SIGNED B-EDGE COVERS OF GRAPHS

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Abstract. We study a signed variant of edge covers of graphs. Let b be a positive integer, and let G be a graph with minimum degree at least b. A *signed b-edge cover* of G is a function $f : E(G) \to \{-1, 1\}$ satisfying $\sum_{e \in E_G(v)} f(e) \geq b$ for every $v \in V(G)$. The minimum of the values of $\sum_{e \in E(G)} f(e)$, taken over all signed b-edge covers f of G, is called the *signed b-edge cover number* and is denoted by $\rho'_b(G)$. For any positive integer b, we show that a minimum signed b-edge cover can be found in polynomial time, using a reduction to b-edge cover, which itself is solved by b-matching. A sharp lower bound for ρ'_b and a sharp upper bound ρ'_2 are given. A sharp upper bound for ρ'_b of Cartesian product graphs is presented. Exact values of ρ'_b for cliques and bicliques are found.

1. Introduction

Structural and algorithmic aspects of covering vertices by edges have been extensively studied in graph theory. An *edge cover* of a graph G is a set C of edges of G such that each vertex of G is incident to at least one edge of C. Let b be a fixed positive integer. A *simple b-edge cover* of a graph G is a set C of edges of G such that each vertex of G is incident to at least b edges of C. Note that a simple b-edge cover of G corresponds to a spanning subgraph of G with minimum degree at least b. Edge covers of bipartite graphs were studied by König [4] and Rado [7], and of general graphs by Gallai [2] and Norman and Rabin [6], and b-edge covers were studied by Gallai [2]. For an excellent survey of results on edge covers and b-edge covers, see Schrijver [8].

We consider a variant of the standard edge cover problem. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, let $E_G(v) = \{uv \in E(G) : u \in V(G)\}$ denote the set of edges of G incident to v. The degree, $d(v)$, of v is $|E_G(v)|$. For a real-valued

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function $f : E(G) \to \mathbb{R}$ and for $X \subseteq E(G)$, we use $f(X)$ to denote $\sum_{e \in X} f(e)$. A function $f : E(G) \to \{-1, 1\}$ is called a *signed b-edge cover* (*SbEC,* for short) of G if $f(E_G(v)) \geq b$ for every $v \in V(G)$. The minimum of the values of $f(E(G))$, taken over all signed b-edge covers f of G, is called the *signed b-edge cover number* of G and is denoted by $\rho'_b(G)$. A *minimum signed b-edge cover* is a signed b-edge cover f satisfying $f(E(G)) = \rho'_b(G)$. For example, $\rho'_2(K_{4.4}) = 8$; see Figure 1. For G to have a signed b-edge cover, it is necessary that the minimum degree of G, denoted $\delta(G)$, be at least b. Hence, *when we discussing ρ'_b, all graphs involved have minimum degree at least b.* In the special case when $b = 1$, ρ'_b is the *signed star domination number* investigated in [9, 11, 12, 13].

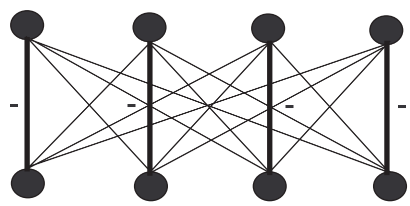

FIGURE 1. A minimum signed 2-edge cover of $K_{4,4}$, with the bold edges labelled -1 and all other edges labelled $+1$.

In Section 2, we investigate the complexity of the minimum signed b-edge cover problem. We prove that a minimum signed b-edge cover can be found in polynomial time. In Section 3, sharp bounds for ρ'_2 are supplied, and upper bounds are conjectured for $b > 2$. In Section 4, a sharp upper bound for ρ'_b of Cartesian product graphs is presented. We finish by determining the exact value of $\rho'_b(G)$ for cliques and bicliques.

All graphs considered in this paper are finite, simple, and undirected. For a general reference on graph theory, the reader is referred to [1, 10]. For a graph G, a vertex $v \in V(G)$ is called *odd (even)* if $d(v)$ is odd (even). For $S \subseteq V(G)$ and $v \in V(G)$, we denote by $G[S]$ and $G - v$ the subgraphs of G induced by S and by $V(G) \backslash \{v\}$, respectively. A *k-factor* of G is a k-regular spanning subgraph of G. In particular, F is a 1-factor of G if and only if $E(F)$ is a *perfect matching* in G. The union of two vertex-disjoint graphs G and H is denoted by $G \cup H$. We use $\mathbb{N}$ to denote the set of nonnegative integers. The *Cartesian product* $G \square H$ has $V(G) = V(G) \times V(H)$, and two vertices (a, b) and (c, d) of G are adjacent if and only if $ac \in E(G)$ and $b = d$ or $a = c$ and $bd \in E(H)$.

