MATCHINGS DEFINED BY LOCAL CONDITIONS

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ABSTRACT. A graph has the neighbour-closed-co-neighbour, or ncc property, if for each of its vertices x, the subgraph induced by the neighbour set of x is isomorphic to the subgraph induced by the closed non-neighbour set of x. Graphs with the ncc property were characterized in [1] by the existence of a locally C_4 perfect matching M: every two edges of M induce a subgraph isomorphic to C_4. In the present article, we investigate variants of locally C_4 perfect matchings. We consider the cases where pairs of distinct edges of the matching induce isomorphism types including P_4, the paw, or the diamond. We give several characterizations of graphs with such matchings. In addition, we supply characterizations of graphs with matchings whose edges satisfy a prescribed parity condition.

1. Introduction

Matchings have been extensively studied in graph theory, and play an important role in combinatorial optimization; see for example, [7, 8]. A *disjoint neighbour perfect* or *dnp matching* M is a perfect matching with the property that no edge of M is in a triangle. For example, every perfect matching in a bipartite graph is dnp, and there is a unique dnp matching in the Cartesian product of an n-vertex clique with K_2, written $K_n \square K_2$.

We only consider graphs which are finite, undirected, and simple. We use the notation $G \restriction S$ for the subgraph of G induced by a set of vertices S, and the notation $G \cong H$ for isomorphic graphs. If x is a vertex of G, then define $N(x)$ to be the set of vertices of G joined to x. Define $N^c[x]$ to be the set $V(G) \setminus N(x)$. R. Nowakowski recently proposed the following vertex partition property as an analogue of similar properties for infinite graphs (such as the infinite random graph): a graph G has the *neighbour-closed-co-neighbour* or *ncc* property, if for all $x \in V(G)$, we have that $G \restriction N(x) \cong G \restriction N^c[x]$. There are many examples of such graphs, such as the bipartite cliques $K_{n,n}$ and the graphs $K_n \square K_2$. There are, however,

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1991 *Mathematics Subject Classification.* 05C70, 05C75, 05C85.

Key words and phrases. Graph, matching, ncc graph, local property, polynomial time algorithm.

The authors gratefully acknowledge the support of NSERC Canada.

many ncc graphs that are not one of these types. The class of ncc graphs were completely characterized in [1] using dnp matchings.

Theorem 1. *A graph G is ncc if and only if there is a positive integer n so that G has $2n$ vertices, G is n-regular, and G has a dnp matching.*

Theorem 1 implies the following.

Theorem 2. *A graph G is ncc if and only if G has a perfect matching M so that every pair of distinct edges of M induce a subgraph isomorphic to C_4.*

A dnp matching in an ncc graph acts "locally" as an isomorphism. This is made precise in the following theorem, which was proved as a claim in the converse of Theorem 2.1 from [1].

Theorem 3. *Let G be an ncc graph with a dnp matching $M = \{a_i b_i : 1 \leq i \leq n\}$. Then the mapping*

$$f : G \restriction \{a_i : 1 \leq i \leq n\} \to G \restriction \{b_i : 1 \leq i \leq n\}$$

defined by $f(a_i) = b_i$ is an isomorphism.

Following [1], we name the mapping f of the theorem an *M-isomorphism.* The conclusion of this theorem holds regardless of what "orientation" the matching is given. Hence, for each edge $xy \in M$, there are two choices for the "a" vertex and two for the "b", giving rise to 2^n distinct M-isomorphisms. In this way, we may view a matching as a mapping (which may not necessarily be an isomorphism), which we refer to as an *M-morphism.* This view leads to a new characterization of ncc graphs.

Theorem 4. *A graph G is ncc if and only if G has n^2 edges, has a perfect matching M so that every M-morphism is an isomorphism, and no two distinct edges of M induce a subgraph isomorphic to K_4.*

Before we prove Theorem 4, we need some notation. Let P_n denote the path with n edges. The graph $2K_2$ consists of two disjoint copies of K_2. The *paw* is K_3 plus one endvertex, and the *diamond* is K_4 minus an edge. See Figure 1. For more on these graphs, the reader is directed to [2].

