$\mathcal{LH}$ has nonempty products

Peter Selinger

March 15, 1994

1 The construction of binary products

We will show that $\mathcal{LH}$, the category of topological spaces with local homeomorphisms, has binary products (in fact, it has arbitrary nonempty limits; see 2.3 below). This has been previously believed to be false, see *e.g.* [1, p. 50]. In the first definition we fix some notation:

Definition For an element x of a topological space X, we will write $\mathcal{U}_x$ for the set of all *open* neigborhoods of x. $f : X \to Y$ is a *local homeomorphism* if for every $x \in X$ there exists a $U \in \mathcal{U}_x$ such that $f|_U$ is a homeomorphism onto an open subset of Y. Denote by $\operatorname{int} A$ the interior of $A \subseteq X$. For any pair of maps $\phi : U \to Y$, $\phi' : V \to Y$ defined on open subsets $U, V \subseteq X$, define the (set) equalizer

$$\operatorname{eq}(\phi, \phi') := \{x \in U \cap V \mid \phi(x) = \phi'(x)\}.$$

In the following, we will only consider *open* equalizers $\operatorname{int}\operatorname{eq}(\phi, \phi')$.

We will now construct the categorical product B of two spaces $X, Y \in \mathcal{LH}$. The basic idea is to consider triples $\langle U, V, \phi\rangle$ consisting of two open sets $U \subseteq X$, $V \subseteq Y$ together with a homeomorphism $\phi : U \to V$. These will be the prototypes for open sets in B. The elements of B will arise as equivalence classes of quintuples (x, y, U, V, ϕ) with U, V and ϕ as before and $x \in U$, $y \in V$ such that $\phi(x) = y$. Let therefore

$$\begin{aligned} A := \{(x, y, U, V, \phi) \quad | \quad & U \subseteq X \text{ open} \\ & V \subseteq Y \text{ open} \\ & \phi : U \to V \text{ a homeomorphism} \\ & x \in U,\ y \in V,\ \phi(x) = y\,\}. \end{aligned}$$

Strictly speaking, the tuple (x, y, U, V, ϕ) is completely determined by x, U and ϕ, but it is more convenient for us to write all the components. On A define $(x, y, U, V, \phi) \sim (x, y, U', V', \phi')$ if there exists $\tilde{U} \in \mathcal{U}_x$ with $\tilde{U} \subseteq U \cap U'$ and

$$\phi|_{\tilde{U}} = \phi'|_{\tilde{U}}.$$

Note that in this case, $\tilde{U} \subseteq \operatorname{int}\operatorname{eq}(\phi, \phi')$, hence without loss of generality $\tilde{U} = \operatorname{int}\operatorname{eq}(\phi, \phi')$, or

$$(x, y, U, V, \phi) \sim (x, y, U', V', \phi') \qquad \text{iff} \qquad x \in \operatorname{int}\operatorname{eq}(\phi, \phi').$$

$\sim$ is an equivalence relation on A: reflexivity and symmetry are obvious; for transitivity notice that if $\phi|_{\tilde{U}} = \phi'|_{\tilde{U}}$ and $\phi'|_{\tilde{U}'} = \phi''|_{\tilde{U}'}$, then $\phi|_{\tilde{U}\cap\tilde{U}'} = \phi''|_{\tilde{U}\cap\tilde{U}'}$. Define $B = A/{\sim}$ and write $[x, y, U, V, \phi]$ for the $\sim$-class of (x, y, U, V, ϕ).

Now consider a topology on B: for any open sets $O \subseteq X, P \subseteq Y$ and any homeomorphism $\psi : O \to P$, define

$$\langle O, P, \psi \rangle := \{[x, y, O, P, \psi] \mid x \in O,\ y \in P,\ \psi(x) = y\}.$$

Note that $\langle O, P, \psi \rangle \subseteq B$ and that $[x, y, U, V, \phi] \in \langle O, P, \psi \rangle$ iff $x \in O$ and $\phi|_{\tilde{U}} = \psi|_{\tilde{U}}$ for some $\tilde{U} \in \mathcal{U}_x$.

Let the topology on B be generated by the sets $\langle O, P, \psi \rangle$.

