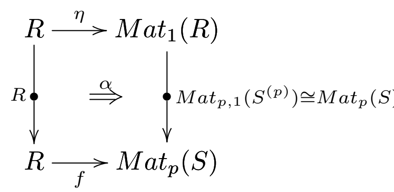
and



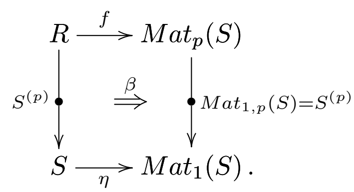
,

are given by

$\alpha(r) = f(r)$



$\beta(\mathbf{s}) = \mathbf{s}$



The verification of the linearity of α and β and the binding equations are left to the reader. □

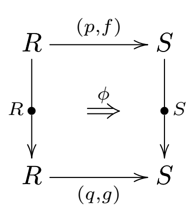
Corollary 16 *In* $\mathbb{R}\mathrm{ing}_{Mat}$*, every horizontal arrow has a conjoint.*

We can now describe the 2-cells in the 2-category $\mathcal{H}or\mathbb{R}\mathrm{ing}_{Mat}$ explicitly in terms of matrices.

Proposition 17 *Given morphisms* (p, f) *and* $(q, g)\colon R \longrightarrow S$ *in* $\mathcal{H}or\mathbb{R}\mathrm{ing}_{Mat}$*, a 2-cell* $\phi\colon (p, f) \longrightarrow (q, g)$ *is a* $q \times p$ *matrix* A *with entries in* S*, such that for every* $r \in R$*,*

$$Af(r) = g(r)A\,.$$

Proof A 2-cell $\phi\colon (p, f) \longrightarrow (q, g)$ is a double cell in $\mathbb{R}\mathrm{ing}_{Mat}$



which is a double cell

$$\begin{array}{ccc} R & \xrightarrow{f} & Mat_p(S) \\ {\scriptstyle R}\,\big\downarrow\!\bullet & \overset{\phi}{\Longrightarrow} & \bullet\!\big\downarrow\,{\scriptstyle Mat_{q,p}(S)} \\ R & \xrightarrow[g]{} & Mat_q(S)\,. \end{array}$$

This is entirely determined by its value at 1

$$\begin{aligned} \phi(r) &= \phi(1)f(r) \\ &= g(r)\phi(1)\,. \end{aligned}$$

Take $A = \phi(1)$. □

Remark 18 We only looked at free modules of finite rank, partly in preparation for the next section, but we can also consider infinite rank ones. Then the matrices are row finite, i.e. for every i, $a_{ij} = 0$ except for finitely many j. Now the distinction between row vectors and column vectors is clear. The first have finite support whereas the second are arbitrary. The double category we would get this way would not be vertically self dual. Every horizontal arrow would still have a companion, the coproduct of copies of the codomain. But conjoints don't always exist. There is a candidate for the conjoint, the product of copies of the codomain, but only one of the conjunctions generalizes.

5 Adjoint bimodules

If, in a double category $\mathbb{A}$, a horizontal arrow $f\colon A \to B$ has a companion f_* and a conjoint f^* then f_* is left adjoint to f^* in $\mathbb{V}\mathrm{ert}\mathbb{A}$. The unit and counit of the adjunction are given by

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle \mathrm{id}_A}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle f_*} \\ A & \xrightarrow{f} & B \\ {\scriptstyle \mathrm{id}_A}\big\downarrow\!\bullet & \psi & \bullet\!\big\downarrow{\scriptstyle f^*} \\ A & \xrightarrow[1_A]{} & A \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \xrightarrow{1_B} & B \\ {\scriptstyle f^*}\big\downarrow\!\bullet & \chi & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}_B} \\ A & \xrightarrow{f} & B \\ {\scriptstyle f_*}\big\downarrow\!\bullet & \beta & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}_B} \\ B & \xrightarrow[1_B]{} & B\,. \end{array}$$

Definition 19 We say that an object B of a double category $\mathbb{A}$ is *Cauchy complete* if every vertical arrow $v\colon A \to\!\!\bullet\!\!\to B$ with a right adjoint is the companion of a horizontal arrow. We say that $\mathbb{A}$ is *Cauchy* if every object is Cauchy complete.

Remark 20 The notion of Cauchy completeness for enriched categories (which we are extending to double categories here) was introduced by Lawvere in [6].

One might ask then, is the double category $\mathbb{R}\mathrm{ing}_{Mat}$ Cauchy? Not quite, but almost. And this leads to a further generalization of morphism of rings.