Proof. The necessity follows by Theorems 1, 2, and 3. For sufficiency, fix distinct edges $e = ab$ and $e' = a'b'$ of M. Up to isomorphism, the graph H induced on the vertices of e and e' is one of $2K_2$, C_4, P_4, the paw, or the diamond. Suppose first that H is the paw, say with edges $ab, aa', ba', a'b'$. But then aa' is an edge, with bb' a non-edge, which violates that every M-morphism is an isomorphism. A similar argument excludes P_4 and the diamond. By Theorem 2, we need only exclude $2K_2$. If $H \cong 2K_2$, then each pair of distinct edges of M distinct from e, e' is joined by at most two

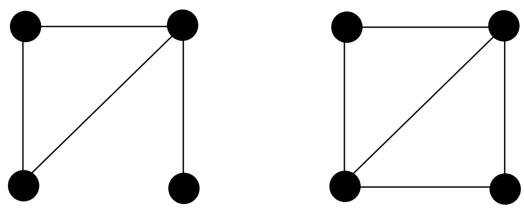

FIGURE 1. The paw and the diamond.

edges (since we have excluded all possibilities for H except $2K_2$ and C_4). But then

$$|E(G)| \leq n + 2\left(\binom{n}{2} - 1\right) < n^2,$$

which contradicts hypothesis. $\square$

Let G have a perfect matching M. We say that M is *locally H* if each pair of distinct edges of M induce a graph isomorphic to H. Hence, a matching may be locally $2K_2$, C_4, P_4, the paw, the diamond, or K_4, with no other possibilities. A graph with a locally $2K_2$ perfect matching consists of n disjoint copies of K_2. Such matchings have been well-studied, and are sometimes called *induced* or *strong*; see [3]. A graph with a locally K_4 perfect matching is a clique. With this notation, we may restate Theorem 2 as follows.

Theorem 5. *A graph is ncc if and only if it has a locally C_4 perfect matching.*

From Theorem 5 and the above discussion, the remaining unexamined choices for H are P_4, the paw or the diamond. In each case, graphs with locally H perfect matchings give rise to an interesting class of graphs. For these graph classes, we prove structural characterizations similar to Theorem 4 in Theorems 6 and 8.

Graphs with locally H perfect matchings have diameter 2 or 3. In Section 3, we present a generalization of locally H perfect matchings to graphs with arbitrary diameter. This gives rise to *parity disjoint* perfect matchings, which are defined via certain distance conditions on the edges of the matching. We characterize such matchings in Theorem 10, and give a polynomial time recognition algorithm for them in Corollary 2.

2. Characterizing graphs with locally H perfect matchings

We now characterize graphs with locally H perfect matchings in a fashion similar to Theorem 4. However, we will use M-morphisms that are not necessarily isomorphisms.

Let $f : V(G) \to V(H)$ be a vertex mapping. We will abuse notation and write $f : G \to H$. The mapping f is a *homomorphism* if $xy \in E(G)$ implies that $f(x)f(y) \in E(H)$; in other words, it sends edges to edges. See the book [6] for more on homomorphisms. The map f is a *cohomomorphism* if $xy \in E(G)$ implies that $f(x)f(y) \notin E(H)$. Cohomomorphisms were first studied in [5]. An *anti-homomorphism* sends edges to non-edges, while an *anti-cohomomorphism* sends non-edges to edges. The mapping f is an *anti-isomorphism* if it is bijective and is both an anti-homomorphism and an anti-cohomomorphism.

