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Graphs with the n-e.c. adjacency property constructed from affine planes

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Abstract

We give new examples of graphs with the n-e.c. adjacency property. Few explicit families of n-e.c. graphs are known, despite the fact that almost all finite graphs are n-e.c. Our examples are collinearity graphs of certain partial planes derived from affine planes of even order. We use probabilistic and geometric techniques to construct new examples of n-e.c. graphs from partial planes for all n, and we use geometric techniques to give infinitely many new explicit examples if $n = 3$. We give a new construction, using switching, of an exponential number of non-isomorphic n-e.c. graphs for certain orders.

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1. Introduction

Adjacency properties of graphs were first studied by Erdős and Rényi in their classic work on random graphs. One such adjacency property is the n-existentially closed property: for a positive integer n, a graph is *n-existentially closed* or *n-e.c.*, if we can extend all n-subsets of vertices in all possible ways; more precisely, if for each n-subset S of vertices, and each subset T of S, there is a vertex not in S joined to each of the vertices of T and to no vertex in $S\backslash T$. The n-e.c. adjacency property and its variants have since been studied by many authors; see, for example, [1–9]. From the results of Erdős and Rényi [10], for an integer m and fixed $p \in (0, 1)$ a random graph $G \in G(m, p)$ with m vertices asymptotically almost surely has the n-e.c. property. Despite this result, relatively few explicit examples of n-e.c. graphs are known.

One such family of n-e.c. graphs are Paley graphs. The *Paley graph* of order q, for a prime power $q \equiv 1 \pmod 4$, is the graph whose vertices are the elements of the finite field $\mathrm{GF}(q)$ in which two distinct vertices x and y are joined if and only if $x - y$ is a square in $\mathrm{GF}(q)$. From the work of [4,5,11], it follows from a non-trivial theorem on character

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sum estimates that Paley graphs of order $q > n^2 2^{2n-2}$ are n-e.c. Until recently, this was the only such family of graphs (other than random graphs) known to contain members which are n-e.c., for arbitrary n.

A k-regular graph G with v vertices, so that each pair of joined vertices has exactly λ common neighbours, and each pair of non-joined vertices has exactly μ common neighbours is called a *strongly regular graph*; we say that G is an $\mathrm{SRG}(v, k, \lambda, \mu)$. Cameron and Stark in [9] give a new family of strongly regular n-e.c. graphs, that are not isomorphic to Paley graphs. They prove the following theorem in [9] via probabilistic methods and by exploiting certain graphs derived from affine designs.

Theorem 1. *Suppose that q is a prime power such that $q \equiv 3 \pmod 4$. There is a function $\varepsilon(q) = \mathrm{O}(q^{-1}\log q)$ such that there exist $2^{\binom{q+1}{2}(1-\varepsilon(q))}$ non-isomorphic $\mathrm{SRG}((q+1)^2, q(q+1)/2, (q^2-1)/4, (q^2-1)/4)$ which are n-e.c. whenever $q \geqslant 16n^2 2^{2n}$.*

In the present article, strongly regular n-e.c. graphs are constructed from certain finite geometries; in particular, finite affine planes of even order. Our approach fundamentally differs from the Paley graph construction of [4,5,11], and from the construction of [9] in two important ways. One difference is our use of geometric methods. The second and perhaps more important difference is that our proofs are *elementary*, in the sense that they do not use any specialized machinery beyond basic properties of affine planes, counting, and probability theory. This is in contrast with the use of the Hasse–Weil character sum estimates to prove that Paley graphs are n-e.c., or the relatively involved probabilistic techniques including Poisson approximation theory used in the proof of Theorem 1.

Our graphs are inspired by the graphs constructed in [3] which were of odd order. For a general n, we use elementary geometric and probabilistic techniques to construct n-e.c. graphs that are non-Paley and non-isomorphic to the graphs constructed in [9]; see Theorem 2. In the case when $n = 3$, the proof of Theorem 4 uses deterministic arguments to find new examples of 3-e.c. graphs. The proof of this theorem uses the coordinatization properties of Desarguesian affine planes; see Section 4. We note that the infinite family of 3-e.c. graphs provided by Theorem 4 includes graphs of much smaller order than those supplied by Theorem 1. (For example, Theorem 4 produces a 3-e.c. graph of order 64, while the order of any 3-e.c. graph produced by Theorem 1 is at least 84,953,089.) In Section 3, we give a new construction using switching, which preserves the n-e.c. property. For certain orders (which will be made more explicit in Theorem 6) the new operation provides an exponential number of non-isomorphic n-e.c. graphs.

2. The Graphs $G(q, \mathscr{U}, A)$ and their adjacency properties

All graphs considered are finite, simple, and undirected. For a graph G, the vertex set of G is written $V(G)$, and the edge set is written $E(G)$. Edges are written xy, and we say that x and y are *joined*. Given a fixed vertex x, the *neighbour set* of x is the set of vertices joined to x, written $N(x)$. A *non-neighbour* of x is a vertex not joined to and not equal to x, and the *co-neighbour set* of x is the set of all non-neighbours of x, written $N^{\mathrm{c}}(x)$. The vertices that are not in a set S of vertices will be written $\overline{S}$ (this should not be confused with the complement of G, which is also written $\overline{G}$). The complete graph, or clique, of order n is written K_n.

Throughout, $q \geqslant 8$ will be a power of 2 (unless otherwise stated) and our affine plane A will be of order q. That is, A is a 2-$(q^2, q, 1)$ design (with "blocks" called "lines"), and hence, satisfies the property that given a point x and a line ℓ, there is a unique line $L(x, \ell)$ parallel to ℓ that goes through x. As is well known, such a plane has q^2 points, $q^2 + q$ lines, and each line contains exactly q points. The relation of parallelism on the set of lines is an equivalence relation, and the equivalence classes are called *parallel classes*. If a point x is on the line ℓ, then we write $x I \ell$. Each pair of non-parallel distinct lines ℓ and m intersect in a unique point, which we will write $\ell \wedge m$. Each pair of distinct points x, y is joined by a unique line that we write as xy. (Although this notation conflicts with our earlier notation for edges of a graph, we keep both notations since they are standard.) If two lines ℓ and m are parallel, then we write $\ell \| m$. Each parallel class contains q lines, and there are $q + 1$ parallel classes.

A *partial plane* results from an affine plane A if we delete some set of lines of A. If P is a partial plane resulting from A, then the *collinearity* (*or point*) *graph* of P is the graph with vertices equal to the points of A, with two points joined if they are joined by a line of P.