2. Complexity

An algorithm is said to run in *strongly polynomial time* if the number of elementary arithmetic and other operations is bounded by a fixed polynomial in the size of the input, where any number in the input is counted only for 1. Strongly polynomial time is of relevance only for algorithms that have numbers among their input; otherwise, strongly polynomial time coincides with the more well-known polynomial time. For more background on strongly polynomial time, the reader is referred to [8]. Our main result for this section is the following theorem.

Theorem 1. *For any positive integer b, a minimum signed b-edge cover can be found in strongly polynomial time.*

For the proof of Theorem 1, we use the following result from [8].

Theorem 2. *If $k : V(G) \to \mathbb{N}$ is a function, then a minimum simple k-edge cover can be found in strongly polynomial time.*

Proof of Theorem 1. The minimum simple k-edge cover problem can be formulated as the following 0-1 linear programming problem:

$$\begin{aligned} &\text{Minimize } \textstyle\sum_{uv\in E(G)} f_{uv} \\ &\sum_{uv\in E_G(u)} f_{uv} \geq k(u), \text{ for every } u \in V(G), \\ &f_{uv} \in \{0,1\}, \text{ for every } uv \in E(G). \end{aligned} \tag{2.1}$$

We may formulate the minimum signed b-edge cover problem as the following problem.

$$\begin{aligned} &\text{Minimize } \textstyle\sum_{uv\in E(G)} f_{uv} \\ &\sum_{uv\in E_G(u)} f_{uv} \geq b, \text{ for every } u \in V(G), \\ &f_{uv} \in \{-1,1\}, \text{ for every } uv \in E(G). \end{aligned} \tag{2.2}$$

It is sufficient to prove that (2.2) may be converted to an instance of (2.1); the proof then follows by Theorem 2. We consider the case only when b is even, as the proof is similar when b is odd. Assuming b is even, if $d(u)$ is odd for some $u \in V(G)$, then $\sum_{uv\in E_G(u)} f_{uv} \geq b$ implies that $\sum_{uv\in E_G(u)} f_{uv} \geq b+1$. Hence, (2.2) is equivalent to

$$\text{(2.3)} \qquad \text{Minimize } \sum_{uv \in E(G)} f_{uv}$$

$$\sum_{uv \in E_G(u)} f_{uv} \geq b, \text{ for each even vertex } u \in V(G),$$

$$\sum_{uv \in E_G(u)} f_{uv} \geq b+1, \text{ for each odd vertex } u \in V(G),$$

$$f_{uv} \in \{-1, 1\}, \text{ for every } uv \in E(G).$$

Now define $g_{uv} = \frac{1}{2}(1 + f_{uv})$ for each $uv \in E(G)$. It is straightforward to see that $g_{uv} \in \{0, 1\}$ for each $uv \in E(G)$. Moreover, (2.3) is equivalent to the following problem.

$$\text{(2.4)} \quad \text{Minimize } 2\sum_{uv \in E(G)} g_{uv} - |E(G)|$$

$$\sum_{uv \in E_G(u)} g_{uv} \geq \frac{1}{2}\left(b + d(u)\right), \text{ for each even vertex } u \in V(G),$$

$$\sum_{uv \in E_G(u)} g_{uv} \geq \frac{1}{2}\left(b + 1 + d(u)\right), \text{ for each odd vertex } u \in V(G),$$

$$g_{uv} \in \{0, 1\}, \text{ for every } uv \in E(G).$$

Define

$$k(u) = \begin{cases} \frac{1}{2}\left(b + d(u)\right), & d(u) \text{ even;} \\ \frac{1}{2}\left(b + 1 + d(u)\right), & d(u) \text{ odd.} \end{cases}$$

To solve (2.4), it is equivalent to minimize $\sum_{uv \in E(G)} g_{uv}$ rather than minimizing $2\sum_{uv \in E(G)} g_{uv} - |E(G)|$, thus (2.4) can be solved as an instance of (2.1). Thus, by Theorem 2, (2.4) is polynomial time solvable. □

We remark that the reduction method in the proof of Theorem 1 applies to more general minimum signed b-edge cover problems, where $b : V(G) \to \mathbb{N}$ is any non-negative integer-valued function.

