Preliminaries to "the social life of generalised Hilbert objects"

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Involutive monoidal categories

Quasi-definition

An IMC is a monoidal category $(\mathcal{V}, \otimes, \mathbb{I})$ which is, in a suitable sense, equivalent to its *reverse*; that is, the monoidal category $(\mathcal{V}, \otimes, \mathbb{I})^{\mathsf{rev}} = (\mathcal{V}, \otimes^{\mathsf{rev}}, \mathbb{I})$, where $p \otimes^{\mathsf{rev}} q := q \otimes p$. In particular, it comes equipped with a (covariant) functor $\overline{(\ )} : \mathcal{V} \to \mathcal{V}$ and coherent natural isomorphisms

$$\overline{p} \otimes \overline{q} \to \overline{q \otimes p} \quad \text{and} \quad \overline{\overline{p}} \to p$$

—for more details see Definition 2.1 of my paper, *On involutive monoidal categories*.

Remark

An IMC satisfies $\mathbb{I} \cong \overline{\mathbb{I}}$; better yet, there is a specific isomorphism $\mathbb{I} \xrightarrow{\sim} \overline{\mathbb{I}}$ that satisfies a number of coherence axioms, and is unique in this respect—see Lemma 2.3 in *op.cit.*

Trivial examples

Any symmetric monoidal category may be regarded as an IMC by choosing $\overline{p} = p$. [But an IMC satisfying $\overline{p} = p$ is not necessarily braided, let alone symmetric—see Example 4.1 in *op.cit.*]

Two relevant examples of symmetric monoidal categories which we choose to regard as IMCs in this way are $(\mathsf{Set}, \times, 1)$ and $(\mathsf{Sup}, \boxtimes, 2)$.

Almost-trivial examples

When dealing with complex vector spaces, one is sometimes led to consider *conjugate-linear transformations* $\varphi : v \to w$; that is, abelian group homomorphisms satisfying

$$\varphi(\lambda \cdot_v \alpha) = \overline{\lambda} \cdot_w \varphi(\alpha)$$

for all $\lambda \in \mathbb{C}$ and $\alpha \in v$. (For instance, conjugation defines a conjugate-linear transformation $\mathbb{C} \to \mathbb{C}$.)

One way of dealing with such maps is to define $\overline{w}$ to be the vector space with the same underlying abelian group as w, but with a different scalar multiplication (namely, $\lambda \cdot_{\overline{w}} \beta = \overline{\lambda} \cdot_w \beta$), so that conjugate-linear transformations $v \to w$ are the same thing as linear transformations $v \to \overline{w}$.

Now the underlying map of a linear transformation $\varphi : v \to w$ also defines a linear transformation $\overline{v} \to \overline{w}$; we choose to denote the latter $\overline{\varphi}$, even though many people would insist that it is the same gadget as φ. In this manner, $v \mapsto \overline{v}$ extends to an endofunctor on the category of vector spaces and linear transformations, here denoted Lin.

Perhaps surprisingly, this endofunctor is not naturally isomorphic to the identity on Lin—this will be explained more fully in the next section. On the other hand, $\overline{\overline{v}} = v$ on the nose; and there is a natural isomorphism $\overline{v} \otimes \overline{w} \cong \overline{v \otimes w}$, which, when combined with the symmetry of $\otimes$, gives us the last remaining datum required to define an IMC structure on Lin. (The induced map $\mathbb{C} \to \overline{\mathbb{C}}$ is simply conjugation.)

In exactly the same way we may (and do!) choose to regard Ban as an IMC with "almost trivial" involution.

A very non-trivial example

The category of operator spaces and linear complete contractions, Ban admits many monoidal structures. One of them, called the *Haagerup tensor product*, and here denoted $\boxtimes$, is not even slightly symmetric. (In the sense that there exist operator spaces c and r for which $c \boxtimes r \ncong r \boxtimes c$.) But it does admit an involution, which we denote $\widetilde{(\ )}$. The underlying vector space of $\widetilde{x}$ is the conjugate of the underlying vector space of x, but the operator space structure of $\widetilde{x}$ is the *opposite* of that of x.