Theorem 6. *Let G be a graph with $2n$ vertices, where n is a positive integer.*

(1) *The graph G has a locally P_4 perfect matching if and only if there is a perfect matching M of G so that every M-morphism is an anti-homomorphism, there are $\frac{n^2+n}{2}$ edges in G, and no two edges of M induce a subgraph isomorphic to $2K_2$.*

(2) *The graph G has a locally paw perfect matching M if and only there is a perfect matching M of G so that every M-morphism is an anti-isomorphism.*

(3) *The graph G has a locally diamond perfect matching M if and only if there is a perfect matching M of G so that every M-morphism is an anti-cohomomorphism, there are $\frac{3n^2-n}{2}$ edges in G, and no two edges of M induce a subgraph isomorphic to K_4.*

Proof. (1) For the forward direction, let M be a locally P_4 perfect matching with $M = \{a_ib_i : 1 \le i \le n\}$. Fix an M-morphism

$$f : G \restriction \{a_i : 1 \le i \le n\} \to G \restriction \{b_i : 1 \le i \le n\}$$

defined by $f(a_i) = b_i$, for $1 \le i \le n$. Since each pair of distinct edges a_ib_i and a_jb_j of M induce a P_4, if say a_ia_j is an edge, then b_ib_j is a non-edge. Hence, by symmetry, f is an anti-homomorphism. As each pair of edges of M are joined by exactly one edge, there are $n + \binom{n}{2} = \frac{n^2+n}{2}$ edges in G. As M is locally P_4 perfect, no two edges of M induce a subgraph isomorphic to $2K_2$.

For the reverse direction, fix distinct edges a_ib_i and a_jb_j of M. Without loss of generality, say $i = 1$ and $j = 2$. By hypothesis, the subgraph H induced by $\{a_1, a_2, b_1, b_2\}$ cannot be isomorphic to $2K_2$. We must therefore exclude the cases when H is C_4, a paw, diamond, or K_4. Suppose for a contradiction that H is C_4. As M is an anti-homomorphism, a_1 is not

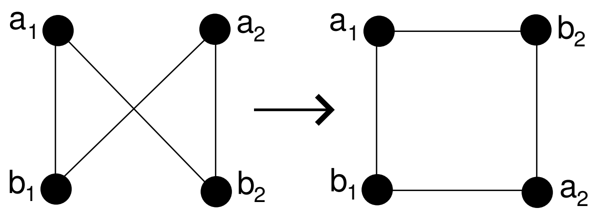

FIGURE 2. Excluding C_4 in the proof of (1).

joined to a_2 and b_1 is not joined to b_2; hence, a_1b_2 and a_2b_1 are edges. Define the M-morphism

$$f' : G \restriction \{a_1, b_2, a_3, \ldots, a_n\} \to G \restriction \{b_1, a_2, b_3, \ldots, b_n\}$$

by $f'(a_i) = \begin{cases} b_i & \text{if } i \neq 2; \\ a_2 & \text{else.} \end{cases}$ See Figure 2. The map f' fails to be anti-homomorphism, as $a_1b_2 \in E(G \restriction \{a_1, b_2, a_3, \ldots, a_n\})$ but $f'(a_1)f'(b_2) \in E(G \restriction \{b_1, a_2, b_3, \ldots, b_n\})$. Hence, H is not C_4. A similar argument excludes the diamond and K_4.

We have shown that each H is either P_4 or a paw. Suppose for a contradiction that some pair of distinct edges of M induces a paw. Let r be the number of pairs of edges of M with exactly 1 edge between them, and let s be the number of pairs of edges with exactly 2 edges between them. Then $r \geq 0$, $s \geq 1$, and $r + s = \binom{n}{2}$. Further,

$$\begin{aligned} |E(G)| &= n + r + 2s \\ &> n + \binom{n}{2} = \frac{n^2 + n}{2}, \end{aligned}$$

which contradicts hypothesis.

(2) For the forward direction, let M be a locally paw perfect matching, and fix an M-morphism $f : G \restriction \{a_i : 1 \leq i \leq n\} \to G \restriction \{b_i : 1 \leq i \leq n\}$ defined by $f(a_i) = b_i$ for $1 \leq i \leq n$. Since each pair of edges a_ib_i and a_jb_j of M induce a paw, if say a_ia_j is an edge, then b_ib_j is a non-edge. Hence, by symmetry, f is an anti-homomorphism. If a_ia_j is a non-edge, then a_ia_j is an edge; by symmetry, f is an anti-cohomomorphism, and thus, f is an anti-isomorphism.

