What is a doubly involutive monoidal category?

(notes on a talk given at Dalhousie on 3 March 2015)

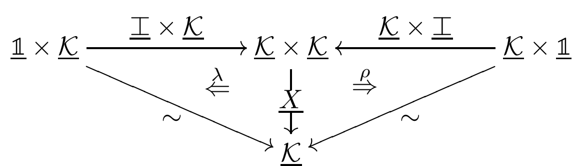
JME

March 5, 2015

1 Recap of Joyal & Street

Let $\underline{\mathcal{K}} = (\mathcal{K}, \otimes, \mathbb{I})$ be a monoidal category. A monoidal functor $\underline{\mathbb{1}} \to \underline{\mathcal{K}}$ is the same thing as a monoid in $\underline{\mathcal{K}}$. In particular, the trivial monoid defines a monoidal functor $\underline{\mathbb{I}} = (\mathbb{I}, \iota, \mathring{\iota}) : \underline{\mathbb{1}} \to \underline{\mathcal{K}}$, which happens to be strong. (In general, our notation for monoidal functors follows a similar pattern: $\underline{M} = (M, \mu, \mathring{\mu}) : \underline{\mathcal{J}} \to \underline{\mathcal{K}}$.) In fact, $\underline{\mathbb{I}}$ is the only strong monoidal functor $\underline{\mathbb{1}} \to \underline{\mathcal{K}}$, up to isomorphism, of course.

Now suppose that $\underline{X} = (X, \chi, \mathring{\chi})$ is a strong monoidal functor $\underline{\mathcal{K}} \times \underline{\mathcal{K}} \to \underline{\mathcal{K}}$, and that λ, ρ are monoidal natural isomorphisms of the form below.

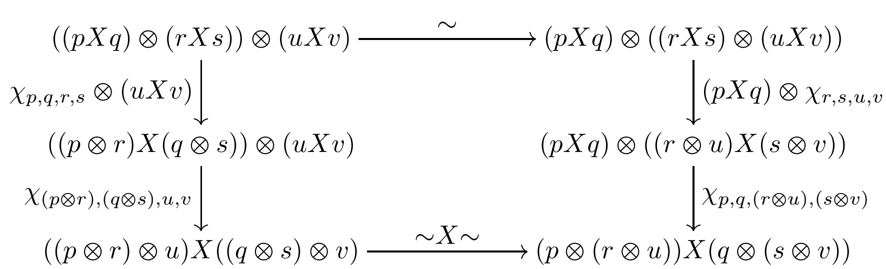


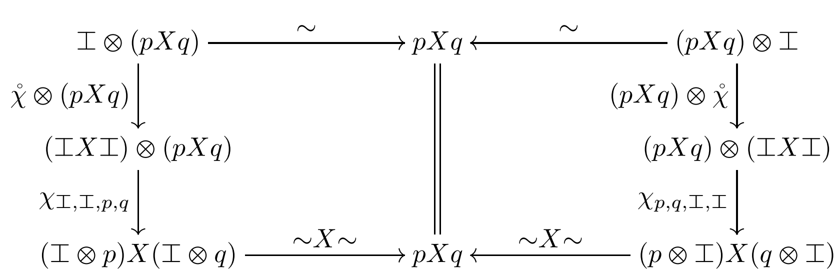
Writing pXq in place of $X(p, q)$, this means we have arrows

$$(pXq) \otimes (rXs) \xrightarrow{\chi_{p,q,r,s}} (p \otimes r)X(q \otimes s) \qquad \mathbb{I} \xrightarrow{\mathring{\chi}} \mathbb{I}X\mathbb{I}$$

$$\mathbb{I}Xp \xrightarrow{\lambda_p} p \qquad pX\mathbb{I} \xrightarrow{\rho_p} p$$

satisfying the diagrams below.