3. Bounds

We now turn to finding sharp lower and upper bounds for the signed b-edge cover numbers in graphs. Sharp lower bounds are easier, while upper bounds are more elusive for general b. We supply a sharp upper bound in the case $b = 2$ and give examples of graphs witnessing all possible values of the parameter ρ_2'. We first consider lower bounds for ρ_b'.

Theorem 3. *Let b be a positive integer. For any graph G of order n and minimum degree at least b,*

$$\rho'_b(G) \geq \lceil bn/2 \rceil.$$

Proof. For a SbEC f of G, for every $v \in V(G)$, we have that

$$f(E_G(v)) \geq b.$$

Hence,

$$\sum_{v \in V(G)} f(E_G(v)) \geq bn.$$

In particular,

$$2f(E(G)) \geq bn.$$

Thus, $\rho'_b(G) \geq bn/2$, and the result follows since $\rho'_b(G)$ is an integer. □

For an upper bound for the signed b-edge cover number, we propose the following conjecture.

Conjecture 4. *Let $b \geq 2$ be an integer. There is a positive integer n_b so that for any graph G of order $n \geq n_b$ with minimum degree b,*

$$\rho'_b(G) \leq (b+1)(n-b-1). \tag{3.1}$$

Since

$$\rho'_b(K_{b+1,n-b-1}) = (b+1)(n-b-1),$$

the upper bound (3.1) would be best possible if the conjecture were true. Conjecture 4 appears difficult to prove even for $b = 3$. However, the case $b = 2$ is the following result.

Theorem 5. *Let G be a graph of order n with minimum degree at least 2. For $n \geq 7$,*

$$\rho'_2(G) \leq 3n - 9.$$

Proof. We proceed by induction on the size $m = |E(G)|$ of G. By a tedious and so omitted argument, it follows that $\rho'_2(G) \leq 12$ if $|V(G)| = 7$. We may therefore assume that $n \geq 8$.

Assume that the theorem is true for all graphs G' with minimum degree at least 2, where $|E(G')| \leq m-1$ and $7 \leq |V(G')| \leq n$. We will prove that $\rho'_2(G) \leq 3n-9$ for a graph G of order n and size m. There are three cases.

Case 1. $\delta(G) = 2$.

Let w be a vertex with degree 2, and two neighbours of w are denoted by u and v.

Subcase 1.1. $uv \notin E(G)$.

Let $G'' = G - w$ and $G' = G'' + uv$. Then G' has order $n-1$ and size $m-1$ with $\delta(G') \geq 2$. By the induction hypothesis, we have that $\rho_2'(G') \leq 3(n-1) - 9 = 3n - 12$. Let f' be a S2EC of G' so that $f'(E(G')) = \rho_2'(G')$. We can form a S2EC f of G by assigning $f(uw) = f(vw) = 1$ and $f(e) = f'(e)$ for each $e \in E(G)\backslash\{uw, vw\}$. Hence,

$$\begin{aligned} f(E(G)) &= f'(E(G')) - f'(uv) + 2 \\ &\leq \rho_2'(G') + 3 \\ &\leq 3n - 9, \end{aligned}$$

implying that $\rho_2'(G) \leq 3n - 9$.

Subcase 1.2. $uv \in E(G)$.

If both u and v have degree 2, then the subgraph induced by u, v and w is an isolated triangle. It is not hard to show that $\rho_2'(G) \leq 3n - 9$ when $n = 8$ or 9. Hence, we may assume that $n \geq 10$. Let $G' = G - \{u, v, w\}$. Then G' is a graph of order $n - 3 \geq 7$ with minimum degree at least 2. By the induction hypothesis, we have that $\rho_2'(G') \leq 3(n-3)-9 = 3n-18$. Hence, $\rho_2'(G) = \rho_2'(G')+3 \leq (3n-18)+3 < 3n-9$. We therefore assume that u or v has degree at least 3.

We obtain a graph G' from G as follows. If both u and v have degree greater than 2, then $G' = G - w$. If one of u and v has degree 2, say u, then G' is obtained from $G - w$ by adding an edge between u and one of its nonadjacent vertices.