Lemma 1 *The intersection of $\langle O, P, \psi \rangle$ and $\langle O', P', \psi' \rangle$ is again of the form $\langle O'', P'', \psi'' \rangle$, with*

$$\begin{aligned} O'' &= \operatorname{int} \operatorname{eq}(\psi, \psi') \\ \psi'' &= \psi|_{O''} = \psi'|_{O''} \\ P'' &= \psi''(O''). \end{aligned}$$

In particular, the sets $\langle O, P, \psi \rangle$ form a basis, and not just a subbasis, of the topology on B, i.e. every open set is a union of basis sets.

Proof: $[x, y, U, V, \phi] \in \langle O, P, \psi \rangle \cap \langle O', P', \psi' \rangle$ iff $x \in O \cap O'$ and $\phi|_{\tilde{U}} = \psi|_{\tilde{U}}$ and $\phi|_{\tilde{U}'} = \psi'|_{\tilde{U}'}$ for some $\tilde{U}, \tilde{U}' \in \mathcal{U}_x$ iff $x \in O \cap O'$ and $\phi|_{\tilde{U}''} = \psi|_{\tilde{U}''} = \psi'|_{\tilde{U}''}$ for some $\tilde{U}'' \in \mathcal{U}_x$, or equivalently $x \in \tilde{U}'' \subseteq \operatorname{int} \operatorname{eq}(\psi, \psi') =: O''$ and $\phi|_{\tilde{U}''} = \psi''|_{\tilde{U}''}$. This finally is the case if and only if $[x, y, U, V, \phi] \in \langle O'', P'', \psi'' \rangle$. $\square$

We now consider the projection maps

$$\begin{aligned} \pi_1 : B \to X &\quad : \quad [x, y, U, V, \phi] \mapsto x \\ \pi_2 : B \to Y &\quad : \quad [x, y, U, V, \phi] \mapsto y. \end{aligned}$$

Lemma 2 *π_1 and π_2 are local homeomorphisms. More specifically π_1 and π_2 map $\langle U, V, \phi \rangle$ homeomorphically onto U and V, respectively.*

Proof: We consider only π_1; the proof for π_2 is entirely symmetric. Take an arbitrary point $b = [x, y, U, V, \phi] \in B$. Then $\langle U, V, \phi \rangle$ is a neighborhood of b and $\pi_1|_{\langle U,V,\phi \rangle}$ is a one-to-one map onto U, with inverse

$$(\pi_1|_{\langle U,V,\phi \rangle})^{-1} : U \to \langle U, V, \phi \rangle \ : \ x \mapsto [x, \phi(x), U, V, \phi].$$

$\pi_1|_{\langle U,V,\phi \rangle}$ is continuous: for open $U' \subseteq U$ we let $V' = \phi(U')$ and have

$$\begin{aligned} \pi_1|^{-1}_{\langle U,V,\phi \rangle}(U') &= \{[x, y, U, V, \phi] \mid x \in U'\} \\ &= \{[x, y, U', V', \phi|_{U'}] \mid x \in U'\} \\ &= \langle U', V', \phi|_{U'} \rangle , \end{aligned}$$

which is open.

π_1 (and hence $\pi_1|_{\langle U,V,\phi \rangle}$) is an open map: by Lemma 1 it suffices to show this for basic open sets, and in fact we have $\pi_1(\langle O, P, \psi \rangle) = O$. $\square$

Lemma 2 establishes the existence of the diagram

$$\begin{tikzcd} & B \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y. \end{tikzcd}$$

We will now show that this is in fact a product diagram. Given local homeomorphisms

$$\begin{tikzcd} & Z \arrow[ddl, "\rho_1"'] \arrow[ddr, "\rho_2"] & \\ & B \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ X & & Y, \end{tikzcd}$$

we construct $f : Z \to B$ as follows: for any $z \in Z$ consider an open neighborhood $W \in \mathcal{U}_z$ on which both ρ_1 and ρ_2 are homeomorphisms.[1] Let

$$\begin{array}{rcl} x & = & \rho_1(z) \\ y & = & \rho_2(z) \\ U & = & \rho_1(W) \\ V & = & \rho_2(W) \\ \phi & = & \rho_2 \circ (\rho_1|_W)^{-1} : U \to V. \end{array} \tag{1}$$

Then $[x, y, U, V, \phi] \in B$. Moreover, $[x, y, U, V, \phi]$ is independent of the particular choice of W, for if we choose, say, $W' \subseteq W$ and we construct $[x', y', U', V', \phi']$ from W', then $x = x'$, $y = y'$, $U' \subseteq U$, $V' \subseteq V$ and $\phi' = \phi|_{W'}$ follow directly from the definition. Hence

$$f(z) = [x, y, U, V, \phi]$$

is well-defined in this way for any $z \in Z$. We clearly have $\rho_1 = \pi_1 \circ f$, $\rho_2 = \pi_2 \circ f$.