Fix A, an affine plane of even order $q \geqslant 8$. If A is Desarguesian, then $q = 2^k$ for some fixed $k \geqslant 3$, and A is coordinatized by $\mathrm{GF}(2^k)$. Consider the partial plane that results from deleting the lines of some fixed set of $\frac{q+2}{2}$ of the parallel classes of A. Then there are $\frac{q}{2}$ parallel classes in the partial plane. We denote the set of lines in the partial plane by $\mathscr{U}$ and the set of deleted lines by $\mathscr{U}'$. Define a graph $G=G(q,\mathscr{U},A)$ to be the collinearity graph of this partial plane. It follows that G is a Latin square graph, and that G is a $\mathrm{SRG}(q^2, \frac{q(q-1)}{2}, \frac{q(q-2)}{4}, \frac{q(q-2)}{4})$. Note that the parameters for the graphs $G(q,\mathscr{U},A)$ appear to be the same as those in Theorem 1. However, some quick checking demonstrates that they are the same if and only if the q in $G(q,\mathscr{U},A)$ has the property that $q=2^p$, where p is prime and 2^p-1 is prime (and hence, 2^p-1 is a *Mersenne prime*). Therefore, for infinitely many values of q, the parameters of our family of graphs are distinct from the parameters of the strongly regular graphs of Cameron and Stark. The graphs $G(q,\mathscr{U},A)$ are defined in an analogous way to the graphs defined in [3], where q was odd, and $\mathscr{U}$ was a set of lines from some $\frac{q+1}{2}$ parallel classes.

We may view the set of all graphs $G(q,\mathscr{U},A)$ as a finite equiprobable probability space $\mathscr{G}(q,\mathscr{U},A)$ of cardinality $\binom{q+1}{\frac{q}{2}}$: each point of the probability space corresponds to a choice of $\mathscr{U}$. With this perspective, we prove the following result. We use logarithms in base 2 (denoted by log) and the notation $\mathbb{R}^+$ for the set of positive real numbers.

Theorem 2. *Let q be a power of 2. For every fixed positive constant $c<\frac{1}{2}$, if $n=\lfloor c\log q\rfloor$, then asymptotically almost surely as $q\to\infty$ a graph $G\in\mathscr{G}(q,\mathscr{U},A)$ is n-e.c.*

As an application of Theorem 2, we obtain a new infinite family of n-e.c. graphs. Before we prove Theorem 2, we introduce some notation that will simplify the discussion. An *n-sequence* σ is a length n binary sequence. An *n-e.c. problem* is a pair (B,σ), where B is an ordered n-subset of vertices and σ is an n-sequence; if $B=(x_1,\dots,x_n)$ and $\sigma=(i_1\dots i_n)$, then a *solution* to (B,σ) is a vertex z not in B so that z is joined to x_j if and only if $i_j=1$. If (B,σ) is an n-e.c. problem, then the *(B,σ)-solution set* is the set of all solutions to the n-e.c. problem (B,σ). If B and σ are clear from context, then we will just say *solution set*. For simplicity, if B is clear from context, we will identify σ with the (B,σ)-solution set. For example, if B consists of three vertices x, y, and z, then (111) consists of $N(x)\cap N(y)\cap N(z)$.

Proof of Theorem 2. Let ℓ_∞ be the line at infinity of A. We identify ℓ_∞ as a set of slopes. For a point p, the *projection from p*, written π_p, is the map from the points of $A\backslash\{p\}$ to the points of ℓ_∞ defined by $\pi_p(x)=px\wedge\ell_\infty$. If X is a set of points, then $\pi_p(X)=\bigcup_{x\in X}\pi_p(x)$.

Choose $\mathscr{U}$ uniformly at random, and fix a set of n vertices $X=\{x_1,\dots,x_n\}$. We choose a fixed constant $d\in\mathbb{R}^+$ so that $0<c<d<\frac{1}{2}$ and let $s=\lceil q^d\rceil$. We inductively construct points p_i in A, where $1\leqslant i\leqslant s$, so that if π_{p_i} is the projection from p_i, then $|\pi_{p_i}(X)|=n$ for all i, and $\pi_{p_i}(X)\cap\pi_{p_j}(X)=\emptyset$, whenever $1\leqslant i<j\leqslant s$. In particular, for each n element set X of points of A, we construct an s element set P_X of points of A with the property that the ns lines xp where $x\in X$ and $p\in P_X$ are (all distinct and) in distinct parallel classes of A. We choose p_1 to be a point that is not on a line joining two points of X. For a fixed $i\leqslant s$, assuming that $p_1,\dots,p_{i-1}$ are chosen, we would like to choose p_i to be a point that is not on a line joining two points of X, and that is not on a line joining a point of X to a point in $\bigcup_{j=1}^{i-1}\pi_{p_j}(X)$. Hence, we may choose a suitable point p_i whenever

$$n+\binom{n}{2}(q-2)+n(i-1)(q-1)+n(n-1)(i-1)(q-2)<q^2. \tag{2.1}$$

As $i\leqslant s$, the condition (2.1) is satisfied with our choice of s for all sufficiently large q, as the reader may verify.

Let $E=\bigcup_{i=1}^{s}\pi_{p_i}(X)$. Then $E\subseteq\ell_\infty$ and $|E|=ns$. Fix a particular n-e.c. problem (X,σ). We estimate the probability from above that none of the vertices of P_X solves (X,σ). The number of n-e.c. problems (X,α) with $\alpha\neq\sigma$ is 2^n-1. Hence, the total possible number of ways that none of the vertices of P_X solves (X,σ) is $(2^n-1)^s$. We say that one of these ways is a *bad adjacency pattern for P_X*. A specific bad adjacency pattern $\mathscr{B}$ may or may not occur with a particular $\mathscr{U}$. Let r denote the number of 1's which occur in all the α's used in $\mathscr{B}$. Then for $\mathscr{B}$ to occur in $\mathscr{U}$, the slopes associated with $\mathscr{U}$ must include the corresponding r slopes in E (and no others in E) and any $q/2-r$ slopes (from the $q+1-ns$ slopes) outside E. Then the probability that $\mathscr{B}$ occurs with a particular $\mathscr{U}$ is

$$\frac{\binom{q+1-ns}{\frac{q}{2}-r}}{\binom{q+1}{\frac{q}{2}}}\leqslant\left(\frac{\frac{q+2}{2}}{q+2-ns}\right)^{ns}=\frac{1}{2^{ns}}\left(\frac{q+2}{q+2-ns}\right)^{ns}, \tag{2.2}$$

where the inequality follows by writing the binomial coefficients as quotients of products, and then estimating the simplified quotient from above by their largest and smallest factors. The number of distinct n-e.c. problems (X, σ) is at most

$$2^n(q^2)^n < q^{3n}, \tag{2.3}$$

since $2 < q$. Therefore, by (2.2) and (2.3), the probability that there is an X and σ so that there is no solution in the graph G to the n-e.c. problem (X, σ) among the vertices of P_X is at most $B(q)C(q)$, where

$$B(q) = q^{3n}\,\frac{(2^n-1)^s}{2^{ns}} = q^{3n}\left(1-\frac{1}{2^n}\right)^s$$