It is clear that G' is a graph of order $n-1$ and size at most $m-1$ with $\delta(G') \geq 2$. By induction hypothesis, we have that $\rho_2'(G') \leq 3n - 12$. Let f' be a S2EC of G' so that $f'(E(G')) = \rho_2'(G')$. We can form a S2EC f of G by assigning $f(uw) = f(vw) = 1$ and $f(e) = f'(e)$ for each $e \in E(G)\backslash\{uw, vw\}$. Then,

$$\begin{aligned} f(E(G)) &= \sum_{e \in E(G)\backslash\{uw,vw\}} f'(e) + f(uw) + f(vw) \\ &\leq (f'(E(G')) + 1) + 2 \\ &= \rho_2'(G') + 3 \\ &\leq 3n - 9, \end{aligned}$$

implying that $\rho_2'(G) \leq 3n - 9$.

Case 2. $\delta(G) = 3$.

Let v be a vertex with degree 3. Then $G' = G - v$ is a graph of order n and size $m - 3$ with $\delta(G') \geq 2$. By the induction hypothesis, we have that $\rho_2'(G') \leq 3n - 12$. Let f' be a S2EC of G' so that $f'(E(G')) = \rho_2'(G')$. We can obtain a S2EC f of G by assigning $f(e) = 1$ for each

$e \in E(G) \backslash E(G')$ and $f(e) = f'(e)$ for each $e \in E(G')$. Observe that

$$\begin{aligned} f(E(G)) &= f'(E(G')) + 3 \\ &= \rho_2'(G') + 3 \\ &\leq 3n - 9. \end{aligned}$$

Hence, $\rho_2'(G) \leq 3n - 9$.

Case 3. $\delta(G) \geq 4$.

In this case, there is an even length cycle in G (see Lemma 3.1 of [12]); denote such a cycle by C. Then $G'' = G - E(C)$ is a graph of order n and size $m - |E(C)|$ with $\delta(G'') \geq 2$. By the induction hypothesis, we have that $\rho_2'(G'') \leq 3n - 9$. Let f'' be a S2EC of G'' so that $f''(E(G'')) = \rho_2'(G'')$. We can obtain a S2EC f of G by assigning 1 and -1 alternately along the cycle C and $f(e) = f''(e)$ for each $e \in E(G'')$. Notice that

$$\begin{aligned} f(E(G)) &= f''(E(G'')) \\ &= \rho_2'(G'') \\ &\leq 3n - 9. \end{aligned}$$

□

As $\rho_2'(K_{3,n-3}) = 3n - 9$, the upper bound given in Theorem 5 is sharp.

By Theorems 3 and 5 we have that

$$n \leq \rho_2'(G) \leq 3n - 9$$

for any graph G of order $n \geq 7$. Are all these possible values of ρ_2' witnessed? The next theorem answers this question in the affirmative.

Theorem 6. *Let $n \geq 7$. For each k satisfying $n \leq k \leq 3n - 9$, there is a connected graph G such that*

$$\rho_2'(G) = k.$$

Proof. We prove the assertion holds by constructing desired graphs for each of the following three cases.

Case 1. $n \leq k < \lceil \frac{3n}{2} \rceil$.

For $k = n$, we can take $G \cong C_n$ and so $\rho_2'(G) = n$. Thus, in the following, we may assume that $k > n$.

We construct a graph G by adding some chords to C_n. Assume that the order of the vertices of the cycle C_n is $v_1, \ldots, v_n$. We add the following chords:

$$\bigcup_{i=1}^{k-n} \{v_i v_{\lfloor n/2 \rfloor + i}\}.$$

See Figure 2 for an example when $n = 8$ and $k = 10$.

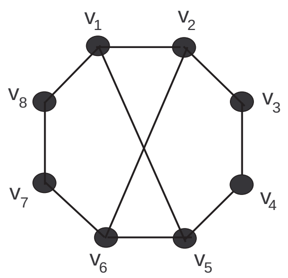

FIGURE 2. A graph G of order 8 with $\rho_2'(G) = 10$.

The graph G has size k and no vertex with degree greater than 3 and so $\rho_2'(G) = |E(G))| = k$.

Case 2. $\lceil \frac{3n}{2} \rceil \leq k \leq 2n - 4$.

In this case, a graph G with $\rho_2'(G) = k$ is the following bipartite graph with some edges subdivided. Let

$$V(G) = \{a, b, u_1, \ldots, u_{k-n+2}, v_1, \ldots, v_{2n-k-4}\}$$

and

$$E(G) = \bigcup_{i=1}^{k-n+2} \{av_i\} \cup \bigcup_{j=1}^{2n-k-4} \{u_j v_j, bu_j\} \cup \bigcup_{j=2n-k-3}^{k-n+2} \{bv_j\}.$$

Every edge of G is incident to a v_i or a u_j, and each such vertex has degree 2, so the only signed 2-edge cover has all edges with weight 1. See Figure 3 for an example of such graph when $n = 10$ and $k = 15$.