For the reverse direction, fix distinct edges a_ib_i and a_jb_j of M. By hypothesis and arguments similar to those given in the proof of (1), the subgraph H induced by $\{a_i, a_j, b_i, b_j\}$ cannot be isomorphic to $2K_2$, P_4, C_4, the diamond, or K_4. Hence, M is locally paw.

(3) For the forward direction, let M be a locally diamond perfect matching, and fix an M-morphism $f : G \restriction \{a_i : 1 \leq i \leq n\} \rightarrow G \restriction \{b_i : 1 \leq i \leq n\}$ defined by $f(a_i) = b_i$. Since each pair of edges a_ib_i and a_jb_j of M induce a diamond, if say a_ia_j is a non-edge, then a_ia_j is an edge; by symmetry f is an anti-cohomomorphism. As each pair of edges of M are joined by exactly three edges, there are $n+3\binom{n}{2} = \frac{3n^2-n}{2}$ edges in G. As M is locally diamond perfect, no two edges of M induce a subgraph isomorphic to K_4.

For the reverse direction, fix a_ib_i and a_jb_j edges of M. By hypothesis and arguments similar to those of (1), the subgraph H induced by $\{a_i, a_j, b_i, b_j\}$ cannot be isomorphic to $2K_2, P_4, C_4$, or K_4. We must exclude the paw. Suppose for a contradiction that H is isomorphic to a paw. Hence, between each pair of edges in M there are either 2 or 3 edges. As in (1) there are integers $r \geq 1$ and $s \geq 0$ so that $r + s = \binom{n}{2}$, and

$$\begin{aligned} |E(G)| &= n + 2r + 3s \\ &< n + 3\binom{n}{2} = \frac{3n^2 - n}{2}, \end{aligned}$$

which is a contradiction. □

Planarity is a strong restriction on graphs with a locally H perfect matching, as witnessed by the following theorem.

Corollary 1. *There are only finitely many non-isomorphic planar graphs which have a locally H matching, where H is one of C_4, P_4, the paw, or the diamond.*

Proof. Fix H as in the statement of the corollary. A graph G with $2n$ vertices and a locally H perfect matching is *dense*, in the sense that $|E(G)| \in O(n^2)$. This fact, Theorem 6, and the well known property that if G is planar then $|E(G)| \leq 3|V(G)| + 6$ complete the proof. □

We now turn to another structural characterization of graphs with a locally H perfect matching. Suppose that G is a graph with perfect matching M, and let ab, $a'b'$ be distinct edges of M. Define an *interchange (with respect to M)* by interchanging the edges and non-edges of $G \restriction \{a, a', b, b\}$, leaving the edges ab and $a'b'$ unchanged, so that the isomorphism type of the subgraph induced by $\{a, a', b, b\}$ is unchanged. We write $G \sim_M G'$ if G' results from G by one C_4-interchange with respect to M. We write $G \sim_M^* G'$ if there is an integer $n \geq 0$, and graphs $G_0 = G, G_1, \ldots, G_n = G'$ so that for all $0 \leq i \leq n-1$, $G_i \sim_M^* G_{i+1}$. See Figure 3.

If G and H are graphs, then we write the *Cartesian product* of G and H as $G \square H$. The following theorem was proved in [1].

Theorem 7. *A graph G is ncc if and only if G has a perfect matching M so that $G \sim_M^* (K_n \square K_2)$.*

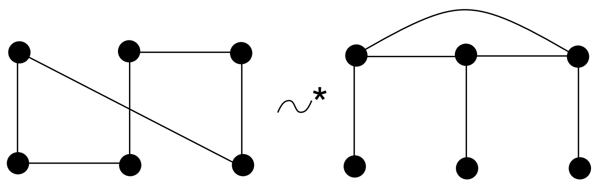

FIGURE 3. A sequence of interchanges in a graph with a locally P_4 perfect matching.

