On Residually Small Varieties

by

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Abstract

This is an expository account of what is currently known about residually small varieties.

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In memory of my sister Paula.

Contents

Introduction

Residually small varieties were first discussed in by Taylor. Taylor discovered many important results, including the "Hanf-number" of subdirect irreducibility in varieties, and a proof of eleven equivalent conditions for a variety to be residually small. Coincident with Taylor's paper, Quackenbush posed a conjecture that has served as an impetus for much of the research on residual smallness. In a recent tour de force, McKenzie refuted both the conjecture of Quackenbush, and the stronger so-called RS conjecture. The refutation of these conjectures ends a search for a resolution to these problems that has spanned some twenty years. This search has served to expand our understanding of subdirect irreducibility in many well-known varieties (including semigroups, groups, and more generally, congruence modular varieties).

The purpose of this paper is to discuss results on residually small varieties up to but not including McKenzie's work of 1993. (The counterexamples of McKenzie, while of relatively simple structure, are technical, and any presentation of such results would only complicate our task of surveying all known results on the subject. The interested reader should read [29] for a discussion of McKenzie's methods). Some twenty papers exist on the subject of residually small varieties, demonstrating a wide range of techniques and theoretical approaches. Apart from pure universal algebra, papers on residually small varieties include tools from model theory

(Mckenzie-Shelah; Baldwin), category theory (Tholen), and set theory (Erdös' theorem in Taylor). Our focus is on the theory of varieties or equational classes [1]; accordingly, our treatments of results from model and category theory are brief. We assume the reader is familiar with the basic results of universal algebra (for example, the first two chapters of [3]). A preliminary section, however, briefly defines the relevant algebraic ideas required. For example, we describe the many equivalent definitions of a subdirectly irreducible algebra used interchangeably throughout the discussion. Two tools of importance in modern universal algebra are tame congruence theory of finite algebras, and commutator theory for congruence modular varieties. We make no assumption of deep knowledge on these subjects. Nevertheless, we introduce some basic terminology, because of the fact that various vital theorems on the RS conjecture make use of both tame congruence theory and commutator theory. The reader will find both the summaries of tame congruence theory in Mckenzie-Valeriote and of the commutator in Mckenzie, McNulty, and Taylor as sufficient for our needs.

Chapter 1 will deal first with the generative work of Taylor on residually small varieties. We discuss the notion of principal congruence formula and state a combinatorial lemma of Erdös that plays a central role in Taylor's determination of the Hanf number for subdirect irreducibility in varieties. An easy corollary of Taylor's theorem (discovered by the author) will be proven. In the second part of the Chapter, we will discuss the results of Quackenbush, including a theorem (proven in the locally finite case), and the conjecture described at the beginning of this introduction. To close the chapter, we state without proof model theoretic results due to Mckenzie- Shelah on the spectra of subdirect irreducibles in varieties.

In the second chapter we will discuss various results concerning varieties with

[1] Gorbunov extended the results of Taylor and McKenzie-Shelah to quasivarieties.

the property of residual smallness. Topics here include: the results of Baldwin and Berman, especially concerning the property of having definable principal congruences; Baldwin's use of infinitary logic; and lastly, we discuss the connection between the categorical property of "having enough injectives" and residual smallness.

As is evidenced in the literature, both the RS and Quackenbush conjectures have generated considerable attention. Accordingly, we devote Chapter 3 to a survey of results on these conjectures before 1993. We describe Ol'shanskii's determination of a syntactic criterion for residual finiteness that proves the Quackenbush conjecture for groups. We summarize the determination of all residually small varieties of semigroups by McKenzie (and independently by Golubov and Sapir). Both Freese and McKenzie's commutator condition for residual smallness in congruence modular varieties, and the results of Hobby-McKenzie using tame congruence theory are central to work on the RS and Quackenbush conjectures. We describe these results only superficially. Lastly, we state open questions related to the RS and Quackenbush conjectures.

Chapter 0

Preliminaries

In this chapter we lay down the basic terminology used in this paper. We assume the reader is familiar with basic Zermelo-Fraenkel (or Gödel-Bernays) set theory, as well as the lattice theory and universal algebra contained in Chapter 1-2 of [3] (or a suitable equivalent). We start with a rigorous definition of an algebra.

Definition 1 *Define a* **language**, $\mathcal{F}$, *of algebras to be a set of* **function symbols** *and assign to each* $f \in \mathcal{F}$ *a unique* $n \in \omega$, *called the* **arity of f**. *An* **algebra A of type** $\mathcal{F}$ *is the ordered pair* $< A, F >$, *where* A *is a nonempty set and* F *is the set of operations on* A *(namely, members of* A^{A^n}, *for some* $n \in \omega$*) indexed by* $\mathcal{F}$ *so that for every n-ary function symbol of* $\mathcal{F}$, *there is a corresponding n-ary operation of* F.

Throughout our discussion, **A**, will always denote an algebra.

The notions of subalgebras, homomorphisms, and products of algebras are as in [3].

Let K be a class of algebras. Define the class operators H, S, and P, as the class of all homomorphic images, subalgebras, and direct products of members of K, respectively (we shall define other class operators along the way). A class of algebras K closed under H,S,P is a **variety**. $V(K)$, or the variety generated by K, equals HSP(K). By a famous theorem of Birkhoff's, varieties are precisely the models of equational theories. Denote the congruence lattice of $\mathbf{A}$ as Con$(\mathbf{A})$. The smallest and largest members of Con$(\mathbf{A})$ are the diagonal relation, $\triangle_A$, and $A \times A$, or $\bigtriangledown_A$, respectively. We say that a variety is **congruence P** for some lattice property P iff every congruence lattice of every algebra in the variety satisfies that property. For example, V is congruence distributive iff all of its members have distributive congruence lattices.

The notion of subdirectly irreducible algebra is fundamental in what follows. We make the following definition:

Definition 2 **A is subdirectly irreducible** *iff for every embedding*

$$\alpha : \mathbf{A} \rightarrow \prod_{i \in I} \mathbf{A}_i$$

where

$$\pi_i \circ \alpha : \mathbf{A} \rightarrow \mathbf{A}_i, \text{ is onto for each } i \in I$$

there exists $j \in I$ *such that* $\pi_j \circ \alpha$ *is an isomorphism.*

Many equivalent definitions of the notion have appeared in the literature. We include a theorem that states numerous equivalent definitions of the subject. As the proof is elementary we omit it.

Theorem 1 *The following are equivalent:*

1. $\mathbf{A}$ *is subdirectly irreducible.*

2. *Con*($\mathbf{A}$) *has a unique* **atom**; *that is, a unique member covering* $\triangle_A$.