and

$$C(q) = \left(\frac{q+2}{q+2-ns}\right)^{ns}.$$

The proof will follow if we demonstrate that

$$\lim_{q\to\infty} B(q)C(q) = 0. \tag{2.4}$$

To verify (2.4), we analyse the asymptotic behaviour of $B(q)$ and $C(q)$ separately. Regarding $B(q)$, we note that

$$\begin{aligned} B(q) &= \exp\left(3n\ln q + s\ln\left(1-\frac{1}{2^n}\right)\right)\\ &\leqslant \exp\left(3n\ln q - \frac{s}{2^n}\right)\\ &\leqslant \exp\left(\frac{3c(\ln q)^2}{\ln 2} - q^{d-c}\right), \end{aligned}$$

where the first inequality follows from properties of the ln function, and the second inequality follows by our definitions of n and s (recall that $n = \lfloor c\log q\rfloor$ and $s = \lceil q^d\rceil$), since $-2^{-n} \leqslant -q^{-c}$, and since $-s \leqslant -q^d$. However, as the constant $d-c>0$, it follows that

$$\lim_{q\to\infty} B(q) = 0. \tag{2.5}$$

For $C(q)$, we first define the function $h : \mathbb{R}^+ \to \mathbb{R}^+$ by $h(x) = \left(\frac{x+1}{x}\right)^{x+1}$. Then h is a strictly decreasing function, and $h(x) > 1$, for all $x \in \mathbb{R}^+$. Observe that with our definitions of n and s,

$$ns \leqslant (q+2)^{1/2} \tag{2.6}$$

for all sufficiently large q. To see this, note that the inequality of (2.6) is equivalent to

$$\frac{\lfloor c\log q\rfloor\lceil q^d\rceil}{(q+2)^{1/2}} \leqslant 1,$$

which holds for all sufficiently large q since the constant $d < \frac{1}{2}$. Hence, for all sufficiently large q, we have that

$$\begin{aligned} C(q) &= \left(h\left(\frac{q+2-ns}{ns}\right)\right)^{(ns)^2/(q+2)}\\ &\leqslant h\left(\frac{q+2-ns}{ns}\right)\\ &\leqslant h(1) = 4, \end{aligned}$$

where the first inequality follows by (2.6), and the second inequality follows by (2.6) and since $h(x)$ is a strictly decreasing function. Hence, for all sufficiently large q,

$$C(q) \leqslant 4.$$

Thus, by (2.5) and this bound for $C(q)$ we have that

$$\lim_{q\to\infty} B(q)C(q) \leqslant 4 \lim_{q\to\infty} B(q) = 0,$$

and so (2.4) follows. □

With some minor changes (specifically to (2.2)), the reader may verify that Theorem 2 also holds for the graphs $G(q, \mathscr{U}, A)$ defined first in [3], where q is an odd prime power, and $\mathscr{U}$ is a set of lines from some set of $\frac{q+1}{2}$ parallel classes. It may be possible to extend our construction by different choices of $\mathscr{U}$, say when the number of parallel classes of each type is not too large or small. As our main goal in the present article is only to present some new families of n-e.c. graphs, we do not pursue this idea here.

While Theorem 2 gives many new examples of n-e.c. graphs, it does not show that *all* choices of $\mathscr{U}$ will give an n-e.c. graph. In [3] it was conjectured that for all n, if q is large enough, then all the graphs in $\mathscr{G}(q, \mathscr{U}, A)$ are n-e.c. We disprove this conjecture in Theorem 3.

Theorem 3. *Let A be a Desarguesian plane of order q. If q is even and $q \geqslant 4$, then for all $n \geqslant 4$ there is a $\mathscr{U}$ such that $G(q, \mathscr{U}, A)$ is not n-e.c.*

Theorem 3 demonstrates that Theorem 4 of this paper is best possible in the sense that 3-e.c. cannot be replaced by n-e.c. for any $n > 3$.

Proof of Theorem 3. We coordinatize A in any fixed way. It is sufficient to prove the theorem for $n = 4$. Let $\mathscr{U}$ consist of the lines with slope in $S = \{w^2 + w : w \in \mathrm{GF}(q)\}$. Then $|S| = q/2$ and $(S, +)$ is a subgroup of the additive group $(\mathrm{GF}(q), +)$. Since q is even, $2s = 0$ for all $s \in \mathrm{GF}(q)$.

Let

$$B = \{(0, 0), (0, 1), (0, a), (0, a + 1)\},$$

where a (and therefore, $a + 1$) $\neq 0, 1$. We use the notation defined just before the proof of Theorem 2. Let $\sigma = (1110)$. There cannot be a solution to (B, σ) of the form $(0, v)$, since the line $(0, v)(0, 0)$ has slope $\infty \notin S$. For a contradiction, suppose that (u, v) with $u \neq 0$ is a solution. Then the slopes of lines between (u, v) and the points of B are $m_1 = vu^{-1}$, $m_2 = (v - 1)u^{-1}$, $m_3 = (v - a)u^{-1}$, and $m_4 = (v - a - 1)u^{-1}$; all of these are in $\mathrm{GF}(q)$. As (u, v) is a solution, m_1, m_2, $m_3 \in S$ and $m_4 \notin S$. But by closure in the additive group S we have that $m_4 = m_1 + m_2 + m_3 \in S$, which is a contradiction. □

It may be proved that if q is odd, then for all $n \geqslant 5$ there is a $\mathscr{U}$ such that $G(q, \mathscr{U}, A)$ is not n-e.c. As the proof is similar to the proof for q even (but letting $S = \{w^2 : w \in \mathrm{GF}(q)\}$), we omit the details.

For all $q \geqslant 8$, the graphs $G(q, \mathscr{U}, A)$, are, however, *always* 3-e.c.

Theorem 4. *Let A be a Desarguesian affine plane of order $q \geqslant 8$, and fix $G \in \mathscr{G}(q, \mathscr{U}, A)$. Then G is 3-e.c. If $q \geqslant 32$ and any set of k vertices are deleted from G, where k is an integer satisfying $0 < k \leqslant \frac{q-12}{8}$, then the resulting graph is 3-e.c.*

Our proof of this result uses the coordinatization properties of Desarguesian affine planes. Owing to its length, we defer the proof of Theorem 4 to Section 4.

In the $q = 8$ case of Theorem 4 all $\binom{9}{4} = 126$ graphs in $\mathscr{G}(8, \mathscr{U}, A)$ are isomorphic (under a graph isomorphism induced by a geometric isomorphism). To see this, note first that an affine plane of order 8 is unique up to geometric isomorphism. Second, it is well known that the full group of automorphisms of A is transitive on ordered triples of points of ℓ_∞. Let $\mathscr{U}_{\mathrm{sl}}$ be the set of slopes of the lines of $\mathscr{U}$. Therefore, we may assume that $\{0, 1, \infty\} \subseteq \mathscr{U}_{\mathrm{sl}}$. The group of automorphisms of the subplane of order 2 fixes the unordered set $\{0, 1, \infty\}$ and (when extended naturally to A) is transitive on the 6-element set $\ell_\infty \backslash \{0, 1, \infty\}$. It follows that all 4-element subsets of ℓ_∞ are geometrically equivalent.