$$
\begin{array}{ccccc}
(\mathbb{I}Xp)\otimes(\mathbb{I}Xq) & \xrightarrow{\chi_{\mathbb{I},p,\mathbb{I},q}} & (\mathbb{I}\otimes\mathbb{I})X(p\otimes q) & \xrightarrow{\iota X(p\otimes q)} & \mathbb{I}X(p\otimes q) \\
\lambda_p\otimes\lambda_q \downarrow & & & & \downarrow \lambda_{p\otimes q} \\
p\otimes q & & = & & p\otimes q \\
\rho_p\otimes\rho_q \uparrow & & & & \uparrow \rho_{p\otimes q} \\
(pX\mathbb{I})\otimes(qX\mathbb{I}) & \xrightarrow{\chi_{p,\mathbb{I},q,\mathbb{I}}} & (p\otimes q)X(\mathbb{I}\otimes\mathbb{I}) & \xrightarrow{(p\otimes q)X\iota} & (p\otimes q)X\mathbb{I}
\end{array}
$$

$$
\begin{array}{ccccc}
\mathbb{I} & \xrightarrow{\mathring{\chi}} & \mathbb{I}X\mathbb{I} & \xrightarrow{\mathbb{I}X\mathring{\iota}} & \mathbb{I}X\mathbb{I} \\
\| & & & & \downarrow \lambda_{\mathbb{I}} \\
\mathbb{I} & & = & & \mathbb{I} \\
\| & & & & \uparrow \rho_{\mathbb{I}} \\
\mathbb{I} & \xrightarrow{\mathring{\chi}} & \mathbb{I}X\mathbb{I} & \xrightarrow{\mathring{\iota}X\mathbb{I}} & \mathbb{I}X\mathbb{I}
\end{array}
$$

(and the same again for ρ).

One obtains natural isomorphisms

$$
p\otimes s \xrightarrow{\rho_p^{-1}\otimes\lambda_s^{-1}} (pX\mathbb{I})\otimes(\mathbb{I}Xs) \xrightarrow{\chi_{p,\mathbb{I},\mathbb{I},s}} (p\otimes\mathbb{I})X(\mathbb{I}\otimes s) \xrightarrow{\sim X\sim} pXs
$$

$$
q\otimes r \xrightarrow{\lambda_q^{-1}\otimes\rho_r^{-1}} (\mathbb{I}Xq)\otimes(rX\mathbb{I}) \xrightarrow{\chi_{\mathbb{I},q,r,\mathbb{I}}} (\mathbb{I}\otimes r)X(q\otimes\mathbb{I}) \xrightarrow{\sim X\sim} rXq
$$

—which we denote $\alpha_{p,s}$ and $\beta_{q,r}$, respectively. Then the composite

$$
q\otimes r \xrightarrow{\beta_{q,r}} rXq \xrightarrow{\alpha_{r,q}^{-1}} r\otimes q
$$

defines a braid on $\otimes$. Furthermore,

$$
\begin{array}{ccc}
(p\otimes q)\otimes(r\otimes s) & \xrightarrow{\alpha_{p,q}\otimes\alpha_{r,s}} & (pXq)\otimes(rXs) \\
\text{braid-induced interchange} \downarrow & & \downarrow \chi_{p,q,r,s} \\
(p\otimes r)\otimes(q\otimes s) & \xrightarrow{\alpha_{(p\otimes r),(q\otimes s)}} & (p\otimes r)X(q\otimes s)
\end{array}
$$

and

$$
\begin{array}{ccc}
\mathbb{I} & = & \mathbb{I} \\
\text{canonical isomorphism} \downarrow & & \downarrow \mathring{\chi} \\
\mathbb{I}\otimes\mathbb{I} & \xrightarrow{\alpha_{\mathbb{I},\mathbb{I}}} & \mathbb{I}X\mathbb{I}
\end{array}
$$

—so α defines a monoidal natural isomorphism $\underline{\otimes}\to\underline{X}$, where $\underline{\otimes}$ is the monoidal functor $\underline{\mathcal{K}}\times\underline{\mathcal{K}}\to\underline{\mathcal{K}}$ comprising the functor $\otimes:\mathcal{K}\times\mathcal{K}\to\mathcal{K}$ together with the braid-induced interchange and canonical isomorphism, as above.

Hence there is no "essential" loss of generality in assuming α to be the identity. Thus, in this case: β is the braid, χ is the braid-induced interchange, and $\mathring{\chi}$ is the canonical isomorphism.