3. $\triangle_A$ *is completely meet irreducible; that is, if* $\triangle_A = \bigcap_{i \in I} \theta_i$, *then* $\triangle_A = \theta_i$ *for some* i *in* I.

4. *There exist distinct* $a, b \in \mathbf{A}$ *such that every morphism* f *of* $\mathbf{A}$ *into any algebra is one-one or satisfies* $f(a) = f(b)$.

5. $\mathbf{A}$ *contains an (a, b)-irreducible pair (that is, a pair of distinct elements contained in every nontrivial congruence).*

6. *The maximal member of Con*($\mathbf{A}$) *separating* a, b *is* $\triangle_A$, *for some distinct* a, $b \in A$.

7. *Any family of morphisms with domain* $\mathbf{A}$ *which separates points of* A *must contain a morphism that is one-one.*

Example 1 It is not hard to check that the only subdirectly irreducible abelian groups are the groups $\mathbb{Z}_{p^k}$, for p a prime, $k \in \omega$, and the Prüfer groups, $\mathbb{Z}_{p^\infty}$.

Example 2 A well-known fact is that the only subdirectly irreducible Boolean algebra (semilattice) is the two-element Boolean algebra (semilattice).

Subdirectly irreducibles form the "building blocks" of algebras, in the sense of the following theorem due to Birkhoff:

Theorem 2 (Birkhoff) *For any algebra* $\mathbf{A}$ *there exist subdirectly irreducible algebras* $\mathbf{A}_{i\{i \in I\}}$ *(of the same type as A), such that there exists a morphism*

$$\alpha : \mathbf{A} \to \prod_{i \in I} \mathbf{A}_i$$

with $\pi_i \circ \alpha$ *onto for all* $i \in I$. *(that is,* $\mathbf{A}$ *is isomorphic to a* **subdirect product** *of subdirectly irreducible algebras).*

It follows from Birkhoff's theorem that $V = ISP(V_{si})$, where I is the isomorphism class operator, and V_{si} are all subdirectly irreducible members of the variety. (the class "V_{si}" is sometimes called a **cogenerating class** of the variety).

When is a variety V cogenerated by a set[1]? If one defines Spec(V_{si}) (the **spectrum of the subdirectly irreducibles in V**) to be the cardinalities of members of V_{si}, then this is equivalent to asking: is there a cardinal η that is larger than every cardinal in Spec(V_{si})? The consideration of this question leads to the following definition:

Definition 3 *If V_{si} is (up to isomorphism) a set, then V is* **residually small**.

By the previous example, the variety of abelian groups is residually small, as are the varieties of Boolean algebras and semilattices. Residually small varieties will be the main focus of our exploration, and we defer any further remarks about them until the next chapter. As will be demonstrated, residual smallness is quite a restrictive property for a variety to possess.

Let $|S|$ be the cardinality of S, and $|S|^+$ be the successor cardinal of $|S|$. Following the convention of elementary real analysis, we let ∞ mean: "larger than any cardinal". We define the following function on varieties which (after McKenzie) we call the **residual character of V**:

Definition 4 *Let V be a variety. Define*

$$\kappa(\mathbf{V}) = \begin{cases} \sup\{|S|^+ : \mathbf{S} \in V_{si}\} & \text{if } V \text{ is residually small} \\ \infty & \text{otherwise} \end{cases}$$

Define $\kappa(\mathbf{A})$ to be $\kappa(V(\mathbf{A}))$.

[1] As opposed to a proper class.

For example, the residual character for the variety of abelian groups is ω_1, and for both the varieties of Boolean algebras and semilattices, the residual character is 3. The variety of groups has arbitrarily large simple groups; therefore, the residual character of that variety is ∞.

To state Taylor's equivalent conditions for residual smallness, we state three definitions necessary for the exposition. We make no use of these concepts above and beyond the description of Taylor's results.

Definition 5 **A** *is* **equationally compact** *iff whenever* Σ *is any set of equations with constants from* A*, and if every finite subset of* Σ *can be satisfied in* **A***, then* Σ *can be satisfied simultaneously in* **A**.

Definition 6 **A** *is an* **absolute retract** *in a variety* V*, iff whenever* $\mathbf{A} \subseteq \mathbf{B} \in V$*, then there exists an epimorphism* $\alpha : \mathbf{B} \to \mathbf{A}$*, which is the identity on* **A**.

Definition 7 **B** *is an* **essential extension** *of an algebra* **A** *iff* $\mathbf{A} \subseteq \mathbf{B}$ *and every proper congruence on* **B** *identifies two points of* **A**.

We next introduce elementary commutator theory, and base our discussion on the exposition found in [15].

Definition 8 *Let* $\alpha, \beta, \gamma \in Con(\mathbf{A})$. α **centralizes** β **modulo** γ *(*$C(\alpha, \beta; \gamma)$*) iff for any n+1-ary term (*$n \geq 1$*), p,*

$$< a, b > \in \alpha, and < c_1, d_1 >, \cdots, < c_n, d_n > \in \beta$$

we have:

$$< p(a, c_1, \cdots, c_n), p(a, d_1, \cdots, d_n) > \in \gamma \leftrightarrow < p(b, c_1, \cdots, c_n), p(b, d_1, \cdots, d_n) > \in \gamma$$

If **A** is an algebra with α, β, and γ_i (i $\in$ I) $\in$ Con(**A**), it is easy to see that if $\forall$i $\in$ I (C($\alpha,\beta;\gamma_i$), then (C($\alpha,\beta;\bigwedge_{i\in I}\gamma_i$). This allows us to make the following definition:

Definition 9 *Let α and $\beta \in$ Con(**A**). Define the* **commutator** *of α and β, with notation $[\alpha,\beta]$, to be the smallest congruence ν for which (C($\alpha,\beta;\nu$)).*

As we need know nothing more than the notation of the commutator, we do not attempt to develop any commutator theory here (which is a deep theory in congruence modular varieties; see [7]).

Another area in which we must defer the reader is in tame congruence theory. In this theory, the congruence lattice of any finite algebra becomes a labelled graph. The patterns and combinations of labelling may influence the possible structure, and conversely the structure of the algebra will determine the possible so-called "type set". For example, finite groups display only types 2 and 3. We merely state what the five types are, and make the assumption (a valid one) that every covering pair of congruences (or prime quotient) of a finite algebra has a unique type. We describe the five possible types in the following definition.

To every prime quotient $<\alpha,\beta>$ we can assign a so-called "minimal" algebra[2]. Each such minimal algebra will have a well-defined type, called the **type of** $<\alpha,\beta>$.

Note that two algebras are **polynomially equivalent** iff they have the same set of **polynomial operations**[3].

Definition 10 *Let α and β be a prime quotient in* **A**, *denoted by $\{\alpha,\beta\}$. Let* **M** *be the minimal algebra of $\{\alpha,\beta\}$.*

[2]See Ch. 4 and 5 of [11].