If we had an isomorphism $c \xrightarrow{\sim} \widetilde{c}$, an isomorphism $r \xrightarrow{\sim} \widetilde{r}$, and an isomorphism $r \boxtimes c \xrightarrow{\sim} \widetilde{r \boxtimes c}$, then we would be able to construct an isomorphism

$$c \boxtimes r \xrightarrow{\quad\sim\quad} \widetilde{c} \boxtimes \widetilde{r} \xrightarrow{\quad\sim\quad} \widetilde{r \boxtimes c} \xrightarrow{\quad\sim\quad} r \boxtimes c$$

—since no such isomorphism exists, we can conclude that there exists an operator space x (one of c, r, or $r \boxtimes c$), for which $x \ncong \widetilde{x}$. [In fact, the example I have in mind satisfies $c \ncong r = \widetilde{c}$, and therefore also $r \ncong \widetilde{r}$; on the other hand $r \boxtimes c = \widetilde{c} \boxtimes \widetilde{r} \cong \widetilde{r \boxtimes c}$.]

Involutive monoidal functors

Quasi-definition

An IMF $(\mathcal{V}, \otimes, \overline{(\ )}, \mathbb{I}) \to (\mathcal{W}, \otimes, \overline{(\ )}, \mathbb{I})$ is a (lax) monoidal functor $(M, \mu, \eta) : (\mathcal{V}, \otimes, \mathbb{I}) \to (\mathcal{W}, \otimes, \mathbb{I})$ which is compatible with involution, in the sense that it comes equipped with a coherent natural transformation of the form $\tau_p : \overline{M(p)} \to M(\overline{p})$—for more details, see Definition 3.2.1 in *op.cit.*

Special case

An IMF $\mathbb{1} \to (\mathcal{V}, \otimes, \overline{(\ )}, \mathbb{I})$ is an *involutive monoid* (or, *dagger monoid*) in $\mathcal{V}$; that is, a monoid in $\mathcal{V}$

$$m \otimes m \xrightarrow{\quad\mu\quad} m \xleftarrow{\quad\eta\quad} \mathbb{I}$$

together with an involution (dagger) $\tau : \overline{m} \to m$ satisfying the axioms below.

$$\begin{array}{ccccc} \overline{m} \otimes \overline{m} & \xrightarrow{\sim} & \overline{m \otimes m} & \xrightarrow{\overline{\mu}} & \overline{m} \\ {\scriptstyle \tau \otimes \tau}\downarrow & & & & \downarrow{\scriptstyle \tau} \\ m \otimes m & & \xrightarrow{\quad\mu\quad} & & m \end{array} \qquad \begin{array}{ccc} \overline{\overline{m}} & \xrightarrow{\overline{\tau}} & \overline{m} \\ & \searrow^{\sim} & \downarrow{\scriptstyle \tau} \\ & & m \end{array}$$

Examples

An involutive monoid in $(\mathsf{Set}, \times, (\ ), 1)$ is indeed a dagger monoid: writing $\alpha \,\&\, \beta := \mu(\alpha, \beta)$ and $\gamma^\dagger := \tau(\gamma)$, the above diagrams boil down to the equations $\alpha^\dagger \,\&\, \beta^\dagger = (\beta \,\&\, \alpha)^\dagger$ and $\gamma^{\dagger\dagger} = \gamma$. Similarly, an involutive monoid in $(\mathsf{Sup}, \Join, (\ ), 2)$ is what is commonly termed an *involutive quantale*.