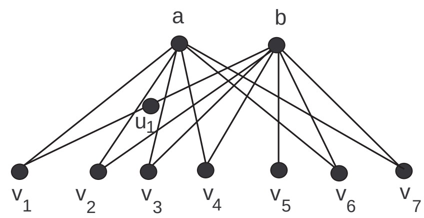

FIGURE 3. A graph G of order 10 and size 15 with $\rho_2'(G) = 15$.

Case 3. $2n - 4 < k \leq 3n - 9$.

In the final case, one graph G with $\rho'_2(G) = k$ is the following bipartite graph. Let

$$V(G) = \{a, b, c\} \cup \{v_1, \ldots, v_{n-3}\}$$

and

$$E(G) = \bigcup_{i=1}^{n-3} \{av_i, cv_i\} \cup \bigcup_{i=1}^{k-2n+6} \{bv_i\}.$$

See Figure 4 for an example when $n = 8$ and $k = 13$.

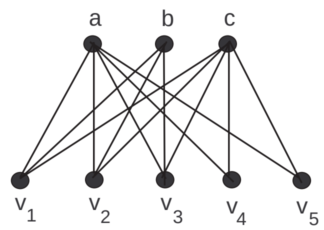

FIGURE 4. A graph G of order 8 with $\rho'_2(G) = 13$.

The graph G is bipartite with vertex classes $X = \{a, b, c\}$ and $Y = \{v_1, \ldots, v_{n-3}\}$. Furthermore, each edge is incident with at least one vertex of degree at most 3. Hence, $\rho'_2(G) = k$. $\square$

4. Cartesian products

We investigate the signed b-edge cover number of Cartesian product graphs. Our main result is the following.

Theorem 7. *Let G be a graph of order n_G and size m_G with k_G odd vertices. Let H be a graph of order n_H and size m_H with k_H odd vertices. For any integer $b \geq 2$, the signed b-edge cover number of $G \square H$ is at most*

$$\min_{1 \leq i \leq b-1} \{n_H \rho'_b(G) + n_G \tau(H), n_G \rho'_b(H) + n_H \tau(G), p(i, G, H)\},$$

where for $J \in \{G, H\}$

$$\tau(J) = k_J + \frac{1 - (-1)^{m_J}}{2},$$

and for $1 \leq i \leq b - 1$,

$$p(i, G, H) = n_H \rho'_i(G) + n_G \rho'_{b-i}(H).$$

The upper bound in Theorem 7 is sharp, as will follow from Corollary 12. To prove Theorem 7, we use the following lemma. We call a function $f : E(G) \to \{-1, 1\}$ *good* if f satisfies that $f(E_G(v)) \geq 0$ for every $v \in V(G)$.

Lemma 8. *For any graph G of order $n \geq 2$ and size m, there is a good function f such that*

$$f(E(G)) = \sum_{e \in E(G)} f(e) \leq k + \frac{1-(-1)^m}{2},$$

where $k \leq 2\lfloor \frac{n}{2} \rfloor$ is the number of odd vertices in G.

Proof. As every graph has an even number of odd vertices, k is even and $k \leq 2\lfloor \frac{n}{2} \rfloor$. Partition the odd vertices of G into $k/2$ pairs, and let H be a graph obtained by adding $k/2$ new vertices $w_1, \dots, w_{\frac{k}{2}}$ to G, and joining each w_i to the two odd vertices of the ith pair. It is clear that H has no odd vertices and so is Eulerian. Let C be an Eulerian circuit of H. We assign values 1 and -1 alternately along C. This defines a function $f' : E(H) \to \{1, -1\}$ such that $\sum_{e \in E_H(v)} f'(e) = 0$ for every $v \in V(H)$ and

$$\sum_{e \in E(H)} f'(e) = \sum_{e \in E(G)} f'(e) = \frac{1-(-1)^m}{2}.$$

Now we modify f' to form a good function f of G as follows: for each odd vertex v of G, change -1 to 1 exactly once on one of the edges incident with v. We need to make such changes at most $k/2$ times, as there are $k/2$ many -1's on edges to the w_i. Hence,

$$f(E(G)) = \sum_{e \in E(G)} f(e) \leq \sum_{e \in E(G)} f'(e) + 2 \times \frac{k}{2} = k + \frac{1-(-1)^m}{2}.$$