3. Switching and the n-e.c. property

Our goal in this section is to construct new non-isomorphic n-e.c. graphs from existing ones. The first tool we need is switching in graphs. If G is a graph and $A \subseteq V(G)$, then the graph G_A is formed by interchanging edges and non-edges between A and $V(G)\backslash A$, and leaving all other edges and non-edges unaltered. We say that G_A is the graph formed from G *by switching on* A. If H is an induced subgraph of G, then we will abuse notation and write G_H for $G_{V(H)}$.

We next introduce a strengthening of the n-e.c. property. Throughout this section, we use the notation defined just before the proof of Theorem 2. For a positive integer n, we say that G is *n-good* if:

(1) There are positive integers r and s such that G is regular of degree r, and $\overline{G}$ is regular of degree s. In addition, either $r = s$, or $2n < s - r + 1$. (Hence, $r \leqslant s$.)
(2) For all n-e.c. problems (B, σ), the solution set determined by B and σ has cardinality at least $n + 1$.

Note that an n-good graph is n-e.c. A Paley graph with sufficiently many vertices will be n-good (by the results of [4,5]), and the graphs in $G(q, \mathscr{U}, A)$, with A Desarguesian and q sufficiently large, are 3-good. (Item (1) follows from using $r = \frac{q(q-1)}{2}$ and $s = \frac{(q+2)(q-1)}{2}$; item (2) follows since, by the proof of Theorem 4, all solution sets where $|B| = 3$ have cardinality at least $\frac{q-4}{8}$.) If n is a fixed positive integer, then it is not hard to show that there is a sufficiently large positive integer $n' \geqslant n$, so that an n'-e.c. graph satisfies item (2) in the definition of n-good. (For example, $n' = n + \lceil \log_2(n+1) \rceil$ works.) From this fact and Theorem 2, for all positive integers n, we may choose a sufficiently large q so that there are graphs in $\mathscr{G}(q, \mathscr{U}, A)$ that are n-good.

The next result demonstrates how switching in n-good graphs leads to new n-e.c. graphs.

Theorem 5. *Let $n \geqslant 2$ be an integer, and let G be an n-good graph. Then for all n-vertex subgraphs $H \leqslant G$, we have that G_H is n-e.c. There exists an n-vertex clique $H \leqslant G$ and for this H we have that G_H is n-e.c. and $G_H \not\cong G$.*

Proof. We show that G_H is n-e.c. Fix an n-subset A of $V(G)$. Let $0' = 1$ and $1' = 0$. Consider the n-e.c. problem (A, σ). Write $A = B \cup C$, where $B = A \cap V(H)$ and $C = A \backslash B$ (note that B may be empty). Let $\sigma_B = (i_1 \ldots i_k)$, where each $i_j \in \{0, 1\}$, be the subsequence of σ that corresponds to the elements of B, and let σ_C be the subsequence of σ that corresponds to the elements of C. Define $\sigma'_B = (i'_1 \ldots i'_k)$. Consider the n-e.c. problem $(B \cup C, \sigma'_B \sigma_C)$ with a solution z in G chosen outside H (which is permissible since G is n-good.) Then z solves (A, σ) in G_H.

Since an n-e.c. graph is $(n + 1)$-universal (that is, each graph of order at most $n + 1$ is isomorphic to an induced subgraph; this fact may be proved by induction on n) we may fix $H \leqslant G$ an n-vertex clique. Since $\overline{G}$ is s-regular, a vertex of H has degree $s + n - 1$ in G_H. Since G is n-e.c. and $|V(H)| = n$, there is a vertex y of G joined to exactly one vertex of H. Then y has degree $r + n - 2$ in G_H. But as $r \leqslant s$, we have that $r + n - 2 < s + n - 1$. Hence, $G_H \not\cong G$ as G_H is not regular. □

We note that it is remarked in the beginning of Section 5 of [9] that for certain parameters, switching with respect to the neighbours of a single vertex then deleting that vertex creates an $(n - 1)$-e.c. strongly regular graph from an n-e.c. strongly regular one. We point out that Theorem 5 does not produce strongly regular (or even regular) n-e.c. graphs.

The degree of a vertex x in G is written $\deg_G(x)$. If G is a graph with n vertices and degrees $d_1 \leqslant \cdots \leqslant d_n$, then the n-tuple $(d_1, \ldots, d_n)$ is called the *degree sequence* of G. If $\alpha = (d_1, \ldots, d_n)$, let $\{\alpha\}$ be the unordered multiset $\{d_1, \ldots, d_n\}$. Two length n degree sequences α and β are *distinct* if $\{\alpha\} \neq \{\beta\}$ as multisets. Note that two distinct length n degree sequences must correspond to non-isomorphic graphs (but the converse may fail). Let $ds(n)$ be the number of distinct degree sequences of order n. We now apply Theorem 5 to give many non-isomorphic examples of n-e.c. graphs.

Theorem 6. *Let $n \geqslant 2$ be any integer and let G be an n-good graph. Then there are at least $ds(n)$-many non-isomorphic n-e.c. graphs of order $|V(G)|$.*

Proof. Fix H any n-vertex graph (which is not necessarily a clique). Since an n-e.c. graph is $(n+1)$-universal, there is an isomorphic copy of H that is an induced subgraph of G. Let $J = G_H$; by Theorem 5, J is n-e.c. and has order $|V(G)|$.

Fix a vertex x in H, and suppose that $\deg_H(x) = k_x \geqslant 0$. Then, by s-regularity of $\overline{G}$, x is joined in J to $s-n+k_x+1$ vertices outside of H in G. Therefore,

$$\deg_J(x) = s - n + 2k_x + 1. \tag{3.1}$$

Now consider all the 2^n distinct solution sets $(i_1, \ldots, i_n)$, where $i \in \{0, 1\}$, in G determined by the n vertices of H (each of which is non-empty since G is n-e.c.). These solution sets partition $V(G)\backslash V(H)$ into 2^n sets. The degree of a vertex in $(1 \cdots 1)$ in J is $r-n = r-n+2n-2n$; the degree of a vertex in $(01 \cdots 1)$ in J is $r-n+2 = r-n+2n-(2(n-1))$; the degree of a vertex in $(0 \ldots 0)$ is $r+n = r-n+2n-0$. In general, the degrees of vertices y in $V(G)\backslash V(H)$ in J are always one of the integers

$$r - n + 2n - 2j,$$

where $0 \leqslant j \leqslant n$.