[3]See Definition 13.3 of [3].

1. **M** *is* **type 1** *or* **unary type** *iff the polynomials of M are the polynomials of the algebra, $< M, \pi >$, for some subgroup π of the group of permutations on M.*

2. **M** *is* **type 2** *or* **affine type** *iff M is polynomially equivalent to a vector space.*

3. **M** *is* **type 3** *or* **Boolean type** *iff M is polynomially equivalent to a two-element Boolean algebra.*

4. **M** *is* **type 4** *or* **lattice type** *iff M is polynomially equivalent to a two-element lattice.*

5. **M** *is* **type 5** *or* **semilattice type** *iff M is polynomially equivalent to a two-element semilattice.*

The type set of any finite algebra is the set of all types of all of it's prime quotients. Hence, it is a subset of $\{1, 2, 3, 4, 5\}$.

Chapter 1

Fundamental Results

This chapter is devoted to the "classical" results on residually small varieties, drawing from the papers of Taylor, Quackenbush, and McKenzie-Shelah.

1.1 Taylor's Results

As mentioned above, we shall compute the Hanf-number for subdirect irreducibility, as well as describe Taylor's eleven equivalent conditions for a variety to be residually small. To do this, we shall need a lemma of Mal'cev's on principal congruences, the notion of a principal congruence formula, and a combinatorial result due to Erdös[1].

Lemma 1 (Mal'cev) *Let a,b,c, $d \in \mathbf{A}$ Then*

$$< a, b > \in \Theta(c, d)$$

iff there exist m terms

$$p_i(x, y_1, \ldots, y_k)_{1 \leq i \leq m}$$

[1]This is also sometimes called the Erdös-Radó Theorem in the literature.

for some $m \in \omega$, *and* $t_1, \ldots, t_k \in \mathbf{A}$ *so that*

$$\begin{aligned} a &= p_1(r_1, t_1, \ldots t_k) \\ p_i(s_i, t_1, \ldots t_k) &= p_{i+1}(r_{i+1}, t_1, \ldots t_k) \text{ for } 1 \leq i < m, \\ p_m(s_m, t_1, \ldots t_k) &= b \end{aligned}$$

where

$$\{r_i, s_i\} = \{c, d\}.$$

Our proof of this theorem follows the proof in [3].

PROOF. Let $p_i(x, y_1, \ldots, y_k)$ be a term of $\mathbf{A}$, and $t_1, \ldots, t_k \in \mathbf{A}$. As congruences are compatible with the operations of $\mathbf{A}$, it follows that

$$< p_i(c, t_1, \ldots t_k), p_i(d, t_1, \ldots t_k) > \in \Theta(c, d).$$

Now if

$$\{r_i, s_i\} = \{c, d\}$$

and

$$p_i(s_i, t_1, \ldots t_k) = p_{i+1}(r_{i+1}, t_1, \ldots t_k),$$

then by the transitivity of $\Theta(c, d)$,

$$< p_1(r_1, t_1, \ldots t_k), p_m(s_m, t_1, \ldots t_k) > \in \Theta(c, d).$$

Thus $\theta \equiv \{< a, b > |$there exist m terms $p_i(x, y_1, \ldots, y_k)_{1 \leq i \leq m}$ for some m $\in \omega$, and $t_1, \ldots, t_k \in \mathbf{A}$ so that $a = p_1(r_1, t_1, \ldots t_k)$, $p_i(s_i, t_1, \ldots t_k) = p_{i+1}(r_{i+1}, t_1, \ldots t_k)$ for $1 \leq i < m$,, and $p_m(s_m, t_1, \ldots t_k) = b$, where $\{r_i, s_i\} = \{c, d\}\}$ is a subset of $\Theta(c, d)$.

θ is a congruence relation:

i) θ is reflexive: let m=2, let p_1 and p_2 be the projections on the second factor, and let $t_1 = t_2 = a$.

ii) θ is symmetric: Let $< a, b > \in \theta$, so that $< a, b >$ is witnessed by some $m \in \omega$, some $t_i \in A$, and some suitable set of terms p_i (as in the definition of θ). Take the same values for m and t_i, but take the reverse sequence of p_i.

iii) θ is transitive: If $< a, b >$, $< b, c > \in \theta$, then the composition of the two sequences of terms witnessing $< a, b >$, $< b, c >$ will give $< a, c > \in \theta$.

iv) θ is compatible with the operations of **A**: Let $< a_k, b_k > \in \theta$ $(1 \leq k \leq n)$. Let f be an operation of **A**.

Let

$$\begin{aligned} a_k &= p_{k1}(r_{k1}, t_1, \ldots t_k) \\ p_{ki}(s_{ki}, t_1, \ldots t_k) &= p_{ki+1}(r_{ki+1}, t_1, \ldots t_k) \text{ for } 1 \leq i < m, \\ p_{km_k}(r_{km_k}, t_1, \ldots t_k) &= b_k. \end{aligned}$$

Then

$$f(b_1, \ldots, b_{k-1}, a_k, \ldots, a_n) = \begin{array}{l} f(b_1, \ldots, b_{k-1}, p_{ki+1}(r_{ki+1}, t_1, \ldots t_k), a_{k+1}, \\ \ldots, a_n), \end{array}$$

and

$$f(b_1, \ldots, b_{k-1}, p_{ki}(s_{ki}, t_1, \ldots t_k), a_{k+1}, \ldots, a_n) = \begin{array}{l} f(b_1, \ldots, b_{k-1}, p_{ki+1}(r_{ki+1}, t_1, \ldots t_k), \\ a_{k+1}, \ldots, a_n), \end{array}$$

for $1 \leq i < m_k$.

Further,

$$f(b_1, \ldots, b_{k-1}, p_{ki}(s_{ki}, t_1, \ldots t_k), a_{k+1}, \ldots, a_n) = f(b_1, \ldots, b_{k-1}, b_k a_{k+1}, \ldots, a_n)$$

Therefore,

$$< f(b_1, \ldots, b_{k-1}, a_k, \ldots, a_n), f(b_1, \ldots, b_k, a_{k+1}, \ldots, a_n) > \in \theta, \text{ for all k;}$$

by definition of θ.

By transitivity,

$$< f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) > \in \theta$$

which establishes compatibility.

Clearly, $< c, d > \in \theta$. As $\Theta(c, d)$ is the smallest congruence containing $< c, d >$, the proof is complete. □

Mal'cev's lemma has the following Corollary:

Corollary 1 *Let a,b,c, and* $d \in A$.
$< a, b > \in \Theta(c, d)$ *iff there exists a four variable formula* $\phi(x, y, z, w)$ *in the first order language of* **A** *such that:*

i) ϕ *is positive (that is, it built up from atomic formulae and does not contain an occurrence of* $\neg$, $\rightarrow$, *or* $\leftrightarrow$*.)*

ii) $\vdash \forall y, z\ [\exists x \phi(x, x, y, z) \rightarrow y \approx z]$

iii) $\mathbf{A} \models \phi(c, d, a, b)$.