Recall that two bimodules $M\colon R \xrightarrow{\bullet} S$ and $N\colon S \xrightarrow{\bullet} R$ are adjoint, or more precisely M is left adjoint to N, if there are an S-S linear map

$$\epsilon\colon M \otimes_R N \longrightarrow S$$

and an R-R linear map

$$\eta\colon R \longrightarrow N \otimes_S M$$

such that

$$\begin{array}{ccccc} & & M \otimes_R N \otimes_S M & & \\ & {\scriptstyle M\otimes_R \eta} \nearrow & & \searrow {\scriptstyle \epsilon \otimes_S M} & \\ M \otimes_R R & \xrightarrow[\cong]{} & M & \xrightarrow[\cong]{} & S \otimes_S M \end{array}$$

and

$$\begin{array}{ccccc} & & N \otimes_S M \otimes_R N & & \\ & {\scriptstyle \eta\otimes_R N} \nearrow & & \searrow {\scriptstyle N \otimes_S \epsilon} & \\ R \otimes_R N & \xrightarrow[\cong]{} & N & \xrightarrow[\cong]{} & M \otimes_S S \end{array}$$

commute.

The following theorem is well-known.

Theorem 21 *A bimodule $M\colon R \xrightarrow{\bullet} S$ has a right adjoint if and only if it is finitely generated and projective as a left S-module.*

It is easier to give a proof than to hunt down a reference which gives it in the precise form we want. We do this after some preliminary remarks.

M is finitely generated, by $m_1, \ldots, m_p$ say, if and only if the S-linear map

$$\tau\colon S^{(p)} \longrightarrow M$$

$\tau(s_1 \ldots s_p) = \sum_{i=1}^{p} s_i m_i$ is surjective. If M is S-projective, then τ splits, i.e. there is an S-linear map

$$\sigma\colon M \longrightarrow S^{(p)}$$

such that $\tau\sigma = 1_M$. In fact, M is a finitely generated and projective S-module if and only if there exist p, τ, σ such that $\tau\sigma = 1_M$.

Let the components of σ be $\sigma_1, \ldots, \sigma_p\colon M \longrightarrow S$. Then $\tau\sigma = 1_M$ means that for every $m \in M$ we will have

$$m = \sum_{i=1}^{p} \sigma_i(m) m_i$$

i.e. the σ_i provide an S-linear choice of coordinates for m relative to the generators $m_1 \dots m_p$. All of this is independent of R.

Proof (Of Theorem 21) Suppose M is left adjoint to N with notation as above. Then $\eta(1) = \sum_{i=1}^{p} n_i \otimes m_i$. The first triangle equation gives for any m

$$\epsilon(m \otimes n_i)m_i = m\,.$$

So we take $\tau\colon S^{(r)} \longrightarrow M$ to be

$$\tau(\mathbf{s}) = \sum_{i=1}^{n} s_i m_i$$

and

$$\sigma_i(m) = \epsilon(m \otimes n_i)\,.$$

Then $\tau\sigma = 1_\mu$ and M is finitely generated and projective.

Conversely, take $N = M^* = Hom_S(M, S)$. We immediately get

$$\epsilon\colon M \otimes_R N \longrightarrow S$$

$$\epsilon(m \otimes f) = f(m)\,.$$

If M is finitely generated and projective, we have $\tau\colon S^{(p)} \longrightarrow M$ and $\sigma\colon M \longrightarrow S^{(p)}$. Define $\eta\colon R \longrightarrow N \otimes_S M$ by

$$\eta(1) = \sum_{i=1}^{p} \sigma_i \otimes \tau(\mathbf{e}_i)\,.$$

The triangle equations are easily checked. □

Given this theorem then, we see that S is Cauchy-complete in $\mathbb{R}\mathrm{ing}_{Mat}$ if and only if every finitely generated projective left S-module is free. Commutative rings with this property are of considerable interest in algebraic geometry having to do with when vector bundles are trivial. If S is a PID or a local ring then it is Cauchy. That polynomial rings are so is the content of the Quillen–Suslin theorem, which is highly non trivial. The fact that Cauchy completeness in $\mathbb{R}\mathrm{ing}_{Mat}$ leads to such questions gives some legitimacy to this double category.

Finitely generated projective is the next best thing to free of finite rank, so how does this relate to the previous sections?

For any r we can write

$$m_i r = \sum_{j=1}^{p} \sigma_j(m_i r) m_j\,.$$

If we let $f_{ij}(r) = \sigma_j(m_i r)$ we get

$$m_i r = \sum_{j=1}^{p} f_{ij}(r) m_j \qquad (*)$$

the same formula as in §3.