Let $r - n = m$, $x \in V(H)$, and $y \in V(G)\backslash V(H)$. Consider in item (1) of the definition of n-good the case that $r = s$. Then $s - n = m$, and by (3.1), $\deg_J(x)$ is always m plus an odd number, while $\deg_J(y)$ is always m plus an even number. Now consider the case when $s - r + 1 > 2n$. In this case, it is not hard to see that for all choices of k_x and j, that $s - n + 2k_x + 1 > r - n + 2n - 2j$. In both cases, for all $x \in V(H)$, $y \in V(G)\backslash V(H)$, we have that

$$\deg_J(x) \neq \deg_J(y).$$

Suppose that H has degree sequence $\alpha = (d_1, \ldots, d_n)$, with $d_1 \leqslant \cdots \leqslant d_n$. By the above discussion, G_H has degree sequence $(\widehat{\alpha}, \sigma)$, where $\widehat{\alpha} = (s-n+2d_1+1, \ldots, s-n+2d_n+1)$ is a subsequence consisting of degrees from vertices $V(H)$ in G_H, and σ is a subsequence containing the degrees $r-n, r-n+2, \ldots, r+n$ from the solution sets

$$(1 \cdots 1), (01 \cdots 1), \ldots, (0 \cdots 0),$$

respectively. Note that the elements of σ depend only on n and r, and not on the degrees in H. Furthermore, for any graph H, by previous discussion, none of the terms of $\widehat{\alpha}$ can equal a term in σ. Suppose that H and H' have distinct degree sequences α and β, respectively. Therefore, by the above discussion, G_H and $G_{H'}$ have degree sequences $(\widehat{\alpha}, \sigma)$ and $(\widehat{\beta}, \sigma)$, respectively. If $\{\widehat{\alpha}\sigma\} = \{\widehat{\beta}\sigma\}$, then $\{\widehat{\alpha}\} = \{\widehat{\beta}\}$. But then $\{\alpha\} = \{\beta\}$, which is contradiction. Hence, $G_H \ncong G_{H'}$ and the result follows. $\square$

A straightforward inductive argument establishes that $2^{n-1} \leqslant ds(n)$. Hence, we obtain the following corollary, which gives an exponential number of non-isomorphic n-e.c. graphs.

Corollary 1. *If there is an n-good graph of order r, then there are at least 2^{n-1} non-isomorphic n-e.c. graphs of order r.*

As we discussed after the definition of the n-good property, for all positive integers n, there are sufficiently large q so that there exist n-good graphs in $\mathcal{G}(q, \mathcal{U}, A)$. Hence, by Corollary 1, for these q there are at least 2^{n-1} non-isomorphic n-e.c. graphs of order q^2.

Corollary 1 does not exhibit strongly regular n-e.c. graphs like the results of [9]. However, we think the n-e.c. preserving operation we present via switching is of interest in its own right. In particular, it is the first such explicit construction that applies to a broad family of n-e.c. graphs.

4. Proof of Theorem 4

Consider a fixed $G \in \mathcal{G}(q, \mathcal{U}, A)$. We use the notation defined just before the proof of Theorem 2. For each triple x, y, z of distinct vertices in $V(G)$ (which are points of A), it is sufficient show that each of the eight solution sets $(i_1 i_2 i_3)$, where $i_j \in \{0, 1\}$, contains at least $\frac{q-4}{8}$ vertices. (As the cardinality of a solution set is an integer, in the case where $q=8$, this proves that there is at least one vertex in each solution set.) If $q \geqslant 32$, we may then delete up to $\frac{q-4}{8}-1=\frac{q-12}{8}$ vertices and the solution sets $(i_1 i_2 i_3)$ will be non-empty in the resulting graph.

Fix three distinct vertices, $x, y, z \in V(G)$. For solutions sets with $B=\{x, y, z\}$, we will always list x, y, and z in that order. We must consider the following six cases:

(1) The vertices x, y, z lie on a line ℓ of A with:
  (1.1) $\ell \in \mathscr{U}$;
  (1.2) $\ell \in \mathscr{U}'$.
(2) The vertices x, y, z form a triangle in A with:
  (2.1) all three sides in $\mathscr{U}$;
  (2.2) two sides in $\mathscr{U}$ and the third in $\mathscr{U}'$;
  (2.3) one side in $\mathscr{U}$ and the other two in $\mathscr{U}'$;
  (2.4) all three sides in $\mathscr{U}'$.

This gives a total of 48 cases to check. By symmetry, we can considerably reduce the number of cases to 28. For example, in Cases 1.1 and 1.2, we need only consider the solution sets (111), (000), and one from each of the solution sets $\{(100), (010), (001)\}$, $\{(110), (011), (011)\}$.

Case 1: The vertices x, y, z lie on a line ℓ.

Case 1.1: Suppose $\ell \in \mathscr{U}$.

The $q-3 \geqslant \frac{q-4}{8}$ vertices of ℓ, different from x, y, z, are in (111). Since $x, y, z \notin N^{\mathrm{c}}(x)$, we have that

$$
\begin{aligned}
|(000)| &= |N^{\mathrm{c}}(x)| - |N^{\mathrm{c}}(x) \cap \overline{N^{\mathrm{c}}(y)}| - |N^{\mathrm{c}}(x) \cap \overline{N^{\mathrm{c}}(z)}| + |N^{\mathrm{c}}(x) \cap \overline{N^{\mathrm{c}}(y)} \cap \overline{N^{\mathrm{c}}(z)}| \\
&= |N^{\mathrm{c}}(x)| - |N^{\mathrm{c}}(x) \cap N(y)| - |N^{\mathrm{c}}(x) \cap N(z)| + |(011)| \\
&= \frac{(q+2)(q-1)}{2} - 2\frac{(q+2)(q-2)}{4} + |(011)| \\
&= \frac{q+2}{2} + |(011)| > \frac{q-4}{8}.
\end{aligned}
$$

Now suppose for a contradiction that there exist $0 \leqslant k < \frac{q-4}{8}$ vertices in (011). From above, $|(000)| = \frac{q+2}{2} + |(011)| = \frac{q+2}{2} + k$. Similarly,

$$
|(000)| = \frac{q+2}{2} + |(101)| = \frac{q+2}{2} + |(110)|,
$$

so $|(101)| = |(110)| = k$. Let

$$
\mathscr{B} = (011) \cup (101) \cup (110) \cup (000).
$$

Then we have that $|\mathscr{B}| = 4k + \frac{q+2}{2} < 4(\frac{q-4}{8}) + \frac{q+2}{2} = q-1$, so there exists a line $\ell' \in \mathscr{U}$ satisfying $\ell' \| \ell$, $\ell' \neq \ell$, and so that ℓ' contains no vertices of $\mathscr{B}$. Therefore,

$$
\begin{aligned}
\frac{q-2}{2} &= |N(x) \cap \ell'| \\
&= |(111) \cap \ell'| + |(110) \cap \ell'| + |(101) \cap \ell'| + |(100) \cap \ell'| \\
&= |(111) \cap \ell'| + |(100) \cap \ell'|.
\end{aligned}
$$

Similarly, $\frac{q-2}{2} = |(111) \cap \ell'| + |(010) \cap \ell'|$ and $\frac{q-2}{2} = |(111) \cap \ell'| + |(001) \cap \ell'|$. Hence,

$$
\begin{aligned}
|(001) \cap \ell'| &= |(010) \cap \ell'| = |(100) \cap \ell'| \\
&= \frac{q-2}{2} - |(111) \cap \ell'|
\end{aligned}
\tag{4.1}
$$

and

$$
\begin{aligned}
|N(x) \cap N(y) \cap \ell'| &= |N(x) \cap N(z) \cap \ell'| \\
&= |N(y) \cap N(z) \cap \ell'| = |(111) \cap \ell'| + |(110) \cap \ell'| = |(111) \cap \ell'|,
\end{aligned}
\tag{4.2}
$$

since $|(110) \cap \ell'| = 0$.