Lemma 3 *$f : Z \to B$ is a local homeomorphism.*
Proof: Starting with a $z \in Z$, choose a neigborhood $W \in \mathcal{U}_z$ on which f is a homeomorphism. Then the image $f(W)$ is contained in $\langle U, V, \phi \rangle$, with $U = \rho_1(W)$, $V = \rho_2(W)$, $\phi = \rho_2 \circ (\rho_1|_W)^{-1}$ as before.

To conclude that f is a homeomorphism, we recall that W is by its choice homeomorphic to U via ρ_1, while U is the homeomorphic image of $\langle U, V, \phi \rangle$ under π_1 (by Lemma 2. Hence $f|_W = (\pi_1|_{\langle U,V,\phi \rangle})^{-1} \circ \rho_1|_W$ is a homeomorphism as well. $\square$

Lemma 4 *f is the unique morphism with*

$$\begin{tikzcd} & Z \arrow[ddl, "\rho_1"'] \arrow[ddr, "\rho_2"] \arrow[d, "f"] & \\ & B \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ X & & Y. \end{tikzcd}$$

Proof: Suppose g is another map making this diagram commute. Let $z \in Z$ be arbitrary and pick

[1]This is where the construction does not generalize to the case of infinite products. However, infinite products do in fact exist; see 2.3 below.

$W \in \mathcal{U}_z$ such that ρ_1, ρ_2, f, g are all homeomorphisms on W. Moreover, pick W small enough such that $g(W)$ is a *basis* open set in B, which can be done by the Lemma 1. Say $g(W) = \langle O, P, \psi\rangle$. Then $g(z) = [x', y', O, P, \psi]$, for some $x' \in O$, $y' \in P$, while $f(z) = [x, y, U, V, \phi]$ and $f(W) = \langle U, V, \phi\rangle$, where x, y, U, V, ϕ are as in (1). By commutativity and Lemma 2 we get

$$\begin{aligned} x = \pi_1(f(z)) = \quad & \rho_1(z) && = \pi_1(g(z)) = x' \\ y = \pi_2(f(z)) = \quad & \rho_2(z) && = \pi_2(g(z)) = y' \\ U = \pi_1(f(W)) = \quad & \rho_1(W) && = \pi_1(g(W)) = O \\ V = \pi_2(f(W)) = \quad & \rho_2(W) && = \pi_2(g(W)) = P. \end{aligned} \tag{2}$$

Finally, for any $w \in W$ we have $g(w) = [\rho_1(w), \rho_2(w), O, P, \psi]$ and $f(w) = [\rho_1(w), \rho_2(w), U, V, \phi]$, establishing

$$\psi(\rho_1(w)) = \rho_2(w) = \phi(\rho_1(w)).$$

But since $\rho_1|_W$ is onto O, this yields $\psi = \phi$. Together with (2) this yields the uniqueness of f:

$$g(z) = [x', y', O, P, \psi] = [x, y, U, V, \phi] = f(z).$$

□

So we have established

Theorem 5 *$\mathcal{LH}$ has binary products.* □

2 Remarks

2.1 Subcategories of $\mathcal{LH}$

Since the maps π_1, π_2 are local homeomorphisms, the product $X \times Y$ inherits from X and, independently, from Y, all local properties (*i.e.* properties that are reflected by local homeomorphisms). Among these are *e.g.* the separation properties T_0 and T_1, local compactness, local metrizability etc. (notice, however, that being Hausdorff is *not* a local property). Hence the respective full subcategories of $\mathcal{LH}$ have products as well.