A formula that satisfies i) and ii) is called a **principal congruence formula**[2].

The next theorem follows from our theorem on subdirect irreducibility.

Corollary 2 **A** *is subdirectly irreducible iff there exists a pair of distinct elements* $a, b \in \mathbf{A}$ *such that for all distinct elements* $c, d \in \mathbf{A}$ *there exists a principal congruence formula* ϕ *with* $\mathbf{A} \models \phi(c, d, a, b)$.

[2]In the sense of [26].

We now state Taylor's main theorem.

Theorem 1 (Taylor) *Let V be a variety of algebras of type $\mathcal{F}$ defined by equations Σ. Let $\eta = \sup(\omega, |\mathcal{F}|)$. Then the following are equivalent:*

1. *for some cardinal μ, every subdirectly irreducible algebra in V has cardinal $\leq \mu$*
2. *every subdirectly irreducible algebra in V has cardinality $\leq 2^\eta$*
3. *there are $\leq 2^{2^\eta}$ non-isomorphic subdirectly irreducible algebras in V*
4. *there is a set K such that $V \subseteq$ ISP(K)*
5. *there is a set $K \subseteq$ with $|K| \leq 2^\eta$ and $|A| \leq 2^\eta$ for all $\mathbf{A} \in K$, such that V =ISP(K)*
6. *each algebra in V has (up to isomorphism) only a set of essential extensions in V.*
7. *if $\mathbf{B} \in V$ is an essential extension of $\mathbf{A}$, then $|B| \leq 2^{\eta+|A|}$*
8. *every algebra in V is a subalgebra of an equationally compact algebra*
9. *every algebra in V is a subalgebra of an equationally compact algebra in V*
10. *every algebra in V is a subalgebra of some absolute retract in V*
11. *for every positive formula $\phi(x, y, z, w)$ in the language of V such that $\vdash \forall y, z$ $[\exists x \phi(x, x, y, z) \rightarrow y \approx z]$ there exists $n \in \omega$ such that*

$$\Sigma \vdash \forall yz[\exists x_i \ldots x_n \bigwedge_{1 \leq i < j \leq n} \phi(x_i, x_j, y, z) \rightarrow y \approx z]$$

All the ingredients for a complete proof of this theorem are in Taylor's paper. Instead of proving the entire result, we focus on proving that residual smallness is equivalent to the second item in the theorem.

We define a concept from Model Theory:

Definition 11 *Let P be a property of models in some fixed language $\mathcal{L}$. If $\mathcal{T}$ is a set of theories, then* **Hanf number of P in** $\mathcal{T}$ *(if it exists) is defined to be the least cardinal η, such that for each $T \in \mathcal{T}$, if T has a model of cardinality η with property P, then T has models with property P of arbitrarily large power.*

Let P be the property "subdirect irreducibility", and let $\mathcal{T}$ be the class of all varieties with language $\mathcal{F}$. Taylor's result says that the Hanf number of P in $\mathcal{T}$ is $\leq (2^{\eta})^{+}$, where η is defined as in the above theorem. We prove this result, and show by example that $(2^{\eta})^{+}$ is exactly the Hanf-number of P in $\mathcal{T}$.

To do this we first state a combinatorial lemma found in [26]. Let $A^{(2)}$ denote all doubletons in A that are not singletons.

Lemma 2 (Erdös) *Let κ be a cardinal $\geq \omega$, and let A be a set with $|A| > 2^{\kappa}$, C a set with $|C| \leq \kappa$. If $f : A^{(2)} \to C$ is a map, then there is a set $B \subset A$, $|B| \geq \omega$ with f constant on B.*

Theorem 2 (Taylor) *The Hanf number for P in $\mathcal{T}$ is $\leq (2^{\eta})^{+}$, where η, P, and $\mathcal{T}$ are defined as above.*

Proof [3]: Let V be a variety with language $\mathcal{F}$, and let $\mathbf{A} \in V_{si}$, with $|A| > 2^{\eta}$. By our theorem on subdirect irreducibility, $\mathbf{A}$ is (a,b)-irreducible, and $\Theta(a,b)$ is the monolith of Con($\mathbf{A}$). Let C be the class of all principal congruence formulae in the language $\mathcal{F}$. It is clear that $|C| \geq \omega$; hence $|C| = \eta$. By Corollary 1, for each $c \neq d$ in A there exists a $\phi \in C$ (depending on c and d) such that $\mathbf{A} \models \phi(c, d, a, b)$. Let $C_{c,d} \equiv \{\phi \in C : \phi(c, d, a, b)\}$. As $D \equiv \{C_{c,d} : \{c, d\} \in A^{(2)}\}$ is a nonempty family of nonempty sets, by the Axiom of Choice there exists a choice function, say g, on D. It follows that $f : A^{(2)} \to C$ defined $\{c, d\} \mapsto g(C_{c,d})$ is a function. By Erdös's lemma, there is a $B \subset A$, $|B| \geq \omega$, such that there exists a $\phi \in C$ such that for all $\{c, d\} \in B^{(2)}$, $\mathbf{A} \models \phi(c, d, a, b)$. Let $\mathcal{N}$ be a set of nullary function symbols, with $|\mathcal{N}| = m \geq \omega$. Let X be an infinite set of variables. If we let $\Sigma(X)$ be the equations of V in variables taken from X, define a new set of first order formulas Φ, to be

$$\{i \not\approx j | i, j \in \mathcal{N} \text{ and } i \neq j\} \cup \Sigma(X) \cup \{\phi(i, j, a, b) | i, j \in \mathcal{N} \text{ and } i \neq j\} \cup \{a \neq b\}.$$

By interpreting each member of $\mathcal{N}$ as a member of B, it follows that $\mathbf{A}$ satisfies every finite subset of Φ. By the Compactness theorem for first order logic, there exists an algebra, N of type $\mathcal{F} \cup \mathcal{N} \cup \{a, b\}$, such that $\mathbf{N} \models \Phi$.