Since $(000) \subseteq \mathscr{B}$, every vertex of ℓ' is in $N(x) \cup N(y) \cup N(z)$. Therefore, by the Principle of Inclusion–Exclusion, (4.1), and (4.2), we have that

$$\begin{aligned} q &= |(N(x) \cup N(y) \cup N(y)) \cap \ell'| \\ &= 3|N(x) \cap \ell'| - |N(x) \cap N(y) \cap \ell'| - |N(x) \cap N(z) \cap \ell'| - |N(y) \cap N(z) \cap \ell'| + |(111) \cap \ell'| \\ &= 3\left(\frac{q-2}{2}\right) - 2|N(x) \cap N(y) \cap \ell'|. \end{aligned}$$

Therefore, $2|N(x) \cap N(y) \cap \ell'| = \frac{3q-6-2q}{2} = \frac{q-6}{2}$, and so $|N(x) \cap N(y) \cap \ell'| = \frac{q-6}{4} \in \mathbb{Z}$. It follows that $q \equiv 2 \pmod 4$, which is a contradiction, so $|(011)| \geqslant \frac{q-4}{8}$.

To complete the remaining cases within Case 1.1, we consider the following argument that we will use repeatedly in the proof. There are $\frac{q-2}{2}$ lines $m \in \mathscr{U}$, $m \neq \ell$, xIm; we call these the *fixed lines*. The *conditions* on a fixed line m are the properties that $m \in \mathscr{U}$, $m \neq \ell$, xIm. Consider the set of $\frac{q+2}{2}$ lines of $\mathscr{U}'$ through y; these are called the *first lines*. The *conditions* on a first line are the properties that it is in $\mathscr{U}'$ and through y. The set of $\frac{q+2}{2}$ lines of $\mathscr{U}'$ through z are the *second lines*. The *conditions* on a second line are the properties that it is in $\mathscr{U}'$ and through z. The first and second lines meet each fixed line m in $\frac{q+2}{2}$ distinct vertices different from x. Since a given fixed line m contains only $q-1$ vertices different from x, by the Pigeonhole property there must be at least three vertices in $(100) \cap m$, and at least $3(\frac{q-2}{2}) \geqslant \frac{q-4}{8}$ vertices in (100).

To apply the argument in the previous paragraph to other cases, we need only specify the fixed, first, and second lines, by the conditions on these lines. We supply the following table which lists the cases where the argument applies. In each case, the total overlap on all fixed lines supplies some number (which we denote by "No." in the last column; each entry of the last column is greater or equal to $\frac{q-4}{8}$) of elements in a solution set. The number of lines and conditions for the various lines are listed in each of the third, fourth, and fifth columns. For example, in the third row (which was discussed in the last paragraph), the entry "$\frac{q-2}{2}, \mathscr{U}, x, \neq \ell$" in the third column, means that "choose the fixed lines in this case to be the $\frac{q-2}{2}$ lines of $\mathscr{U}$ incident with x and distinct from ℓ." We note that $\frac{q-6}{2} \geqslant 1$ so long as $q \geqslant 8$.

Case	3-Set	Fixed lines	1st lines	2nd lines	No.
1.1	(100)	$\frac{q-2}{2}, \mathscr{U}, x, \neq \ell$	$\frac{q+2}{2}, \mathscr{U}', y$	$\frac{q+2}{2}, \mathscr{U}', z$	$3\left(\frac{q-2}{2}\right)$
	(000)	$\frac{q+2}{2}, m \in \mathscr{U}', x$	$\frac{q}{2}, \mathscr{U}', y, \nparallel m$	$\frac{q}{2}, \mathscr{U}', z, \nparallel m$	$\frac{q+2}{2}$
1.2	(110)	$\frac{q}{2}, \mathscr{U}', z, \neq \ell$	$\frac{q}{2}, \mathscr{U}, x$	$\frac{q}{2}, \mathscr{U}, y$	$\frac{q}{2}$
	(100)	$\frac{q}{2}, \mathscr{U}, x$	$\frac{q}{2}, \mathscr{U}', y, \neq \ell$	$\frac{q}{2}, \mathscr{U}', z, \neq \ell$	$\frac{q}{2}$
2.1	(100)	$\frac{q-6}{2}, \mathscr{U}, x, \neq xy, \neq xz, L(x, yz)$	$\frac{q+2}{2}, \mathscr{U}', y$	$\frac{q+2}{2}, \mathscr{U}', z$	$4\left(\frac{q-6}{2}\right)$
	(000)	$\frac{q+2}{2}, m \in \mathscr{U}', x$	$\frac{q}{2}, \mathscr{U}', y, \nparallel m$	$\frac{q}{2}, \mathscr{U}', z, \nparallel m$	$q+2$
2.2	(100)	$\frac{q-4}{2}, \mathscr{U}, x, \neq xy, L(x, yz)$	$\frac{q+2}{2}, \mathscr{U}', y$	$\frac{q}{2}, \mathscr{U}', z, \neq xz$	$3\left(\frac{q-4}{2}\right)$
	(101)	$\frac{q-4}{2}, m \in \mathscr{U}, x, \neq xy, L(x, yz)$	$\frac{q+2}{2}, \mathscr{U}', y$	$\frac{q-4}{2}, \mathscr{U}, z, \neq yz, L(z, m)$	$\frac{q-4}{2}$
2.3	(010)	$\frac{q-2}{2}, \mathscr{U}, y, \neq L(y, xz)$	$\frac{q}{2}, \mathscr{U}', x, \neq xy$	$\frac{q}{2}, \mathscr{U}', z, \neq yz$	$q-2$
	(110)	$\frac{q-2}{2}, m \in \mathscr{U}, x, \neq xz$	$\frac{q-2}{2}, \mathscr{U}, y, \neq L(y, m)$	$\frac{q+2}{2}, \mathscr{U}', z$	$\frac{q-2}{2}$
2.4	(110)	$\frac{q-2}{2}, \mathscr{U}', z$	$\frac{q}{2}, \mathscr{U}, x$	$\frac{q}{2}, \mathscr{U}, y$	$\frac{q-2}{2}$

Case 1.2: Suppose that $\ell \in \mathscr{U}'$.

