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Waves and total distributivity

Following joint work with Francisco Marmolejo and Bob Rosebrugh on "Completely and totally distributive categories", I reported at CT2013 on the *wave functor* $W : \mathscr{K} \to \widehat{\mathscr{K}} = \mathbf{set}^{\mathscr{K}^{\mathrm{op}}}$ of a totally cocomplete category $\mathscr{K}$. Indeed, if the defining property of $\mathscr{K}$ is given by an adjunction $\bigvee \dashv Y : \mathscr{K} \to \widehat{\mathscr{K}}$, where Y is the Yoneda functor, then W is given for objects A and K in $\mathscr{K}$ by

$$W(A)(K) = \mathbf{set}^{\widehat{\mathscr{K}}}(\mathscr{K}(A,-).\bigvee, [K,-])$$

where $[K,-]$ denotes evaluation of an object of $\widehat{\mathscr{K}}$ at K. **set** denotes the category of *small sets* and it requires some work to show that W is well-defined. W arises along with natural transformations $\beta : W\bigvee \to 1_{\widehat{\mathscr{K}}}$ and $\gamma : \bigvee W \to 1_{\mathscr{K}}$ that satisfy $\bigvee\beta = \gamma\bigvee$ and $\beta W = W\gamma$. A total $\mathscr{K}$ is said to be *totally distributive* if $\bigvee$ has a left adjoint. It was shown that $\mathscr{K}$ is totally distributive iff γ is invertible iff $W \dashv \bigvee$.

For any total $\mathscr{K}$ there is a well-defined, associative composition of waves. If we write $\widetilde{\mathscr{K}} : \mathscr{K} \nrightarrow \mathscr{K}$ for the small profunctor determined by W then composition becomes an arrow $\circ : \widetilde{\mathscr{K}} \circ_{\mathscr{K}} \widetilde{\mathscr{K}} \to \widetilde{\mathscr{K}}$, although $\widetilde{\mathscr{K}} \circ_{\mathscr{K}} \widetilde{\mathscr{K}}$ is not in general small. Moreover, there is also an augmentation $(\bar{-}) : \widetilde{\mathscr{K}}(-,-) \to \mathscr{K}(-,-)$, corresponding to a natural transformation $\delta : W \to Y$ constructed via β. We will show that if $\mathscr{K}$ is totally distributive then $\circ : \widetilde{\mathscr{K}} \circ_{\mathscr{K}} \widetilde{\mathscr{K}} \to \widetilde{\mathscr{K}}$ is invertible, meaning that composition of waves is *interpolative*, and $\widetilde{\mathscr{K}}$ thus supports an idempotent comonad structure. In fact, $\widetilde{\mathscr{K}} \circ_{\mathscr{K}} \widetilde{\mathscr{K}} = \widetilde{\mathscr{K}} \circ_{\widetilde{\mathscr{K}}} \widetilde{\mathscr{K}}$ so that $\widetilde{\mathscr{K}}$ becomes a *taxon* structure, in the sense of Koslowski, on the objects of $\mathscr{K}$. In the paper with Marmolejo and Rosebrugh we showed that, for any small taxon $\mathscr{T}$, the category of taxon functors $\mathbf{Tax}(\mathscr{T}^{\mathrm{op}}, \mathbf{set})$ is totally distributive. To this we now add, for any totally distributive $\mathscr{K}$, there is an equivalence of categories $\mathscr{K} \to \mathbf{Tax}(\widetilde{\mathscr{K}}^{\mathrm{op}}, \mathbf{set})$.