□

Proof of Theorem 7. To show that

$\rho_b'(G \square H) \leq \min\{n_H \rho_b'(G) + n_G \tau(H), n_G \rho_b'(H) + n_H \tau(G), p(i, G, H)\}$,

it suffices to show

$$\rho_b'(G \square H) \leq n_H \rho_b'(G) + n_G \tau(H), \tag{4.1}$$

$$\rho_b'(G \square H) \leq n_G \rho_b'(H) + n_H \tau(G), \tag{4.2}$$

and for each $1 \leq i \leq b-1$,

$$\rho_b'(G \square H) \leq p(i, G, H). \tag{4.3}$$

The statement (4.2) will follow by symmetry from (4.1). To show that (4.1), it suffices to construct a SbEC f of $G\Box H$ so that

$$\sum_{e\in E(G\Box H)} f(e) \leq n_H\rho'_b(G) + n_G\tau(H).$$

Let f^G and f^H be SbECs of G and H such that $f^G(E(G)) = \sum_{e\in E(G)} f^G(e) = \rho'_b(G)$ and $f^H(E(H)) = \sum_{e\in E(H)} f^H(e) = \rho'_b(H)$, respectively. By Lemma 8, there exists a good function g^H of H such that

$$g^H(E(H)) = \sum_{e\in E(H)} g^H(e) \leq k_H + \frac{1-(-1)^{m_H}}{2}.$$

Note that for any vertex $v \in V(H)$, the subgraph $(G\Box H)[S_v]$ induced by $S_v = \{(u,v) : (u,v) \in V(G\Box H)\}$ is isomorphic to G. Similarly, for any vertex $u \in V(G)$, the subgraph $(G\Box H)[T_u]$ induced by $T_u = \{(u,v) : (u,v) \in V(G\Box H)\}$ is isomorphic to H.

We define f as follows. For every $v \in V(H)$, if $u_1u_2 \in E(G)$, then

$$f\left((u_1,v)(u_2,v)\right) = f^G(u_1u_2).$$

For every $u \in V(G)$, if $v_1v_2 \in E(H)$, then

$$f\left((u,v_1)(u,v_2)\right) = g^H(v_1v_2).$$

Hence, for each $(u,v) \in V(G\Box H)$, we have that

$$\begin{aligned}\sum_{e\in E_{G\Box H}((u,v))} f(e) &= \sum_{e\in E_G(u)} f^G(e) + \sum_{e\in E_H(v)} g^H(e)\\ &\geq b+0\\ &= b,\end{aligned}$$

and

$$\begin{aligned}\sum_{e\in E(G\Box H)} f(e) &= n_H\sum_{e\in E(G)} f^G(e) + n_G\sum_{e\in E(H)} g^H(e)\\ &\leq n_H\rho'_b(G) + n_G\left(k_H + \frac{1-(-1)^{m_H}}{2}\right)\\ &= n_H\rho'_b(G) + n_G\tau(H).\end{aligned}$$

Thus,

$$\rho'_b(G\Box H) \leq n_H\rho'_b(G) + n_G\tau(H),$$

and (4.1) follows.

We now prove (4.3). For each $1 \leq i \leq b-1$, let f_i^G be SiEC of G such that $\sum_{e\in E(G)} f_i^G(e) = \rho'_i(G)$ and let f_{b-i}^H be S$(b-i)$EC of H

such that $\sum_{e\in E(H)} f^H_{b-i}(e) = \rho'_{b-i}(H)$. Define f as follows. For every $v \in V(H)$, if $u_1u_2 \in E(G)$, then

$$f\left((u_1,v)(u_2,v)\right) = f^G_i(u_1u_2).$$

For every $u \in V(G)$, if $v_1v_2 \in E(H)$, then

$$f\left((u,v_1)(u,v_2)\right) = f^H_{b-i}(v_1v_2).$$

Hence, for each $(u,v) \in V(G\Box H)$, we have that

$$\begin{aligned}\sum_{e\in E_{G\Box H}((u,v))} f(e) &= \sum_{e\in E_G(u)} f^G_i(e) + \sum_{e\in E_H(v)} f^H_{b-i}(e)\\ &\geq i + (b-i)\\ &= b,\end{aligned}$$

and

$$\begin{aligned}\sum_{e\in E(G\Box H)} f(e) &= n_H \sum_{e\in E(G)} f^G_i(e) + n_G \sum_{e\in E(H)} f^H_{b-i}(e)\\ &\leq n_H\rho'_i(G) + n_G\rho'_{b-i}(H)\\ &= p(i,G,H).\end{aligned}$$