The $q-3 \geqslant \frac{q-4}{8}$ points of ℓ different from x, y, and z are in (000). Based on previous work in the table, it remains in this case to consider the 3-set (111). Suppose for a contradiction that $|(111)| = k$, where $0 \leqslant k \leqslant \frac{q-4}{4}$. Then

$$\begin{aligned}|(110)| &= |N(x) \cap N(y)| - k \\ &= \frac{q(q-2)}{4} - k,\end{aligned}$$

$$\begin{aligned}|(100)| &= |N(x) \cap N^{\mathrm{c}}(z)| - |(110)| \\ &= \frac{q^2}{4} - \frac{q(q-2)}{4} + k \\ &= \frac{q}{2} + k.\end{aligned}$$

By the table in Case 1.2 for the 3-set (100), each line of $\mathscr{U}$ through x must contain at least one vertex of (100). So there exist at least $\frac{q}{2} - k > \frac{q+4}{4}$ lines of $\mathscr{U}$ through x which contain exactly one vertex of (100). A straightforward check shows that such a line would also contain x itself plus $\frac{q}{2} - 1$ vertices of (110), $\frac{q}{2} - 1$ vertices of (101), and therefore, no vertices of (111). Similarly, there exist at least $\frac{q}{2} - k \geqslant \frac{q+4}{4}$ lines of $\mathscr{U}$ through y which contain exactly one vertex of (010) and no vertices of (111). Since all the lines of $\mathscr{U}$ are from $\frac{q}{2} < q - 2k$ parallel classes, there must exist parallel classes $\pi_{j_1}, \pi_{j_2} \subseteq \mathscr{U}$ so that $L(x, j_1)$ and $L(x, j_2)$ each contain exactly one vertex of (100), and $L(y, j_1)$ and $L(y, j_2)$ each contain exactly one vertex of (010).

Now we consider the triangle $x, y, L(x, j_1) \wedge L(y, j_2)$. Since the automorphism group of A is transitive on triangles, we can coordinatize A so that these vertices have the coordinates (0, 0), (0, 1), (1, 0), respectively, and each vertex of G is of the form (u, v), where $u, v \in \mathrm{GF}(q)$. We use the notation $[m, a]$, where $m, a \in \mathrm{GF}(q)$ to represent the line with y-intercept a and with slope m. A vertical line (with slope ∞) is written $[a]$, where a is the x-intercept.

Then $[0] = (0,0)(0,1) = xy = \ell \in \mathscr{U}'$, and $[0,0] = (0,0)(1,0) = x(L(x, j_1) \wedge L(y, j_2)) = L(x, j_1) \in \mathscr{U}$. Since the coordinates come from $\mathrm{GF}(2^k)$, we have that

$$[1,1] = [-1,1] = (0,1)(1,0) = y(L(x, j_1) \wedge L(y, j_2)) = L(y, j_2) \in \mathscr{U}$$

so all vertical lines (which we say have slope ∞) are in $\mathscr{U}'$, while all lines with slope 0 or 1 are in $\mathscr{U}$. Recall that $\mathscr{U}_{\mathrm{sl}}$ is the set of slopes of the lines of $\mathscr{U}$; we use a similar notation $\mathscr{U}'_{\mathrm{sl}}$ for the set of slopes of $\mathscr{U}'$. Since $z \neq x, y$ and $zI\ell = [0]$, z must have coordinates $(0, d)$ for some $d \neq 0, 1$.

Consider $L(x, j_1) = [0, 0] \in \mathscr{U}$ which contains no vertices of (111). For any $u \in \mathscr{U}_{\mathrm{sl}} \backslash \{0\}$,

$$(u^{-1}, 0)I[0,0], [u,1], [ud, d],$$

$$(u^{-1}d, 0)I[0,0], [ud^{-1}, 1], [u, d].$$

Thus, if $u \in \mathscr{U}_{\mathrm{sl}} \backslash \{0\}$, since $(111) \cap [0,0] = \emptyset$, we have that

$$\{ud : u \in \mathscr{U}_{\mathrm{sl}} \backslash \{0\}\}, \quad \{ud^{-1} : u \in \mathscr{U}_{\mathrm{sl}} \backslash \{0\}\} \subseteq \mathscr{U}'_{\mathrm{sl}} \backslash \{\infty\}. \tag{4.3}$$

If we let $u = 1 \in \mathscr{U}_{\mathrm{sl}}$, then (4.3) implies that $d, d^{-1} \in \mathscr{U}'_{\mathrm{sl}} \backslash \{\infty\}$.

The lines $L(x, j_2) = [1, 0]$, $L(y, j_1) = [0, 1]$, and $L(y, j_2) = [1, 1]$ in $\mathscr{U}$ also contain no vertices of (111), so the relations

$$\begin{aligned}&(1,1)I[1,0], [0,1], [1+d, d], \\ &(d,d)I[1,0], [d^{-1}+1, 1], [0, d], \\ &(d+1, 1)I[0,1], [1, d], [(d+1)^{-1}, 0], \\ &(d+1, d)I[1,1], [(1+d^{-1})^{-1}, 0], [0, d]\end{aligned}$$

imply that $1+d$, $d^{-1}+1$, $(d+1)^{-1}$ and $(1+d^{-1})^{-1}$ are in $\mathscr{U}'_{\mathrm{sl}}$.

In addition,

$$(d+1,d+1)I[1,0],[(1+d^{-1})^{-1},1],[(d+1)^{-1},d],$$
$$((d^{-1}+1)^{-1},(d^{-1}+1)^{-1})I[1,0],[d^{-1},1],[d,d].$$

Thus, $(d+1,d+1),((d^{-1}+1)^{-1},(d^{-1}+1)^{-1})\in(100)$. However, since $|[1,0]\wedge(100)|=1$, we have that $(d^{-1}+1)^{-1}=d+1$. A straightforward calculation gives that $d+1=d^{-1}$. Then $1=dd^{-1}=d(d+1)$, and $d^2=d+1=d^{-1}$.

Since $|\mathscr{U}_{\rm sl}\backslash\{0\}|=\frac{q-2}{2}=|\mathscr{U}'_{\rm sl}\backslash\{\infty\}|-1$, we have by (4.3) that

$$|\{ud^{-1}:u\in\mathscr{U}_{\rm sl}\backslash\{0\}\}\cap\{ud:u\in\mathscr{U}_{\rm sl}\backslash\{0\}\}|\geqslant\frac{q-4}{2}.$$

Hence, there exist distinct $u_1,u_2\in\mathscr{U}_{\rm sl}\backslash\{0\}$ such that $u_1d^{-1}=u_2d$. Then by (4.3) and since $d^2=d^{-1}$, we have that $u_1=u_2d^2=u_2d^{-1}\in\mathscr{U}'_{\rm sl}\backslash\{\infty\}$, which is a contradiction.

Case 2: The vertices x,y,z are not collinear and so form a triangle.

Case 2.1: Suppose that $xy,xz,yz\in\mathscr{U}$.