Theorem 22 *(1) The functions f_{ij} define a non-unitary homomorphism $f\colon R \longrightarrow Mat_p(S)$.*
(2) Any such homomorphism comes from a bimodule which is finitely generated and projective as a left S-module.

Proof (a) f is clearly additive.
Multiply $(*)$ by r' on the right and apply σ_k to get

$$\sigma_k(m_i r r') = \sum_{j=1}^{p} f_{ij}(r)\sigma_k(m_j r')$$

i.e.

$$f_{ik}(rr') = \sum_{j=1}^{p} f_{ij}(r) f_{jk}(r')\,.$$

Thus f is a homomorphism $R \longrightarrow Mat_p(S)$. $f_{ij}(1) = \sigma_j(m_i)$ and so corresponds to the linear map $\sigma\tau\colon S^{(p)} \longrightarrow S^{(p)}$, which is not the identity unless the m_i form a basis.
(b) Given a homomorphism $f\colon R \longrightarrow Mat_p(S)$, define M by

$$M = \left\{ \mathbf{s} \in S^{(p)} \mid \mathbf{s} f(1) = \mathbf{s} \right\}\,.$$

M is an S-R-bimodule. First of all it's clearly a sub left S-module of $S^{(p)}$. Define the right action of R by

$$\mathbf{s} \bullet r = \mathbf{s} f(r)\,.$$

$\mathbf{s}f(r)f(1) = \mathbf{s}f(r1) = \mathbf{s}f(r)$ so $\mathbf{s}f(r) \in M$. The bimodule equations automatically hold because f is a homomorphism: the only thing to check is $\mathbf{s} \bullet 1 = \mathbf{s}$, i.e. $\mathbf{s}f(1) = \mathbf{s}$, which is in the definition of M.

Define $\tau\colon S^{(p)} \longrightarrow M$ by $\tau(\mathbf{s}) = \mathbf{s}f(1)$ and let $\sigma\colon M \longrightarrow S^{(p)}$ be the inclusion. Clearly $\tau\sigma = 1_M$, so M is finitely generated projective as a left S-module. The generators are $\tau(\mathbf{e}_i) = \mathbf{e}_i f(1)$. Now let $g\colon R \longrightarrow Mat_p(S)$ be the homomorphism defined by

$$g_{ij}(r) = \sigma_j(\mathbf{e}_i f(1) \bullet r)$$

as in the discussion just before the statement of the theorem. Then

$$g_{ij}(r) = \sigma_j(\mathbf{e}_i f(1) f(r)) = \sigma_j(\mathbf{e}_i f(r))$$

which is the j^{th} component of the i^{th} column of $f(r)$, i.e. $g_{ij}(r) = f_{ij}(r)$. $\square$

Homomorphisms $R \longrightarrow Mat_p(S)$ have already appeared in the quantum field theory literature (see e.g. [7]) where they are called *amplifying homomorphisms* or *amplimorphisms* for short.

Let's define the double category $\mathbb{A}$mpli whose objects are rings (with 1), horizontal arrows are amplimorphisms $R \longrightarrow S$, i.e. non-unitary homomorphisms $R \longrightarrow Mat_p(S)$ for some p. Composition is like for $\mathbb{R}\mathrm{ing}_{Mat}$: first apply f to an element $r \in R$ to get a $p \times p$ matrix in S, and then apply g to each entry separately to get a $p \times p$ block matrix of $q \times q$ matrices, and then consider this as a $(pq) \times (pq)$ matrix.

Vertical arrows are bimodules $M \colon R \longrightarrow\!\!\!\!\bullet\; S$ and cells

$$\begin{array}{ccc} R & \xrightarrow{(p,f)} & R' \\ {\scriptstyle M}\big\downarrow\bullet & \overset{\phi}{\Rightarrow} & \bullet\big\downarrow{\scriptstyle M'} \\ S & \xrightarrow[(q,g)]{} & S' \end{array}$$

are cells

$$\begin{array}{ccc} R & \xrightarrow{f} & Mat_p(R') \\ {\scriptstyle M}\big\downarrow\bullet & \overset{\phi}{\Rightarrow} & \bullet\big\downarrow{\scriptstyle Mat_{q,p}(M')} \\ S & \xrightarrow[g]{} & Mat_q(S') \end{array}$$

i.e. additive functions $\phi \colon M \longrightarrow Mat_{q,p}(M')$ such that for every $m \in M$, $r \in R$, $s \in S$ we have

$$\phi(mr) = \phi(m)f(r)$$

$$\phi(sm) = g(s)\phi(m)\,.$$

Here we have taken the definition of cells to be the same as for $\mathbb{R}\mathrm{ing}_{Mat}$, which doesn't refer to identities at all and doesn't need modification. One could instead define cells as S-R-linear maps from M into "$Mat_{q,p}(M')$ with scalars restricted" along f and g. For non-unitary homomorphisms, restriction of scalars doesn't work exactly as for unitary ones. A modification is required to insure the unit conditions for the action. One has to look at S-R-linear maps from M into

$$\{A \in Mat_{q,p}(M') \mid Af(1) = A = g(1)A\}\,.$$

However this is easily seen to be equivalent to the definition we have given.