Indeed, in general,

$$\begin{array}{ccc} \mathbb{I}\otimes p & \xrightarrow{\alpha_{\mathbb{I},p}} & \mathbb{I}Xp \\ & {\scriptstyle\sim}\searrow \quad \swarrow{\scriptstyle\lambda_p} & \\ & p & \end{array} \qquad \begin{array}{ccc} p\otimes \mathbb{I} & \xrightarrow{\alpha_{p,\mathbb{I}}} & pX\mathbb{I} \\ & {\scriptstyle\sim}\searrow \quad \swarrow{\scriptstyle\rho_p} & \\ & p & \end{array}$$

so α is even an isomorphism between the ensemble $(\otimes, \sim, \sim)$ and (X, λ, ρ).

2 Monoidal IMCs

For definitions of all things involutive monoidal: see *On involutive monoidal categories*.

Let $\underline{\mathcal{K}} = (\mathcal{K}, \otimes, \overline{(\ )}, \mathbb{I})$ be an involutive monoidal category. An involutive monoidal functor $\underline{\mathbb{1}} \to \underline{\mathcal{K}}$ is the same thing as a dagger monoid in $\underline{\mathcal{K}}$. There is a trivial dagger monoid which defines a strong involutive monoidal functor $\underline{\mathbb{I}} = (\mathbb{I}, \iota, \dot{\iota}, \mathring{\iota}) : \underline{\mathbb{1}} \to \underline{\mathcal{K}}$. (In general, our notation for involutive monoidal functors follows this pattern: $\underline{M} = (M, \mu, \dot{\mu}, \mathring{\mu}) : \underline{\mathcal{J}} \to \underline{\mathcal{K}}$.) I believe that this is the only strong involutive monoidal functor $\underline{\mathbb{1}} \to \underline{\mathcal{K}}$, up to isomorphism, of course.

As in the previous section, almost all canonical isomorphisms, including now $\overline{p} \otimes \overline{q} \to \overline{q \otimes p}$, $\overline{\overline{p}} \to p$, and $\mathbb{I} \to \overline{\mathbb{I}}$, will be denoted simply $\sim$; the occasional exceptions are ι, $\dot{\iota}$, and $\mathring{\iota}$, all of which are all canonical.

Suppose that $\underline{X} = (X, \chi, \dot{\chi}, \mathring{\chi})$ is a strong involutive monoidal functor $\underline{\mathcal{K}} \times \underline{\mathcal{K}} \to \underline{\mathcal{K}}$, and that λ, ρ are involutive monoidal natural isomorphisms of the form below.

$$\begin{array}{ccccc} \underline{\mathbb{1}} \times \underline{\mathcal{K}} & \xrightarrow{\underline{\mathbb{I}} \times \underline{\mathcal{K}}} & \underline{\mathcal{K}} \times \underline{\mathcal{K}} & \xleftarrow{\underline{\mathcal{K}} \times \underline{\mathbb{I}}} & \underline{\mathcal{K}} \times \underline{\mathbb{1}} \\ & \overset{\lambda}{\Leftarrow} & \downarrow \underline{X} & \overset{\rho}{\Rightarrow} & \\ & {\scriptstyle\sim}\searrow & \underline{\mathcal{K}} & \swarrow{\scriptstyle\sim} & \end{array}$$

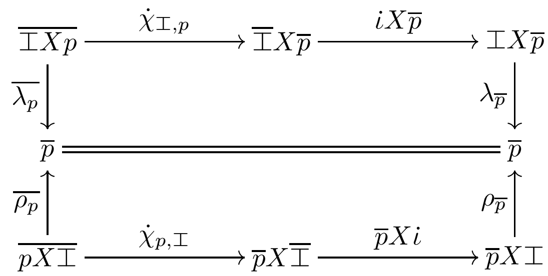
Again writing pXq in place of $X(p,q)$, this means we have arrows

$$\overline{pXq} \xrightarrow{\dot{\chi}_{p,q}} \overline{p}X\overline{q}$$

—in addition to those encountered in the previous section; moreover, they satisfy the diagrams