Define the graph K_n' by adding an endvertex joined to each vertex of K_n. Let the vertices of K_n be labelled $\{x_j : 1 \leq j \leq n\}$. Define the graph K_n'' by adding a set of n independent vertices y_i, so that for each $1 \leq i \leq n$, y_i is joined to all x_j with $j \geq i$. We use the notation $\overline{K_n}$ for the complement of K_n. Define the graph K_n''' by adding all edges between K_n and $\overline{K_n}$. The proof of the following theorem, which extends Theorem 7 to locally H matchings, follows from the definitions.

Theorem 8. *Let G be a graph.*

(1) *The graph G has a locally P_4 perfect matching M if and only if it has a matching M so that $G \sim_M^* K_n'$.*
(2) *The graph G has a locally paw perfect matching M if and only if it has a matching M so that $G \sim_M^* K_n''$.*
(3) *The graph G has a locally diamond perfect matching M if and only if it has a matching M so that $G \sim_M^* K_n'''$.*

Locally H graphs, where H is one of P_4, the paw, or the diamond are in a certain sense *universal.* We make this precise in the following theorem.

Theorem 9. *Let G be a fixed graph, and suppose that H is isomorphic to one of P_4, the paw, or the diamond. Then G is isomorphic to the induced subgraph of a graph G' with a locally H perfect matching, so that $|V(G')| \leq 2|V(G)|$.*

Proof. We give the construction for $H \cong P_4$, since the cases of the paw and diamond are handled analogously. Let $V(G) = \{x_1, \ldots, x_n\}$. To form G'', add to G vertices $\{y_1, \ldots, y_n\}$ so that for all i, y_i is only joined to x_i. Form G' as follows: if x_i is not joined to x_j in G'', then add an edge between y_i and y_j; add no other edges. It is straightforward to check that $\{x_iy_i : 1 \leq i \leq n\}$ is a locally P_4 perfect matching in G'. □

We do not know if the problems of recognizing a locally H perfect matching, where H is P_4, the paw, or the diamond, are polynomial time.

3. Parity disjoint matchings and pairings

All graphs in this section are connected. It is not hard to see that a graph with a locally H perfect matching, where H is connected, has diameter 2 or 3. In this section, we consider a variation of locally H perfect matchings to include graphs of arbitrary diameter. We denote by $d_G(u,v)$ the distance between u and v; we may drop the subscript G if it is clear from context.

A *pair* in a graph is an unordered set of two distinct vertices. A *parity disjoint* or *pd* pair is a pair $\{a,b\}$ of vertices with the property that for all vertices x

$$d(a,x) \equiv d(b,x) + 1 \pmod 2.$$

In other words, a pair is pd if every vertex of even (odd) distance to a is odd (even) distance to b. A *pd edge* is a pd pair that is an edge. For instance, an ncc graph G is diameter 2, so by Theorem 5 each edge in a dnp matching of G is pd. All edges in a bipartite graph is pd.

A *pairing* P is a set of pairwise disjoint pairs. In particular, a pairing is a matching if each pair forms an edge of the graph. A *pd pairing* is a pairing P so that

(1) for all $x \in V(G)$, there is a unique pair $p \in P$ so that $x \in p$;
(2) for each pair $\{a,b\} \in P$, $d(a,b)$ is odd;
(3) each pair in P is pd.

A *pd matching* is a pd pairing P where each pair in P is an edge. For example, an ncc graph or a balanced bipartite graph (that is, a bipartite graph whose vertex classes have the same cardinality) have dnp pairings.

Before we give a characterization of graphs with pd matchings and pairings, we need a few definitions. Define the graph G^{+odd} by joining all pairs of non-joined vertices of G that are an odd distance apart. See Figure 4 for an example of G^{+odd}.