We get for example a nice result for posets by specializing to the full subcategory of $\mathcal{LH}$ consisting of T_0 spaces in which arbitrary intersections of opens are open (this is a local property, too). This category is equivalent [1, 1.372] to the category $\mathcal{POS}^{l.i.}$ of posets with local isomorphisms ($f : P \to Q$ is a *local isomorphism* of posets if for any $a \in P$, $f|_{\downarrow a} : \downarrow a \xrightarrow{\sim} \downarrow b$). In this category, the product of two posets P and Q has elements $[a, b, f]$ where $a \in P$, $b \in Q$ and $f : \downarrow a \xrightarrow{\sim} \downarrow b$, ordered by $[a, b, f] \le [a', b', f']$ iff $a \le a'$, $b \le b'$ and $f = f'|_{\downarrow a}$.

2.2 Almost a regular category (and more!)

Since $\mathcal{LH}$ also has pullbacks, equalizers and images and pullbacks transfer covers, it has all the properties of a regular category [1, 1.5]—with one important exception: it doesn't possess a terminator. Despite the lack of a terminator, certain constuctions can be carried out as in a regular category, *e.g.* composition of relations is well-defined and associative, and we have a notion of a *graph* of a function as being an entire and simple relation [1, 1.564].

The latter comes out rather naturally in $\mathcal{LH}$: the graph of $f : X \to Y$ is given by the subobject of $X \times Y$

$$\operatorname{graph} f = \{[x, y, U, V, f|_U] \in X \times Y\}.$$

Conversely, a subobject $G \subseteq X \times Y$ is a graph iff all $\langle U, V, \phi\rangle, \langle U', V', \phi'\rangle \in G$ satisfy $\phi|_{U\cap U'} = \phi'|_{U\cap U'}$ (*i.e.* π_1 simple), and $X = \bigcup_{\langle U,V,\phi\rangle \in G} U$ (*i.e.* π_1 entire). The function f can then be recovered by 'pasting' all the local ϕ's together.

2.3 $\mathcal{LH}$ is "conditionally complete"

As David Feldman has pointed out, the product of X and Y, when regarded as a sheaf over X, is given by the functor that associates to an open set U of X all local homeomorphisms from U to Y. While the description of binary products in section 1 avove does not immediately generalize to infinite products (see the footnote before Lemma 3), the sheaf description does. But unlike in the finite case, in the general (infinite) case there is no obvious no *symmetric* description of the product. One of the factors has to be preferred as the base space of the constructed sheaf. This asymmetry seems to be the deeper reason for the absence of an *empty* product, *i.e.* a terminator.

The presence of equalizers implies that every nonempty diagram in $\mathcal{LH}$ has a limit. Peter Freyd proposes to call this property "conditional completeness".

One might hope that $\mathcal{LH}$ is also "conditionally exponential", *i.e.* that certain exponential objects exist (clearly B^0 doesn't exist for any B, for it would be a terminator). However, upon closer inspection it turns out that there aren't *any* exponentials in $\mathcal{LH}$: given objects A, B, suppose there exists B^A. Now choose a nonempty C which has no open neighborhood homeomorphic to an open neighborhood in either A or B^A (*e.g.* choose C of cardinality bigger than that of A and B^A with the indiscrete topology). Then both (C, B^A) and $C \times A$ are empty and $(C \times A, B)$ is a singleton.

2.4 The right adjoint of the pullback functor

Instead of exponentials, we get a right adjoint for pullbacks in $\mathcal{LH}$. This is generally true for categories whose slices are cartesian closed:

For any category $\mathbf{A}$ with pullbacks, and any $f : A \to B$, consider the functor $f^\# : \mathbf{A}/B \to \mathbf{A}/A$ defined by pullbacks:

$$\begin{tikzcd} D \arrow[r, "f^\# x"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ C \arrow[r, "x"] & B \end{tikzcd}$$

Proposition 6 *$\mathbf{A}/B$ is cartesian closed iff $f^\#$ has a right adjoint for all $f : A \to B$.*

Proof: This is a consequence of 1.854 in [1] and the fact that $\mathbf{A}/A \simeq (\mathbf{A}/B)/(f)$. $\square$

Since $\mathcal{LH}/Y = \mathcal{S}h(Y)$ is cartesian closed, we know that in $\mathcal{LH}$ the pullback functor has a right adjoint. Will it help us if we understand what this adjoint is? ...to be continued ...

References

[1] P. J. Freyd and A. Scedrov. *Categories, Allegories*, volume 39 of *North-Holland Mathematical Library*. North-Holland, 1989.