$$\begin{array}{ccc} \overline{pXq} \otimes \overline{rXs} & \xrightarrow{\sim} & \overline{(rXs) \otimes (pXq)} \\ {\scriptstyle \dot{\chi}_{p,q} \otimes \dot{\chi}_{r,s}}\downarrow & & \downarrow{\scriptstyle \overline{\chi_{r,s,p,q}}} \\ (\overline{p}X\overline{q}) \otimes (\overline{r}X\overline{s}) & & \overline{(r\otimes p)X(s\otimes q)} \\ {\scriptstyle \chi_{\overline{p},\overline{q},\overline{r},\overline{s}}}\downarrow & & \downarrow{\scriptstyle \dot{\chi}_{(r\otimes p),(s\otimes q)}} \\ (\overline{p}\otimes\overline{r})X(\overline{q}\otimes\overline{s}) & \xrightarrow{\sim X \sim} & \overline{r\otimes p}X\overline{s\otimes q} \end{array}$$

$$\begin{array}{ccc} \overline{\overline{pXq}} & \xrightarrow{\sim} & pXq \\ {\scriptstyle \overline{\dot{\chi}_{p,q}}}\downarrow & & \| \\ \overline{\overline{p}X\overline{q}} & & \| \\ {\scriptstyle \dot{\chi}_{\overline{p},\overline{q}}}\downarrow & & \| \\ \overline{\overline{p}}X\overline{\overline{q}} & \xrightarrow{\sim X \sim} & pXq \end{array}$$



—in addition to those encountered in the previous section.

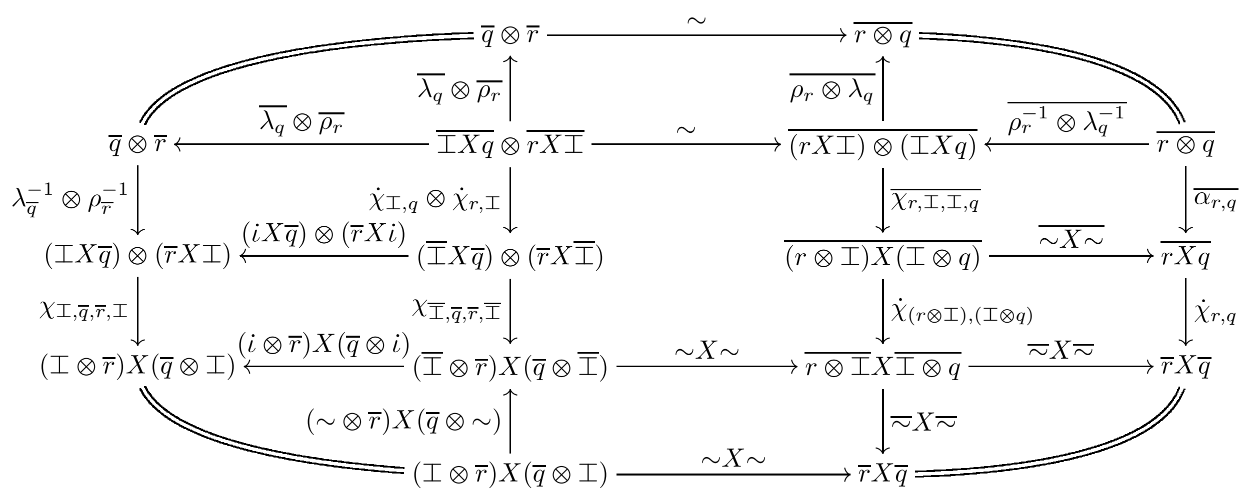
As before, one obtains natural isomorphisms

$$p \otimes s \xrightarrow{\rho_p^{-1} \otimes \lambda_s^{-1}} (pX\mathbb{I}) \otimes (\mathbb{I}Xs) \xrightarrow{\chi_{p,\mathbb{I},\mathbb{I},s}} (p \otimes \mathbb{I})X(\mathbb{I} \otimes s) \xrightarrow{\sim X \sim} pXs$$