Theorem 23 *(1)* $\mathbb{A}$mpli *is a double category.*
(2) $\mathbb{A}$mpli *is vertically self dual.*
(3) Every horizontal arrow has a companion and a conjoint.
(4) Every adjoint pair of vertical arrows is represented by a horizontal one, i.e. $\mathbb{A}$mpli *is Cauchy.*

Proof (1) Horizontal composition of arrows and cells is the same as for $\mathbb{R}\text{ing}_{Mat}$, as is vertical composition. We check that the vertical composition of cells, as given in $\mathbb{R}\text{ing}_{Mat}$, is well-defined even if g is not unitary. Consider the composition of

$$\begin{array}{ccc} R & \xrightarrow{(p,f)} & R' \\ {\scriptstyle M}\bullet\downarrow & \overset{\phi}{\Rightarrow} & \downarrow\bullet{\scriptstyle M'} \\ S & \xrightarrow{(q,g)} & S' \\ {\scriptstyle N}\bullet\downarrow & \overset{\psi}{\Rightarrow} & \downarrow\bullet{\scriptstyle N'} \\ T & \xrightarrow[(l,h)]{} & T' \,. \end{array}$$

It is given by

$$\begin{array}{ccc} R & \xrightarrow{f} & Mat_p(R') \\ {\scriptstyle N\otimes_S M}\bullet\downarrow & \overset{\psi\otimes_g\phi}{\Longrightarrow} & \downarrow\bullet{\scriptstyle Mat_{l,q}(N')\otimes_{Mat_q(S')}Mat_{q,p}(M')} \\ T & \xrightarrow[h]{} & Mat_l(T') \end{array}$$

followed by the canonical

$$Mat_{l,q}(N') \otimes_{Mat_q(S')} Mat_{q,p}(M') \longrightarrow Mat_{l,p}(N' \otimes_{S'} M')\,.$$

$\psi \otimes_g \phi$ is defined by

$$(\psi \otimes_g \phi)(n \otimes_S m) = \psi(n) \otimes_{Mat_q(S')} \phi(m)$$

and the only place that g enters is in the equation

$$(\psi \otimes_g \phi)(ns \otimes_S m) = (\psi \otimes_g \phi)(n \otimes_S sm)$$

i.e.

$$\psi(ns) \otimes_{Mat_q(S')} \phi(m) = \psi(n) \otimes_{Mat_q(S')} \phi(sm)$$

or

$$\psi(n)g(s) \otimes_{Mat_q(S')} \phi(m) = \psi(n) \otimes_{Mat_q(S')} g(s)\phi(m)$$

which clearly holds. “Unitarity” does not enter into it.
(2) The vertical duality is the same as for $\mathbb{R}\text{ing}_{Mat}$, i.e. taking the opposite ring and adjusting the horizontal arrows by the use of transpose.
(3) Given an amplimorphism $(p, f)\colon R \longrightarrow S$, we've already constructed its companion in Theorem 22:

$$(p, f)_* = \{\mathbf{s} \in S^{(p)} \mid \mathbf{s}f(1) = \mathbf{s}\}\,.$$

We have just to show that it's actually a companion. Define the binding cells as follows:

$$
\begin{array}{ccc}
R & \xrightarrow{(1,1_R)} & R \\
{\scriptstyle R}\bullet\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow\bullet{\scriptstyle (p,f)_*} \\
R & \xrightarrow[(p,f)]{} & S
\end{array}
\qquad
\begin{array}{ccc}
R & \xrightarrow{(p,f)} & S \\
{\scriptstyle (p,f)_*}\bullet\downarrow & \overset{\beta}{\Rightarrow} & \downarrow\bullet{\scriptstyle S} \\
S & \xrightarrow[(1,1_g)]{} & S
\end{array}
$$