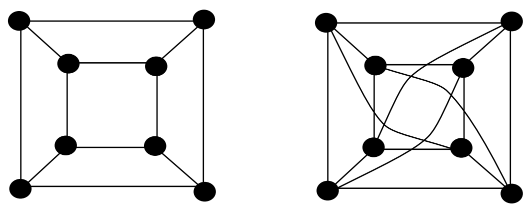

FIGURE 4. A graph G and G^{+odd}.

Let $f : G \to H$ be a vertex mapping. We say that f *preserves parity* if for all $x, y \in V(G)$,

$$d_G(x,y) \equiv d_H(f(x), f(y)) \pmod 2.$$

Define $e(x)$ to be the set of vertices of even distance to x in G (including x); the set $o(x)$ is defined analogously. A perfect matching M of G is *co-dnp* if for each edge $ab \in M$, there is no $x \in V(G)$ that is non-joined to both a and b.

Theorem 10. *Let G be a graph with $2n$ vertices.*

(1) *A graph G has a pd pairing if and only if G^{+odd} is ncc.*
(2) *A graph G has a pd matching if and only there is a perfect matching M of G so that every M-morphism preserves parity, and for all $x \in V(G)$, $|e(x)| = n$.*

Proof. (1) For the forward direction, assume that G has a pd pairing $P = \{\{a_i, b_i\} : 1 \leq i \leq n\}$. Since $d(a_i, b_i)$ is odd by hypothesis, a_ib_i is an edge of G^{+odd}. Hence, $M = \{a_ib_i : 1 \leq i \leq n\}$ is a perfect matching in G^{+odd}. By Theorem 5 we need only check that any two distinct edges of M induce C_4. Suppose that a_i and b_i have either a common neighbour or common non-neighbour z. In either case, $d_G(a_i, z) \equiv d_G(b_i, z) \pmod 2$, which is a contradiction. The result follows since a matching which is dnp and co-dnp is locally C_4.

For the reverse direction, suppose that G^{+odd} is ncc. Let $M = \{a_ib_i : 1 \leq i \leq n\}$ be a locally C_4 matching in G^{+odd}, and so $P = \{\{a_i, b_i\} : 1 \leq i \leq n\}$ is a pairing in G (some of the edges a_ib_i of G^{+odd} may not be present in G). If $z \in V(G)$ has the property that $d_G(z, a_i)$ and $d_G(z, b_i)$ have the same parity, then this would contradict that a_i and b_i has no common neighbour nor non-neighbour in G^{+odd}.

(2) For the forward direction, let G have a pd matching $M = \{a_ib_i : 1 \leq i \leq n\}$. We prove that the M-morphism f mapping a_i to b_i preserves parity. Now,

$$d(a_i, a_j) \equiv d(a_j, b_i) + 1 \equiv d(b_i, b_j) + 2 \equiv d(b_i, b_j) \pmod 2.$$

As f was arbitrary, every M-morphism preserves parity.

For all i and j, each edge a_jb_j of M has exactly one of a_j or b_j in $e(a_i)$. The same holds for $e(b_i)$. Hence, for all vertices x of G, we have that $|e(x)| = n$.

For the reverse direction, fix $M = \{a_ib_i : 1 \leq i \leq n\}$ a matching of G with the prescribed property. Consider the edge a_1b_1. Since $|e(a_1)| = n$, by relabelling if necessary, we may assume that $e(a_1) = \{a_1, \ldots, a_n\}$ and $o(a_1) = \{b_1, \ldots, b_n\}$. As every M-morphism preserves parity and since $|e(b_1)| = n$, we have that $e(b_1) = \{b_1, \ldots, b_n\}$ and $o(b_1) = \{a_1, \ldots, a_n\}$ Hence, $o(a_1) = e(b_1)$ and $e(a_1) = o(b_1)$. In particular, $a_1b_1 \in M$ is a pd edge.