Let $N \subset \mathbf{N}$ be members of $\mathbf{N}$ corresponding to $\mathcal{N}$, and let a', b' denote the members of $\mathbf{N}$ corresponding to a,b. Hence, $|N| = \eta$ and $a' \neq b'$. Let θ be a maximal congruence of $\mathbf{N}$ with respect to not containing $< a', b' >$. Then $\mathbf{N}/\theta$ is subdirectly irreducible. Clearly, $< i, j > \notin \theta$, for i,j $\in N$, and i$\neq$j, since $\mathbf{N} \models \phi(i, j, a', b')$. As there are at least as many congruences classes in θ as there are members of N, it follows that $|N/\theta| \geq m$. As m was an arbitrary infinite cardinal, this establishes the result. □

Example 3 (Taylor) We construct a subdirectly irreducible algebra of cardinality 2^η, with η unary operations, where η is any infinite cardinal. Define an algebra $\mathbf{A} \equiv < A, F >$, where A is the set 2^η, F is the set of η unary operations f_i, for $i < \eta$, defined for all $j < \eta$ as

$$f_i(a)(j) = a(i)$$

where $a \in 2^\eta$. If we let $a \in 2^\eta$ be the constant function a(i) = 0, for all $i \in \eta$, and $b \in 2^\eta$ be the constant function b(i) = 1, for all $i \in \eta$, then it's not hard to show that **A** is (a, b)-irreducible. Hence, **A** is subdirectly irreducible. However, any variety of unary type is residually small[3].

Example 3 and Theorem 2 show that 2^η is indeed the Hanf number of P in $\mathcal{T}$. Taylor also shows in the proof of his main Theorem (Theorem 1), that the bounds for items iii), v), and vii) are best possible.

If we enrich our set theory enough to allow the existence of strongly inaccessible cardinals (uncountable cardinals η that are regular and obey: $\rho < \eta \rightarrow 2^\rho < \eta$) then the following is an easy Corollary of Taylor's result:

Corollary 3 *(ZFC + the axiom of inaccessibles) Let* $\mathbf{A}$ *be an algebra,* η *an inaccessible cardinal* $> \omega$, $\omega \leq |A| < \eta$. *If* $\kappa(\mathbf{A}) \geq \eta$, *then* $\kappa(\mathbf{A}) = \infty$.

PROOF. Let $\mathcal{F}$ be the type of **A**. **A** is term equivalent with the algebra $\mathbf{A}'$, which shares the same universe as A, but whose language, $\mathcal{F}'$, consists of representatives from the "natural partition" on $\mathcal{F}$. By this we mean the following: There are $|\mathcal{F}|$ function symbols but only $s \equiv m \leq \bigcup_{i\in\omega} |A|^{|A|^i}$ possible n-ary functions on A. In general $|\mathcal{F}| \geq s$, so that by the "pigeonhole principle" some function symbols may represent the same function. This will induce a partition on $\mathcal{F}$, where two function symbols are related iff they represent the same function. It is not hard to see that $V(\mathbf{A}) = V(\mathbf{A}')$; hence, without loss of generality, we can work with $\mathcal{F}'$.

Let $m = \sup(\omega, |\mathcal{F}'|)$. Then $m \leq \bigcup_{i\in\omega} |A|^{|A|^i}$. Fix $i \in \omega$. Then it is easy to check that $|A|^{|A|^i} = 2^{|A|}$. As η is strongly inaccessible, and since $|A| < \eta$, it follows

[3]see [26], p.39.

that $m < \eta$. Hence, $\eta > 2^m$ so that $\eta > (2^m)^+$. By Theorem 2, $\kappa(\mathbf{A}) = \infty$. □

1.2 Quackenbush's Theorem and Conjecture

A paper of Quackenbush, published near the time of Taylor's paper, established another result concerning subdirectly irreducibility in varieties. Quackenbush's Theorem (as he first proved it) pertains only to finitely generated varieties, but is true for any locally finite variety (proved first by Baldwin and Berman, and later by Dziobiak). We state and prove this fact, with a proof taken from [3]. One can find the relevant theorems on ultraproducts in that book as well.

P_U denotes the class operator of all ultraproducts of a class.

Theorem 3 (Quackenbush) *If V is a locally finite variety, with (up to isomorphism) only finitely many finite subdirectly irreducibles, then $\kappa(V) < \omega$.*

Proof [3]: Let W be the class of finite subdirectly irreducible members of V. Fix $\mathbf{A} \in V$. Let K_A be the set of finitely generated subalgebras of A. By Theorem IV 2.14 of [3],

$$\mathbf{A} \in ISP_U(K_A).$$

As finitely generated algebras are finite in locally finite varieties, every member of K_A is a subdirect product of members of W. Thus,

$$K_A \subseteq IP_S(W) \subseteq ISP(W).$$

This gives

$$\mathbf{A} \in ISP_USP(W) \subseteq ISPP_U(W) \text{ by Thm. IV 2.23 of [3]}$$

As an ultraproduct of a finite set of finite algebras is isomorphic to one of those algebras

$$\mathbf{A} \in ISP(W)$$

so that **A** is a subdirect product of members of W. Hence, if **A** was subdirectly irreducible, $\mathbf{A} \in W$. □

Quackenbush includes an example of a variety V generated by an infinite algebra **A** with the property that V has infinitely many finite subdirectly irreducible algebras, but no infinite one.

Example 4 (Quackenbush) Consider a family of algebras $\{\mathbf{A}_i\}_{i\in\omega}$, with $\mathbf{A}_i$ defined to be $< \{0,1\}, F_{i_{i\in\omega}} >$, where $F_i = \{m_i\} \cup \{f_{ij}\}$, with

(a) $m_i(x,x,y) = m_i(x,y,x) = m_i(y,x,x) = x$,

(b) $f_{ij}(x) = \begin{cases} 1-x & \text{if i=j} \\ x & \text{otherwise} \end{cases}$

Let V be generated by $\{\mathbf{A}_i\}_{i\in\omega}$. As each $|A_i| = 2$, $\mathbf{A}_i$ is simple, and clearly $\mathbf{A}_i \not\cong \mathbf{A}_j$ for $i \neq j$. As members of V have "majority terms", V is congruence distributive[4]. By Jónsson's theorem[5], the subdirectly irreducible algebras in V are contained in $HSP_U(\{\mathbf{A}_i\}_{i\in\omega})$. However, an ultraproduct of two-element algebras is a two-element algebra.

Quackenbush conjectured the existence of a finite algebra with $\kappa(\mathbf{A}) = \omega$. He also asked: "Can the algebra be of finite type?"[6]. These questions have turned out

[4]see Thm. 12.3 of [3].

[5]Thm. IV 6.8 of [3]

[6]p. 265 of [25] .

to be very difficult, with the majority of the literature leaning towards a negative answer to both of them. Hence, we formally state them as conjectures.

Conjecture 1 (Quackenbush) unrestricted Conjecture *There does not exist a finite algebra* $\mathbf{A}$ *with* $\kappa(\mathbf{A}) = \omega$.

restricted Conjecture *There does not exist a finite algebra* $\mathbf{A}$ *of finite type with* $\kappa(\mathbf{A}) = \omega$.