are the linear maps

$$
\begin{array}{ccc}
R & \xrightarrow{1_R} & R \\
{\scriptstyle R}\bullet\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow\bullet{\scriptstyle Mat_{p,1}((p,f)_*)} \\
R & \xrightarrow[f]{} & Mat_p(S)
\end{array}
\qquad
\begin{array}{ccc}
R & \xrightarrow{f} & Mat_p(S) \\
{\scriptstyle (p,f)_*}\bullet\downarrow & \overset{\beta}{\Rightarrow} & \downarrow\bullet{\scriptstyle Mat_{l,p}(S)} \\
S & \xrightarrow[1_S]{} & S
\end{array}
$$

$$\alpha(r) = f(r) \qquad\qquad \beta(\mathbf{s}) = \mathbf{s}\,.$$

In the definition of α, note that $Mat_{p,1}((p,f)_*)$ is a $p \times 1$ matrix of $1 \times p$ matrices so can be identified with a $p \times p$ matrix. We have to check that the rows satisfy the defining property of $(p,f)_*$, $\mathbf{s}f(1) = \mathbf{s}$. This is done simultaneously for all rows by $f(r)f(1) = f(r)$ (the i^{th} row is $\mathbf{e}_i f(r)f(1) = \mathbf{e}_i f(r)$).

The existence of conjoints is dual.

(4) This is just a formal summary of the preceding discussion: M has a right adjoint if and only if it is finitely generated and projective and this holds if and only if it is induced by a non-unitary homomorphism into a matrix ring. □

We can now describe explicitly, in terms of matrices, the 2-category $\mathcal{A}mpli = \mathcal{H}or\mathbb{A}\text{mpli}$. The objects are rings with 1, the morphisms are pairs (p,f) where p is a positive integer and $f\colon R \longrightarrow Mat_p(S)$ is a (not necessarily unitary) homomorphism. Composition $(p',f')(p,f)$ is $(p'p,h)$ where

$$h = (R \xrightarrow{f} Mat_p(S) \xrightarrow{Mat_p(f')} Mat_p Mat_{p'}(T) \cong Mat_{p'p}(T))\,.$$

A 2-cell $t\colon (p,f) \Rightarrow (q,g)$ is an R-R linear map

$$
\begin{array}{ccc}
R & \xrightarrow{f} & Mat_p(S) \\
{\scriptstyle R}\bullet\downarrow & \overset{\phi}{\Rightarrow} & \downarrow\bullet{\scriptstyle Mat_{q,p}(S)} \\
R & \xrightarrow[g]{} & Mat_q(S)
\end{array}
$$

which is uniquely determined by its value on 1, as

$$\begin{aligned}\phi(r) &= \phi(1 \cdot r) = \phi(1)f(r) \\ &= \phi(r \cdot 1) = g(r)\phi(1)\,.\end{aligned}$$

Given any $q \times p$ matrix A such that $Af(r) = g(r)A$ for all r, the function $\phi(r) = Af(r)$ gives such a cell. However, different A's may give the same ϕ. Indeed $(Af(1))f(r) = Af(r)$. To get uniqueness we have only to impose the extra condition $Af(1) = A$. Note that the vertical identity transformation on f is not I_p, the identity $p \times p$ matrix, which obviously doesn't satisfy this last condition, but rather $f(1)$ itself.

We summarize this discussion in the following.

Proposition 24 *The* 2*-category* $\mathcal{A}mpli$ *of amplifying homomorphisms has unitary rings as objects, amplimorphisms* $(p, f)\colon R \longrightarrow S$ *as morphisms and as* 2*-cells* $\phi\colon (p, f) \Rightarrow (q, g)$*,* $q \times p$ *matrices* A *such that*
(1) $Af(1) = A$
(2) *for every* $r \in R, Af(r) = g(r)A\,.$
The identity 2*-cell on* (p, f) *is the* $p \times p$ *matrix* $f(1)$*.*

Corollary 25 *Two representations* (p, f) *and* (q, g) *of the same* S*-*R*-bimodule are related as follows: There is a* $q \times p$ *matrix* A *and a* $p \times q$ *matrix* B *such that*
(1) $Af(1) = A$ *and* $Af(r) = g(r)A$
(2) $Bg(1) = B$ *and* $Bg(r) = f(r)B$
(3) $AB = g(1)$ *and* $BA = f(1)\,.$

Postscript

Double category considerations have naturally led to generalizing homomorphisms to amplimorphisms which arose independently in connection with quantum field theory. We also discovered a natural notion of 2-cell allowing us to compare parallel amplimorphisms. These are called intertwiners in the physics literature. Even if we restrict to actual homomorphisms the 2-cells are not trivial and provide a good unifying notion.

Amplimorphisms of degree 1 are non-unitary homomorphisms. I don't know what Jim would make of that, but later in life he had turned his attention to quantum mechanics, so I like to believe that he would be pleased with these developments.

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