Define a pair of distinct vertices x, y of G to be *even twins* if $e(x) = e(y)$. Since every M-morphism preserves parity, every even twin of a_1 among the a_i is mapped by M to an even twin of b_1 among the b_i. Further, there are the same number of even twins of a_1 among the a_i as even twins of b_1 among the b_i. Therefore, each even twin of a_1 is matched by M to an even twin of b_1. List the even twins of a_1 and b_1 as $u_1 = a_1, u_2, \ldots, u_k$ and $v_1 = b_1, v_2, \ldots, v_k$, respectively, so u_i is matched to v_i by M. Let $M_{u_1} = \{u_i v_i : 1 \leq i \leq k_{u_1}\}$, and let $M_{u_1} = M_{u_i} = M_{v_i}$ for all $1 \leq i \leq k_{u_1}$. Note that each edge $u_i v_i$ is a pd edge.

Define

$$M = \bigcup_{z \in V(G)} M_z.$$

We now prove that M is a pd matching. To see that M is a matching, suppose to the contrary that there are two edges uv and uv' in M. But then v and v' are in the set $\{v_i : 1 \leq i \leq k_u\}$. But u is matched by M with a unique element of $\{v_i : 1 \leq i \leq k_u\}$, which is a contradiction. The matching M is pd by construction. $\square$

Theorem 10 (1) implies that if G has a pd matching, then G^{+odd} is ncc, but the converse is false. Consider the graph G formed from $K_3 \square K_2$ by deleting one edge in its unique dnp matching. The graph $G^{+odd} \cong G$ is ncc, but G has no pd matching.

We now demonstrate how to recognize graphs with pd matchings and pairings in polynomial time. To form the graph G^{-odd}, delete all edges ab with the property that there is a vertex x such that $d(a, x) \equiv d(b, x) \pmod 2$. The graph G^{-odd} may be constructed from G in polynomial time; the same is true with G^{+odd}. The graph G has a pd matching (pairing) if and only if G^{-odd} (G^{+odd}) has a perfect matching (is ncc). This gives rise to the following corollary of Theorem 10.

Corollary 2. *There is a polynomial-time algorithm to determine whether a graph has a pd matching (pairing).*

We conclude with a discussion of operations preserving pd matchings. If G and H are graphs (whose vertex set may intersect non-trivially), then we write $G \cup H$ for the graph with vertices $V(G) \cup V(H)$ and edges $E(G) \cup E(H)$.

Corollary 3.
1. *If G has a pd matching and H is any graph, then $G \square H$ has a pd matching.*
2. *If G is any graph, then the graph G' formed by joining an endvertex to each vertex of G has a pd matching.*
3. *If G has a pd matching, then the graph G'' formed by joining a path of length two to a fixed vertex has a pd matching.*
4. *Let G and H have pd matchings M and M', respectively. If $V(G) \cap V(H) = \{a, b\}$, where ab is pd edge in M and M', then $M \cup M'$ is a pd matching of $G \cup H$.*

Proof. (1) Let $M = \{a_i b_i : 1 \leq i \leq n\}$ be a pd matching of G. Define

$$M_\square = \{(a_i, x)(b_i, x) : 1 \leq i \leq n, x \in V(H)\}.$$

It is straightforward to verify that $M_\square$ is a perfect matching of $G \square H$. Now fix $i \in \{1, \ldots n\}$, and $(u, v) \in V(G \square H)$. Then working (mod 2) we have that

$$\begin{aligned} d_{G\square H}((a_i, x), (u, v)) &= d_G(a_i, u) + d_H(x, v) \\ &\equiv d_G(b_i, u) + 1 + d_H(x, v) \\ &= d_{G\square H}((b_i, x), (u, v)) + 1, \end{aligned}$$

where the first and second equality follows by properties of distance in $G \square H$, and the congruence follows since M is a pd matching. As i and (u, v) were arbitrary, we have that $M_\square$ is a pd matching of $G \square H$.