An involutive monoid in $(\mathsf{Lin}, \otimes, \overline{(\ )}, \mathbb{C})$ is a unital $*$-algebra; that is, a unital algebra equipped with a conjugate-linear dagger. Similarly, an involutive monoid in $(\mathsf{Ban}, \Join, \overline{(\ )}, \mathbb{C})$ is a unital Banach $*$-algebra; that is, a unital Banach algebra equipped with an isometric (*i.e.*, norm-preserving) conjugate-linear dagger.

Remark

IMFs compose. Hence an IMF $(\mathcal{V}, \otimes, \overline{(\ )}, \mathbb{I}) \to (\mathcal{W}, \otimes, \overline{(\ )}, \mathbb{I})$ allows us to convert involutive $(\mathcal{V}, \otimes, \overline{(\ )}, \mathbb{I})$-monoids into involutive $(\mathcal{W}, \otimes, \overline{(\ )}, \mathbb{I})$-monoids.

Examples

The forgetful functor $\mathsf{Lin} \to \mathsf{Set}$ underlies an IMF $(\mathsf{Lin}, \otimes, \overline{(\ )}, \mathbb{C}) \to (\mathsf{Set}, \times, (\ ), 1)$: since the underlying set of $\overline{v}$ is, by definition, the same as that of v, we take τ to be the identity. This IMF carries a $*$-algebra to its underlying dagger monoid.

More interestingly, the free functor $\mathsf{Set} \to \mathsf{Lin}$, here denoted span, also underlies an IMF $(\mathsf{Set}, \times, (\ ), 1) \to (\mathsf{Lin}, \otimes, \overline{(\ )}, \mathbb{C})$: $\mathsf{span}\, b$ has a canonical basis, namely the range of the unit map $b \to |\mathsf{span}\, b|$; we define $\tau : \overline{\mathsf{span}\, b} \to \mathsf{span}\, b$ to be the (conjugate-)linear transformation which acts as the identity for preserves that basis. This IMF carries dagger monoids to $*$-algebras in the way one expects.

Similarly, both the unit-ball functor $\mathsf{Ban} \to \mathsf{Set}$ and its left adjoint $\ell^1 : \mathsf{Set} \to \mathsf{Ban}$ underlie IMFs.

Aside

Assuming the axiom of choice, every vector space v has a basis b; therefore we obtain isomorphisms

$$v \xrightarrow{\sim} \mathsf{span}\, b \xrightarrow{\sim} \overline{\mathsf{span}\, b} \xrightarrow{\sim} \overline{v}$$

—this is why we wrote earlier that the non-existence of a natural transformation $(\ ) \to \overline{(\ )}$ might be surprising.

Question

Is this use of the axiom of choice strictly necessary? I.e., is it possible to build a model of ZF containing a complex vector space which is not isomorphic to its conjugate? I would find that delightful.

A key example

For a Banach space x, let $Q(x)$ denote the complete lattice of closed linear subspaces of x. Given Banach spaces x and y, and a continuous linear transformation $\omega : x \to y$, inverse image defines an inf-homomorphism $\omega^* : Q(y) \to Q(x)$.

Let $Q(\omega)$ denote the left adjoint of ω^*; then Q underlies an IMF $(\mathsf{Ban}, \boxtimes, \overline{(\ )}, \mathbb{C}) \to (\mathsf{Sup}, \boxtimes, (\ ), 2)$. This IMF carries Banach $*$-algebras to an involutive quantales in exactly the way quantale theorists like.

Question

For an arbitrary topos $\mathcal{E}$, can we construct a similar IMF $\mathsf{Ban}(\mathcal{E}) \to \mathsf{Sup}(\mathcal{E})$? This might be harder than it sounds, but I certainly hope it is the case.

Involutive monoidal natural transformations

Where there are IMFs, there are obviously also IMNTs—for details, see Definition 3.2.2 in *op.cit.*

In the case of IMFs of the form $\mathbb{1} \to (\mathcal{V}, \otimes, \overline{(\ )}, \mathbb{I})$, an IMNT boils down to the obvious notion of homomorphism between involutive monoids—equivalently, of dagger functor between dagger monoids.