Ralph McKenzie has refuted the unrestricted Quackenbush conjecture in his unpublished manuscript from January of 1994. However, the "restricted" question is still open, and McKenzie states in his manuscript that it may be one of the more difficult of the problems in this area. McKenzie goes on to say,

> *For a long time, I hoped that the non-existence of such an algebra could be proved using the tame congruence theory of [7]. But tame congruence theory is insensitive to the number of basic operations of an algebra ...[and] if the answer [to the conjecture] is yes, then new methods will be required to prove it. (p. 2, [21])*

1.3 The Spectrum of Subdirectly Irreducible Algebras in a Variety

We start with a definition.

Definition 12 *Let* K *be a class of algebras in a variety* V. *Let* η *be as in Taylor's theorem. Define* $S(K)^{\eta}(S(K)_{\eta})$, *the* **upper (lower) spectrum of K**, *to be the* *class of cardinals $\beta \geq \eta$ ($\leq \eta$), such that there is a member of K with cardinality β. Define the* **spectrum of K** *to be $S(K) = S(K)^{\eta} \cup S(K)_{\eta}$.*

McKenzie and Shelah analyzed the spectrum of the subdirectly irreducible members of a variety.

Theorem 4 (McKenzie-Shelah) *Let V be a variety, V_{si} the subdirectly irreducible members of V. Let η be defined as in Theorem 1.*

i) *(ZF+GCH) $S(V_{si})$ omits 0 and 1; $S(V_{si})^{\eta} = \emptyset, \{\eta\}, [\eta, 2^{\eta}]$, or $\{\beta | \beta \geq \eta\}$; there are no other restrictions.*

ii) *(ZFC) The characterization of i) is true if $\eta = \omega$. For every η, $S(V_{si})_{\eta}$ can be any set of cardinals $\leq \eta$ that excludes 0 or 1. For uncountable η, when $S(V_{si})_{\eta}$ does not include all $\beta \geq \eta$, then it is an interval $[\eta, \lambda)$, where $\lambda \leq (2^{\eta})^{+}$.*

We note that McKenzie-Shelah proved more than this, essentially extending these results to T-simple models, where T is a universal theory (that is, equivalent to a set of universal sentences). For example, if T is an equational theory, then a T-simple model is just a simple algebra. Hence, the spectral results of McKenzie-Shelah will apply to the class of simple algebras in a variety. These results are beyond the scope of this paper.

Chapter 2

Other Results

In this second chapter of our treatise we examine results from various sources, which although important, are not central. These include: the results of Baldwin and Berman on definable principal congruences; Baldwin's use of infinitary logic; a categorical result, in which "having enough injectives" in a variety is shown to imply residual smallness; lastly, we include a chart of varieties which lists if they are residually small, inspired by a similar chart in [13].

2.1 The Results of Baldwin-Berman

The paper of Baldwin-Berman [1] contains many results of interest, including a "near" counterexamples to Quackenbush's conjecture. Besides this "near" counterexample, we present their work on the relationship between definable principal congruences and residual smallness, as well as their syntactic criterion for residual finiteness.

Before we begin, we state two theorems of model theory which we shall use during our discussion.

Theorem 5 (Upward Löwenheim-Skolem-Tarski Theorem) *Let T be a theory in a language $\mathcal{L}$. If T has models of infinite power, it has models in all powers $\alpha \geq \max(|\mathcal{L}|, \omega)$.*

Theorem 6 (Downward Löwenheim-Skolem-Tarski Theorem) *Let $\mathbf{A}$ be a model of power α, and let $|\mathcal{L}| \leq \beta \leq \alpha$. Then $\mathbf{A}$ has an elementary submodel*[1] *of power β.*

We start with a definition.

Definition 13 *Let V be a variety. V has* **definable principal congruences**[2] *or* **DFP** *iff there is a first order formula $\phi(x, y, z, w)$ (in the language of the type V) such that for every a, $b \in \mathbf{A} \in V$, $\{< c, d >: \phi(a, b, c, d)\} = \Theta(a, b)$.*

For example, commutative rings with unit have definable principal congruences: in such a ring $\mathbf{R}$, $< c, d > \in \Theta(a, b)$ iff $(\exists z \in R)(c - d = z(a - b))$ [8]. Every locally finite variety with the congruence extension property (every member $\mathbf{A}$ of the variety V has the property that if $\mathbf{A} \subseteq \mathbf{B} \in V$, then every congruence of $\mathbf{A}$ is the restriction of a congruence of $\mathbf{B}$) has DFP [1]. We do not prove this, but prove instead the following result:

Theorem 7 (Baldwin-Berman) *Let V be a variety with DFP. Then V is residually small iff for some $n \in \omega$, $\kappa(V) = n$.*

[1] See Definition 2.2.16 of [4].

[2] In the sense of [1].

PROOF. The reverse direction is obvious.

We make some definitions. Let Σ_V be the set of equations defining V, let Φ_n be the formula stating that "there are n distinct elements," and let Ψ be the formula defining principal congruences (which exists by hypothesis) in V.

Let α be:

$$\exists x_0 \exists x_1 [\exists x_2 \exists x_3 (\Psi(x_0, x_1, x_2, x_3) \wedge x_2 \neq x_3)$$
$$\wedge (\forall x_4 \forall x_5 \forall x_6 \forall x_7 (\Psi(x_0, x_1, x_6, x_7) \vee x_4 \neq x_5 \rightarrow \Psi(x_4, x_5, x_6, x_7))].$$

Let Γ_n be the formula $\Phi_n \wedge \alpha$.

It should be clear that $\mathbf{A} \models \alpha$ iff $\mathbf{A}$ has an (a,b)-irreducible pair, that is, $\mathbf{A}$ is subdirectly irreducible. Define

$$\Lambda \equiv \Sigma_V \cup \{\Gamma_n : n \in \omega\}.$$

Assume (to prove the contrapositive of the forward direction), that for no $n \in \omega$, $\kappa(V) = n$. Then Λ is consistent. By Theorem 5, Λ has models of any infinite cardinality. Hence, V is residually large. □

This result has the following immediate Corollary:

Corollary 4 (Baldwin-Berman) *Let V be a variety of finite type with DFP. If V has infinitely many finite subdirectly irreducible algebras, V is residually large.*

The reader will note that with Corollary 4, any algebra that refutes Quackenbush's restricted conjecture cannot have definable principal congruences.

We state a remark of [1] as a Lemma.