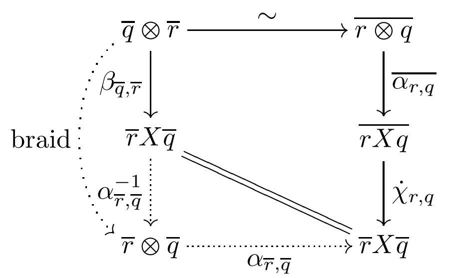
$$q \otimes r \xrightarrow{\lambda_q^{-1} \otimes \rho_r^{-1}} (\mathbb{I}Xq) \otimes (rX\mathbb{I}) \xrightarrow{\chi_{\mathbb{I},q,r,\mathbb{I}}} (\mathbb{I} \otimes r)X(q \otimes \mathbb{I}) \xrightarrow{\sim X \sim} rXq$$

—which we continue to denote $\alpha_{p,s}$ and $\beta_{q,r}$, respectively.

Now it is natural to conjecture that $\dot{\chi}$ is somehow related to the braid, and this is indeed true.

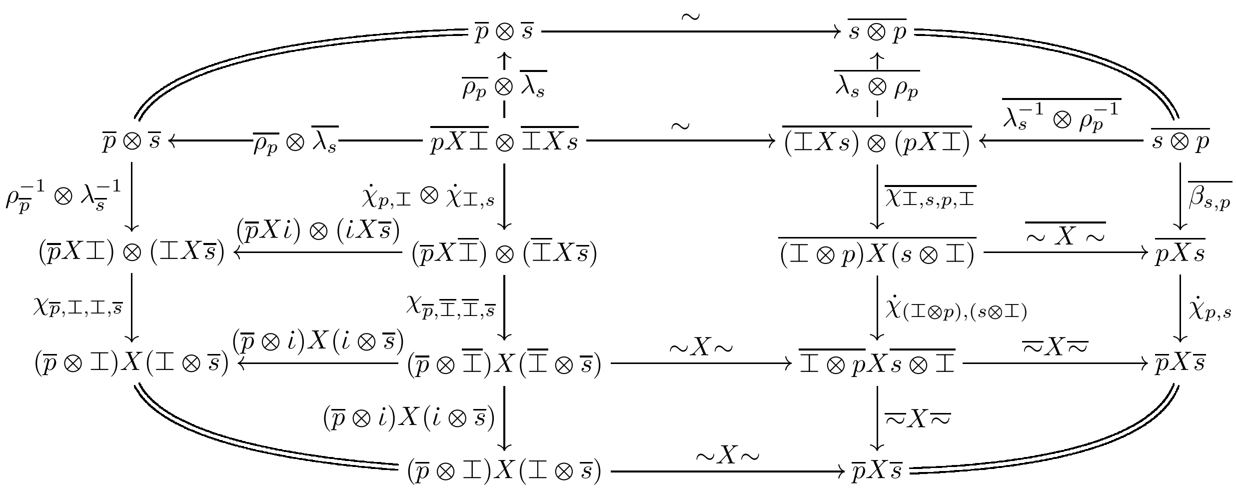


can be summarised as

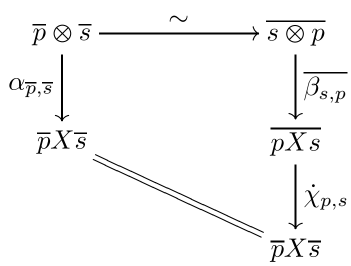


—so $\dot{\chi}$ is related to the braid by isomorphisms which can be assumed to be identities without "essential" loss of generality.

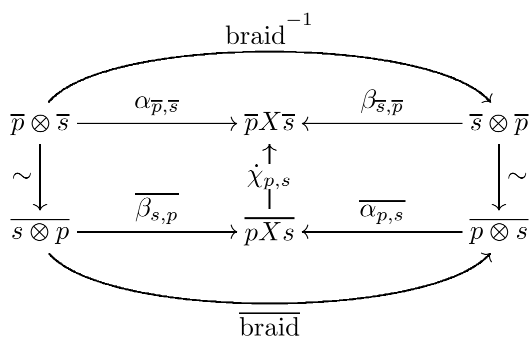
But are there any restrictions on the braid?



can be summarised as



—which, when combined with the previous characterisation of $\dot{\chi}$, results in the "anti-real" axiom of Beggs and Majid.



Is this all? I think so.