Thus,

$$\rho'_b(G\Box H) \leq p(i,G,H),$$

and (4.3) follows. □

Corollary 9. *Let G be an Eulerian graph of order n_G and size m_G. For any graph H of order n_H, and for any integer $b \geq 2$,*

$$\left\lceil \frac{bn_Gn_H}{2} \right\rceil \leq \rho'_b(G\Box H) \leq n_G\rho'_b(H) + \frac{1-(-1)^{m_G}}{2}n_H.$$

Proof. The lower bound follows by Theorem 3. By hypothesis, G has no odd vertices and so $\tau(G) = \frac{1-(-1)^{m_G}}{2}$. The second inequality holds by Theorem 7. □

Many other graph products exist, such as the categorical, strong, and lexicographic products; see the book [3] for more on graph products. We will investigate how the parameter ρ'_b acts with respect to these products in future work.

5. Cliques and bicliques

As the parameter ρ'_b is new, it is important to determine its values for some familiar graphs. The exact values of ρ'_b for cliques, K_n, and bicliques, $K_{n,n}$, are found in this final section. In both cases, we use results on graph factors, 1-factorable graphs, and hamiltonian factorable graphs.

Theorem 10. *Fix $b \geq 1$ an integer. For any integer $n \geq b+2$, we have the following.*

$$\rho'_b(K_n) = \begin{cases} bn/2, & n \equiv 0 \text{ (mod 2)},\ b \equiv 1 \text{ (mod 2)}; \\ (b+1)n/2, & n-b \equiv 2 \text{ (mod 4)},\ b \equiv 1 \text{ (mod 2)}; \\ (b+1)n/2+1, & n-b \equiv 0 \text{ (mod 4)},\ b \equiv 1 \text{ (mod 2)}; \\ bn/2, & n-b \equiv 1 \text{ (mod 4)},\ b \equiv 0 \text{ (mod 2)}; \\ bn/2+1, & n-b \equiv 3 \text{ (mod 4)},\ b \equiv 0 \text{ (mod 2)}; \\ (b+1)n/2, & n \equiv 0 \text{ (mod 2)},\ b \equiv 0 \text{ (mod 2)}. \end{cases}$$

Proof. We only prove the case when b is odd, as the proof is similar when b is even. Consider the following three cases.

Case 1. $n \equiv 0$ (mod 2).

By Theorem 9.19 on page 273 of [1], K_n is 1-factorable into $(n-1)$ 1-factors. So, we assign $x(e) = 1$ for each edge e of $\frac{1}{2}(n+b-1)$ 1-factors, and $x(e') = -1$ for each edge e' of the remaining $\frac{1}{2}(n-b-1)$ 1-factors. It is straightforward to verify that f is a SbEC of K_n and $f(E(K_n)) = bn/2$. Hence, $\rho'_b(K_n) \leq bn/2$. It follows by Theorem 3 that $\rho'_b(K_n) = bn/2$.

Case 2. $n-b \equiv 2$ (mod 4).

In this case, $n = b+4k+2$ for some integer $k \geq 0$. Every vertex of K_{b+4k+2} has even degree $b+4k+1$, and b is odd. Thus, for any SbEC f and for each $v \in V(K_{b+4k+2})$, $f\left(E_{K_{b+4k+2}}(v)\right) \geq b$ implies that $f\left(E_{K_{b+4k+2}}(v)\right) \geq b+1$. Summing $f\left(E_{K_{b+4k+2}}(v)\right)$ over all vertices, we get

$$f(E(K_{b+4k+2})) \geq (b+1)(b+4k+2)/2 = (b+1)n/2.$$

Hence,

$$\rho'_b(K_{b+4k+2}) \geq (b+1)n/2.$$

To show that the equality holds, we need to obtain a SbEC f of K_{b+4k+2} such that $f(E(K_{b+4k+2})) = (b+1)n/2$. By Theorem 9.21 on page 275 of [1], K_{b+4k+2} can be factored into $2k+(b+1)/2$ hamiltonian cycles $C_1, \cdots, C_{2k+(b+1)/2}$. Note that the graph $H = K_{b+4k+2} - \bigcup_{i=1}^{(b+1)/2} E(C_i)$ is Eulerian with $2kn$ edges. Assigning 1 to each edge of $C_1, \cdots, C_{(b+1)/2}$, and assigning +1 and -1 alternately along an Eulerian circuit of H, we obtain a SbEC f of K_{b+4k+2} such that $f(E(K_{b+4k+2})) = (b+1)n/2$.