Lemma 3 *Let V be a residually large variety. Then V has subdirectly irreducible members of every infinite cardinality.*

PROOF. Define **B** to be a **pure subalgebra** of **A** iff every finite set of equations with coefficients from B satisfied in **A** is satisfied in **B**. Now, let **A** be (a,b)-irreducible. Then any pure-subalgebra **B** of **A** containing a, b is (a,b)-irreducible [1]. If we also assume **A** is of infinite power, Theorem 6 implies the existence of a pure subalgebra of **A** of every infinite cardinal less $|A|$, which establishes the result. □

We present next a "near" counterexample to Quackenbush's unrestricted conjecture. It is "near" in the sense that it is an example of a locally finite variety V with $\kappa(V) = \omega$, generated by ω many algebras rather than one.

Example 5 (Baldwin-Berman) The language $\mathcal{L}$ of our variety V will consist of a ternary operation symbol t, a constant symbol 0, and for each prime p, a p+1-ary function symbol g_p. Let $K = \{\mathbf{Z}_p : \text{p prime}\}$, where $\mathbf{Z}_p$ is an algebra defined as follows: the universe of $\mathbf{Z}_p$ is just $\{{c_0}^p, \ldots, {c_{p-1}}^p\}$. In $\mathbf{Z}_p$, ${c_0}^p$ denotes "0", and we interpret "t" as the ternary majority function that takes value ${c_0}^p$ if its arguments are all distinct. Now we define g_p: if $p \neq q$ then $g_q(y_1, \ldots, y_{q+1}) = {c_0}^p$, for $y_i \in Z_p$. For $< a_1, \ldots, a_p > \in {Z_p}^p$,

$$g_p(a_1, \ldots a_p, {c_j}^p) = \begin{cases} c^p_{j+1(\text{mod p})} & \text{if the } a_i \text{ are all distinct} \\ {c_0}^p & \text{otherwise} \end{cases}$$

Let $V = V(K)$. As V is congruence distributive, Jónsson's Theorem states that $V_{si} \subseteq HSP_U(K)$. Following the proof in [1], we show that $V_{si} = K$. It is not hard to see that every member of K is subdirectly irreducible (in fact, simple).

Next, for all primes p, define

$$\alpha_p \equiv (\forall x_0, \ldots, \forall x_p \bigvee_{0 \le i < j \le p} x_i = x_j) \vee (\forall x_1, \ldots, \forall x_p \forall y g_p(x_1, \ldots, x_p, y) = 0)$$

and let

$$\beta \equiv \forall x_0 \forall x_1 \forall x_2 ((\bigvee_{0 \leq i < j \leq 2} x_i = x_j) \vee t(x_0, x_1, x_2) = 0)$$

Clearly, $K \models \alpha_p \wedge \beta$, for all primes p. Both the α_p and β are positive universal sentences; hence, every member of $HSP_U(K)$ satisfies them. If **A** is an infinite model in K, we must have that:

$$\mathbf{A} \models (\forall x_1, \ldots, \forall x_p \forall y g_p(x_1, \ldots, x_p, y) = 0) \wedge (\beta)$$

Then for a,b $\in$ **A**,

$$\Theta(a,b) = \{<a,b>, <a,0>, <b,0>, <0,a>, <0,b>, <b,a>\} \cup \triangle_A$$

From this equality, clearly no infinite member **A** of V can be (a,b)-irreducible for any distinct $a, b \in A$. Hence, there can be no infinite members of V_{si}.

By a property of ultraproducts, every member **B** of $P_U(K)$ with cardinality p is isomorphic to $\mathbf{Z}_p$. Therefore, if **A**$\in V_{si}$, then **A**$\in HS(\mathbf{B})$, for some **B** isomorphic to $\mathbf{Z}_p$ for some p. For a fixed p, it is easy to check that every subset of $\mathbf{Z}_p$ containing 0 is a subalgebra. Further, such a subalgebra, if different from **B**, is term equivalent to an algebra **M**$\equiv <M, t, 0>$, where M is a nonempty set . Define the class L of algebras to be the class of all such algebras **M**. L is axiomatized by the following positive universal sentences

$$\forall x \forall y \forall z (x = y \vee x = z \vee y = z \vee t(x,y,z) \approx 0)$$

and

$$\forall x \forall y (t(x,x,y) \approx t(x,y,x) \approx t(y,x,x) \approx x).$$

Therefore, L is closed under H. We conclude that V_{si} contains K and possibly some members of L. It is not hard to see that, up to isomorphism, the two element member of L is the only subdirectly irreducible algebra in L (which is not isomorphic to $\mathbf{Z}_2$)[3].

To show that V is locally finite, we will show that $|\mathbf{F}_V(\overline{p})| < \omega$, for each prime p (here, $\mathbf{F}_V(\overline{p})$ is the free algebra in V with p generators). $\mathbf{F}_V(\overline{p})$ is isomorphic to a subdirect product of subdirectly irreducibles generated by p or fewer elements. This follows as $q > p$ implies that $\mathbf{Z}_q$ is not generated by p or fewer elements. Therefore, $\mathbf{F}_V(\overline{p})$ is a subdirect product of finitely many finite algebras.

We describe a syntactic criterion for a variety to have residual character $< \omega$. We first make two definitions.

Definition 14 *A positive existential formula* $\phi(x, y, z, w)$ *is a* **weak congruence formula** *iff* $\vdash \forall y \forall z[\exists \phi(x, x, y, z) \rightarrow y = z]$.

Definition 15 *Let* a, $b \in \mathbf{A}$. *Let* ϕ *be a weak congruence formula. Define*

$$S_\phi = \{\{x, y\} \in A^{(2)} : \mathbf{A} \models \phi(x, y, a, b) \vee \phi(y, x, a, b)\}.$$

We now state a theorem of [1] which answers Problem 1.25 of [26].

Theorem 8 (Baldwin-Berman) *Let* V *be a variety defined by equations* Σ, *and let* $n \in \omega$. *The following are equivalent.*

i) V *is residually* $< n$.

ii) *For every weak congruence formula* ϕ

$$\Sigma \vdash \forall y \forall z[(\exists x_1, \ldots, \exists x_n \bigwedge_{1 \leq i < j \leq n} \phi(x_i, x_j, y, z)) \rightarrow y = z].$$

[3]We note that in [1], it is claimed (erroneously) that $K = V_{si}$.

iii) *If* $\mathbf{A} \in V$ *is (a,b)-irreducible and* $X^{(2)} \subseteq S_\phi$ *for some weak congruence formula* ϕ*, then* $|X| < n$.