Summary Given an involutive monoidal category $\underline{\mathcal{K}} = (\mathcal{K}, \otimes, \overline{(\ )}, \mathbb{I})$, and a braid for $\otimes$ which satisfies the "anti-real" axiom above, we can make $\otimes$ into a strong involutive monoidal functor $\underline{\mathcal{K}} \times \underline{\mathcal{K}} \to \underline{\mathcal{K}}$, by equipping it with: the braid-induced interchange $(p \otimes q) \otimes (r \otimes s) \to (p \otimes r) \otimes (q \otimes s)$, the braid-induced involution

$$\overline{p \otimes q} \xrightarrow{\sim^{-1}} \overline{q} \otimes \overline{p} \xrightarrow{\text{braid}} \overline{p} \otimes \overline{q}$$

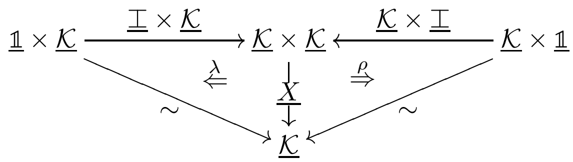
and the canonical isomorphism $\mathbb{I} \to \mathbb{I} \otimes \mathbb{I}$. Moreover, the canonical isomorphisms $\mathbb{I} \otimes p \to p$ and $p \otimes \mathbb{I} \to \mathbb{I}$ are involutive monoidal, wrt $\underline{\otimes}$ and $\underline{\mathbb{I}}$.

Conversely, given a strong involutive monoidal functor $\underline{X} : \underline{\mathcal{K}} \times \underline{\mathcal{K}} \to \underline{\mathcal{K}}$, and involutive monoidal natural isomorphisms λ, ρ of the relevant type, we can induce a braid on $\otimes$, and an involutive monoidal natural isomorphism $\underline{\otimes} \to \underline{X}$.

3 Involutive monoidal IMCs—take 1

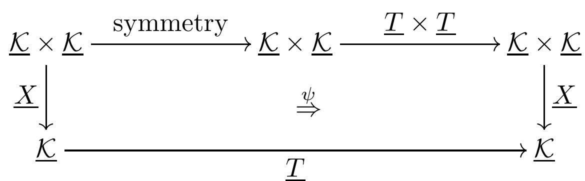
Now suppose that—in addition to the data given above:

1. an involutive monoidal category $\underline{\mathcal{K}} = (\mathcal{K}, \otimes, \overline{(\ )}, \mathbb{I})$,
2. a strong involutive monoidal functor $\underline{X} = (X, \chi, \dot{\chi}, \mathring{\chi}) : \underline{\mathcal{K}} \times \underline{\mathcal{K}} \to \underline{\mathcal{K}}$
3. involutive monoidal natural isomorphisms

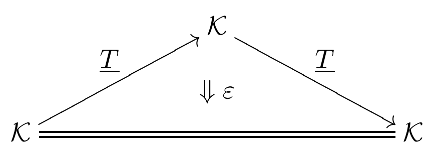


we also have

4. an involutive monoidal functor $\underline{T} = (T, \tau, \dot{\tau}, \mathring{\tau}) : \underline{\mathcal{K}} \to \underline{\mathcal{K}}$
5. an involutive monoidal natural isomorphism



6. an involutive monoidal natural isomorphism



satisfying various expected equations.

What then? As before, one can induce a natural isomorphism $\otimes \to X$, but one cannot induce an analogous natural isomorphism $\overline{(\ )} \to T$, as we shall see in the next talk.

However, in the case where $T = \overline{(\ )}$, one can derive a further structure on $\underline{\mathcal{K}}$, namely a *balance* ξ for the previously constructed braid, which also satisfies an "anti-real" axiom

$$\overline{\xi_p} = \xi_{\overline{p}}^{-1}$$

and I claim that that is all.

In other words, I claim that, given an involutive monoidal category $\underline{\mathcal{K}}$, and a braid β for $\otimes$, and a balance ξ for β, each satisfying the corresponding "anti-real" axiom, we can construct data as above with $T = \overline{(\ )}$.

But it now seems to me that the general case is of more interest than I originally thought, and I intend to explore it in my next talk.