Case 3. $n - b \equiv 0 \pmod 4$.

In this case, $n = b + 4k$ for some integer $k \geq 1$. By an argument similar to that in Case 2, we have that $\rho'_b(K_{b+4k}) \geq (b+1)n/2$.

We show by contradiction that the equality does not hold. Suppose that there is a SbEC f of K_{b+4k} such that $f(E(K_{b+4k})) = \rho'_b(K_{b+4k}) = (b+1)n/2$. Let p and q be the numbers of edges in K_{b+4k} with values 1 and -1, respectively. Then,

$$p + q = (b + 4k - 1)n/2$$

and

$$p - q = (b + 1)n/2.$$

Adding the last two equations, we obtain that

$$2p = bn + 2kn,$$

which is impossible since both n and b are odd. So, $\rho'_b(K_{b+4k}) \geq (b+1)n/2 + 1$.

To finish our proof in this case, we need to find a SbEC f of K_{b+4k} such that $f(E(K_{b+4k})) = (b+1)n/2 + 1$. By Theorem 9.21 on page 275 of [1], K_{b+4k} can be factored into $2k + (b-1)/2$ hamiltonian cycles $C_1, \cdots, C_{2k+(b-1)/2}$. Note that the graph $H = K_{b+4k} - \bigcup_{i=1}^{(b+1)/2} E(C_i)$ is Eulerian with $(2k-1)n$ edges. Assigning 1 to each edge of $C_1, \cdots, C_{(b+1)/2}$, and assigning +1 and -1 alternately along an Eulerian circuit of H starting with 1, we obtain a SbEC f of K_{b+4k} such that $f(E(K_{b+4k})) = (b+1)n/2 + 1$. □

We finish by determining ρ'_b for the bicliques $K_{n,n}$.

Theorem 11. *Fix b a positive integer. For any integer $n \geq b + 1$,*

$$\rho'_b(K_{n,n}) = \begin{cases} bn, & n\text{-}b \equiv 0 \pmod 2; \\ (b+1)n, & n\text{-}b \equiv 1 \pmod 2). \end{cases}$$

Proof. By [5] (see also Theorem 9.18 on page 272 of [1]), the graph $K_{n,n}$ can be factored into n 1-factors. By Theorem 3, $\rho'_b(K_{n,n}) \geq bn$. If $n - b \equiv 0 \pmod 2$, then it suffices to show that $\rho'_b(K_{n,n}) \leq bn$. We can do so by constructing a SbEC f for which $f(E(K_{n,n})) = bn$. Since $K_{n,n}$ can be factored into n 1-factors, we can assign 1 to each edge of $\frac{n+b}{2}$ 1-factors and -1 to each edge of $\frac{n-b}{2}$ 1-factors. This defines a SbEC f of $K_{n,n}$ satisfying $f(E(K_{n,n})) = bn$.

If $n - b \equiv 1 \pmod 2$, then for any SbEC f of $K_{n,n}$ and each $v \in V(K_{n,n})$ we have that

$$\sum_{e \in E_{K_{n,n}}(v)} f(e) \geq b + 1.$$

Thus,

$$f\left(E(K_{n,n})\right) \geq (b+1)n,$$

which proves that $\rho'_b(K_{n,n}) \geq (b+1)n$.

To show that $\rho'_b(K_{n,n}) \leq (b+1)n$, it suffices to construct a SbEC f such that $f\left(E(K_{n,n})\right) = (b+1)n$. We assign 1 to each edge of $\frac{1}{2}(n+b+1)$ 1-factors and -1 to each edge of the remaining $\frac{1}{2}(n-b-1)$ 1-factors. This defines a SbEC f of $K_{n,n}$ satisfying $f(E(K_{n,n})) = (b+1)n$. □

Corollary 12. *Let G be an Eulerian graph of order n_G and size m_G. For all positive integers $n \geq b \geq 2$ satisfying $n \equiv b$ (mod 2), if m_G is even, then*

$$\rho'_b(G \Box K_{n,n}) = bnn_G.$$

Proof. Under the hypotheses, $\tau(G) = 0$. By Theorem 11, $\rho'_b(K_{n,n}) = bn$. So, by Theorem 7, we have $\rho'_b(G \Box K_{n,n}) \leq bnn_G$. The reverse inequality follows by Theorem 3. □

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