Choose any $m\in\mathscr{U}$, $m\neq xy,xz,L(x,yz)$ with xIm. This is possible since $q\geqslant 8$. Then $m\wedge yz\in(111)$, so there are $\frac{q-6}{2}\geqslant\frac{q-4}{8}$ vertices of (111) on yz. There are also $\frac{q-6}{2}$ such vertices on each of xy and xz.

Now fix any line $n\in\mathscr{U}'$, xIn. Then $n\wedge yz\in(011)$, so $|(011)|\geqslant\frac{q+2}{2}\geqslant\frac{q-4}{8}$. The remaining cases are contained in the table.

Case 2.2: Suppose that $xy,yz\in\mathscr{U}$ and $xz\in\mathscr{U}'$.

Consider any line $m\in\mathscr{U}$, $m\neq xy,L(x,yz)$, and xIm. Then $m\wedge yz\in(111)$ and $L(y,m)\wedge xz\in(010)$, so $|(111)|,|(010)|\geqslant\frac{q-4}{2}\geqslant\frac{q-4}{8}$.

Fix any line $n\in\mathscr{U}'$, $n\neq xz$ with xIn. Then $n\wedge yz\in(011)$ and $L(y,n)\wedge xz\in(000)$, so $|(011)|,|(000)|\geqslant\frac{q}{2}\geqslant\frac{q-4}{8}$. The remaining cases are contained in the table.

Case 2.3: Suppose that $xy,yz\in\mathscr{U}'$ and $xz\in\mathscr{U}$.

Consider any $m\in\mathscr{U}\backslash\{L(y,xz)\}$ through y. Then $m\wedge xz\in(111)$ and $L(x,m)\wedge yz\in(100)$, so $|(111)|,|(100)|\geqslant\frac{q-2}{2}\geqslant\frac{q-4}{8}$.

Now fix any $n\in\mathscr{U}'\backslash\{xy,yz\}$, incident with y. Then $n\wedge xz\in(101)$ and $L(x,n)\wedge yz\in(000)$, so $|(101)|,|(000)|\geqslant\frac{q-2}{2}$. The remaining cases are contained in the table.

Case 2.4: Suppose that $xy,xz,yz\in\mathscr{U}'$.

First, consider any line n of $\mathscr{U}'\backslash\{xy,xz,L(x,yz)\}$ through x. Then $n\wedge yz\in(000)$ and $|(000)|\geqslant\frac{q-4}{2}$. Now take any line ℓ of $\mathscr{U}$ through x. Then $\ell\wedge yz\in(100)$, so $|(100)|>\frac{q}{2}$. Similarly, $|(010)|,|(001)|>\frac{q}{2}$. All vertices on ℓ, except x, are in $N(x)$. Since $|N(y)\cap\ell|=|N(z)\cap\ell|=\frac{q-2}{2}$, we have

$$\begin{aligned}q-1&=|(100)\cap\ell|+|(110)\cap\ell|+|(101)\cap\ell|+|(111)\cap\ell|\\&=|(111)\cap\ell|+|N(y)\cap\ell|+|N(z)\cap\ell|-|(111)\cap\ell|\\&=|(100)\cap\ell|+q-2-|(111)\cap\ell|.\end{aligned}$$

Therefore, $|(111)\cap l|=|(100)\cap l|-1$. By a similar argument, for a line m of $\mathscr{U}$ through y, and a line n of $\mathscr{U}$ through z, we have that

$$\begin{aligned}&|(111)\cap m|=|(010)\cap m|-1,\\&|(111)\cap n|=|(001)\cap n|-1.\end{aligned}\tag{4.4}$$

Now suppose for a contradiction that $|(111)|=k$, where $0\leqslant k\leqslant\frac{q-4}{4}$. Then there are at least $\frac{q}{2}-k\geqslant\frac{q}{4}+1$ parallel classes in $\mathscr{U}$ so that the lines through x in these classes contain no vertices of (111). A similar result holds for y, so there exists at least one parallel class from $\mathscr{U}$, say π_m, such that

$$L(x,m)\cap(111)=L(y,m)\cap(111)=\emptyset.\tag{4.5}$$

We can assign coordinates using the triangle $L(x,m)\wedge yz, x, y$ as the coordinate frame so that these vertices have coordinates (0,0), (1,0), (0,1), respectively. Hence,

$$[0]=(0,0)(0,1)=(L(x,m)\wedge yz)y=yz\in\mathscr{U}',$$
$$[0,0]=(0,0)(1,0)=(L(x,m)\wedge yz)x=L(x,m)\in\mathscr{U},$$
$$[1,1]=[-1,1]=(1,0)(0,1)=xy\in\mathscr{U}'.$$

Since $zIyz=[0]$, z has coordinates $(0,d)$, for some $d\neq 0,1$. Let $\mathscr{U}_{\mathrm{sl}}$ ($\mathscr{U}'_{\mathrm{sl}}$) be the set of slopes of lines of $\mathscr{U}$ ($\mathscr{U}'$). Recall by (4.5) that $[0,0]$ contains no vertices of (111). Take any vertex $(a,0)I[0,0]$ with $a\neq 0$. Then $(a,0)\in N(x)$. Since

$$(a,0)(0,1)=[a^{-1},1],$$
$$(a,0)(0,d)=[a^{-1}d,d],$$

if $a^{-1}\in\mathscr{U}_{\mathrm{sl}}$, then $a^{-1}d\in\mathscr{U}'_{\mathrm{sl}}$.

By (4.5), the line $[0,1]=L(y,m)$ contains no vertices of (111). Each vertex $(a+1,1)$ with $a\neq 1$ is on $[0,1]$ and so is in $N(y)$. Since $(a+1,1)(1,0)=[a^{-1},a^{-1}]$ and $(a+1,1)(0,d)=[(a+1)^{-1}(d+1),d]$, if $a^{-1}\in\mathscr{U}_{\mathrm{sl}}$, then $(a+1)^{-1}(d+1)\in\mathscr{U}'_{\mathrm{sl}}$.

Consider the set $\mathscr{S}=\{a : a\neq 0; a^{-1}\in\mathscr{U}_{\mathrm{sl}}\}$. Then $|\mathscr{S}|=\frac{q-2}{2}$. For each element $a\in\mathscr{S}$, we have that $a^{-1}d$, $(a+1)^{-1}(d+1)\in\mathscr{U}'_{\mathrm{sl}}$, and the vertex $(a+1,d)$ is on the lines $[0,d]$, $[a^{-1}d,a^{-1}d]=(a+1,d)(1,0)$ and $[(a+1)^{-1}(d+1),1]=(a+1,d)(0,1)$, which implies that $(a+1,d)\in(001)$. Therefore, the line $[0,d]$ contains $\frac{q-2}{2}$ vertices of (001) and so by (4.4) it contains $\frac{q-4}{2}$ vertices of (111). Since $\frac{q-4}{2}>\frac{q-4}{4}\geqslant k$, this is a contradiction.

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