(2) Let $a \in V(G') \setminus V(G)$ be an endvertex of G' joined to b. If z is vertex of G', then $d(a, z) = d(b, z) + 1$, so ab is a pd edge. Hence, $M = \{a_i b_i : 1 \leq i \leq n, b_i \in V(G),\ a_i \in V(G') \setminus V(G)$ is an endvertex joined to $b_i\}$ is a pd matching of G'. The proof of (3) is similar to the one given for (2), and so is omitted.

For (4), let $M = \{a_i b_i : 1 \leq i \leq m\}$ and $M' = \{a_i' b_i' : 1 \leq i \leq n\}$. Without loss of generality, let $a = a_m = {a_1}'$ and $b = b_m = {b_1}'$. To see that $M \cup M'$ is a pd matching of $G \cup H$, we show that $a_1 b_1$ is a pd edge in $G \cup H$ (the other cases are similar). Fix $z \in V(G) \cup V(H)$. If z is in $V(G)$, then a shortest path from z to a_1 or b_1 must have all of its vertices in G. Since $a_1 b_1$ is a pd edge in G, the distances in $G \cup H$ from z to a_1 and to b_1 are of opposite parities.

Now let $z \in V(H) \setminus V(G)$. Any shortest path connecting z to a_1 or b_1 must go through one of a or b.

Case 1: The shortest paths P from z to a_1 and Q from z to b_1 both traverse through a. (The case when P and Q traverse through b is similar and so is omitted.)

Hence, if x is a_1 or b_1 then

$$d_{G\cup H}(x, z) = d_G(x, a) + d_H(a, z). \tag{3.1}$$

Let P' be the subpath of P in G from a_1 to a and Q' the subpath of Q in H from b_1 to a. See Figure 5.

The *parity* of a path is even (odd) if its number of edges is even (odd). Then P' and Q' have opposite parities since ab is pd in G. It follows by (3.1) that P and Q have opposite parities in $G \cup H$.

Case 2: The path P traverses through a and Q through b. (The case when P goes through b and Q through a is analogous and so is omitted.)

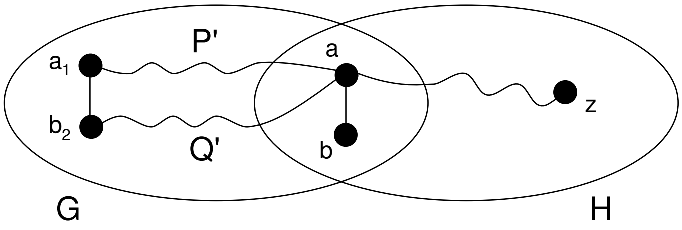

FIGURE 5. Case 1.

Then

$$d_{G\cup H}(a_1,z) = d_G(a_1,a)+d_H(a,z), \tag{3.2}$$

$$d_{G\cup H}(b_1,z) = d_G(b_1,b)+d_H(b,z). \tag{3.3}$$

Let P' be the subpath of P in G from a_1 to a and Q' the subpath of Q in G from b_1 to b. Let P'' be the subpath of P in H from a to z, and let Q'' be the subpath of Q in H from b to z. See Figure 6.

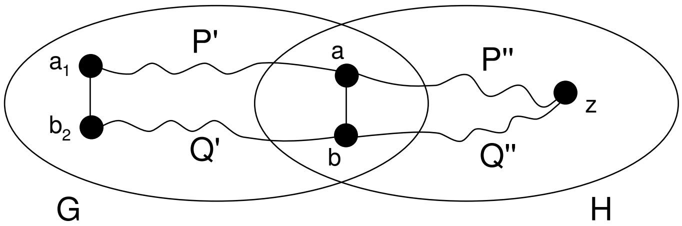

FIGURE 6. Case 2.

Then P'' and Q'' have opposite parities, since ab is pd in H. As ab and a_1b_1 are pd in G, we have that P' and Q' have the same parities. Hence, by (3.2) and (3.3) P and Q have opposite parities in $G\cup H$. □

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