PROOF [1]: i) $\rightarrow$ ii): If ii) is false for some weak congruence formula ϕ, then for some $\mathbf{A} \in V$,

$$\mathbf{A} \models \exists y \exists z[\exists x_1, \ldots, \exists x_n(\bigwedge_{1 \leq i < j \leq n} \phi(x_i, x_j, y, z) \wedge y \neq z)].$$

Let $a, b, c_1, \ldots, c_n \in A$ witness this formula. Let θ be a maximal congruence in $\mathbf{A}$ not containing $< a, b >$. Let $f : \mathbf{A} \rightarrow \mathbf{A}/\theta$ be the canonical map. Then $\mathbf{A}/\theta$ is subdirectly irreducible. We show that $|A/\theta| \geq n$. As ϕ is positive, if $i \neq j$ then

$$\mathbf{A}/\theta \models \phi(f(c_i), f(c_j), f(a), f(b)).$$

As ϕ is a weak congruence formula, $f(c_i) \neq f(c_j)$ if $i \neq j$.

ii)$\rightarrow$iii): Assume iii) fails. Then there exists $X \subseteq A$, where $\mathbf{A} \in V$ is (a, b)-irreducible, $X^{(2)} \subseteq S_\phi$, and $|X| \geq n$. Let $\{x_1, \cdots, x_n\}$ be a set of n distinct elements in X. Define ϕ' to be the weak congruence formula $\phi(x, y, z, w) \vee \phi(y, x, z, w)$. Then $a, b, x_1, \cdots, x_n$ will witness the following formula

$$\exists y_1, \cdots, \exists y_n(\bigwedge_{1 \leq i < j \leq n} \phi'(x_i, x_j, v, w) \wedge v \neq w).$$

Hence, ii) fails, as desired.

iii)$\rightarrow$ i): If i) is false, there exists $\mathbf{A} \in V$ such that $\mathbf{A}$ is (a,b)-irreducible and $|A| \geq n$. Let $S = \{x_1, \ldots, x_n\}$ be a subset of A of cardinality n. As $\mathbf{A}$ is (a,b)-irreducible, there is a congruence formula ϕ_{ij} such that

$$\mathbf{A} \models \phi_{ij}(x_i, x_j, a, b)$$

for all $1 \leq i < j \leq n$. Let $\phi = \bigvee_{1 \leq i < j \leq n} \phi_{ij}$. As ϕ is a weak congruence formula, by iii) if $X^{(2)} \subseteq S_\phi$ then $|X| < n$. But $S^{(2)} \subseteq S_\phi$ and $|S| \geq n$, which gives a contradiction. $\square$

2.2 Infinitary Logic

We now turn to Baldwin's second work on subdirectly irreducible algebras. In [2], he first describes an example of a locally finite variety of finite type with exactly one infinite subdirectly irreducible algebra. Baldwin then derives some consequences on the number of subdirectly irreducibles in a variety by using infinitary logic. We describe the latter analysis.

An infinitary language $\mathcal{L}_{\alpha,\beta}$ consists of the expansion of the language $\mathcal{L}$ of usual first-order logic to include two new kinds of logical operations. Define the operations as follows:

1. Let $(\phi_\epsilon)_{\epsilon<\gamma}$ be a sequence of formulas in $\mathcal{L}_{\alpha,\beta}$ with $\gamma < \alpha$. Then

   $$\bigwedge_{\epsilon<\gamma}(\phi_\epsilon) \text{ and } \bigvee_{\epsilon<\gamma}(\phi_\epsilon)$$

   are formulas of $\mathcal{L}_{\alpha,\beta}$.

   We interpret $\bigwedge_{\epsilon<\gamma}(\phi_\epsilon)$ to hold in a structure for a given assignment in free variables iff each (ϕ_ϵ) holds; similarly, $\bigvee_{\epsilon<\gamma}(\phi_\epsilon)$ holds iff (ϕ_ϵ) holds for some $\epsilon < \gamma$.

2. Let $(x_\epsilon)_{\epsilon<\gamma}$ be a sequence of variables with $\gamma < \beta$. Let ϕ be a formula of $\mathcal{L}_{\alpha,\beta}$. Then

   $$\forall(x_\epsilon)_{\epsilon<\gamma}\phi \text{ and } \exists(x_\epsilon)_{\epsilon<\gamma}\phi$$

   are formulas of $\mathcal{L}_{\alpha,\beta}$. We interpret that $\forall(x_\epsilon)_{\epsilon<\gamma}\phi$ holds in a structure for a given assignment to the remaining free variables iff each ϕ holds for every assignment to the variables $(x_\epsilon)_{\epsilon<\gamma}$, but holding the assignment fixed for the remaining free variables; similarly, $\exists(x_\epsilon)_{\epsilon<\gamma}\phi$ holds iff ϕ holds for some assignment.

We show that for any variety V in a countable language, V_{si} is definable in $\mathcal{L}_{\omega_1,\omega}$.

Let Σ be the equations defining the variety, and for a fixed distinct set of variables x, y, let $\Phi_{x,y}$ be all congruence formulas $\phi(z, w, x, y)$ for V. Then $\mathbf{A} \in V$ is subdirectly irreducible in V iff

$$\mathbf{A} \models \exists x \exists y[(x \neq y) \wedge \forall z \forall w(z \neq w \rightarrow \bigvee \Phi_{x,y})] \wedge \bigwedge \Sigma.$$

Consider the following two theorems on $\mathcal{L}_{\omega_1,\omega}$:

Theorem 9 (Harnik, Makkai) *If ϕ is a sentence of $\mathcal{L}_{\omega_1,\omega}$, and ϕ has η countable models, $\aleph_1 \leq \eta < 2^{\aleph_0}$, then ϕ has a model of cardinality $\aleph_1$.*

Theorem 10 (Shelah) *(GCH) If a sentence of $\mathcal{L}_{\omega_1,\omega}$ has at least one but strictly less than $2^{\aleph_1}$ models of power $\aleph_1$ then it has a model of power $\aleph_2$.*

We apply Theorems 9 and 10 to obtain the following immediate Corollaries:

Corollary 5 (Baldwin) *If V has countable type and $\kappa(V) = \aleph_1$, then it has $\leq \aleph_0$ or $2^{\aleph_0}$ countable subdirectly irreducible algebras.*

Corollary 6 (Baldwin) *(GCH) If a residually small variety V of countable type has a subdirectly irreducible member of power $\aleph_1$, it has $2^{\aleph_1}$ such.*

2.3 A Categorical Result

There are two ways that category theory has influenced the study of residually small varieties. The first way, known to Taylor (and Banachewski before him), is that for a variety of algebras to have the categorical property of "having enough injectives" is equivalent to the variety being residually small and some other properties. The second way, used by Tholen, is a generalization of "subdirectly irreducible" (and hence, residual smallness) to fairly general categories. We describe the first approach. As our main subject is universal algebra, we shall defer the proofs in this section to other sources.

Definition 16 *Let V be a variety. An algebra $\mathbf{A} \in V$ is* **V-injective** *iff whenever $\mathbf{B} \subseteq \mathbf{C} \in V$ and $h{:}\mathbf{B} \to \mathbf{A}$ is a morphism, then there exists a morphism $g{:}\mathbf{C} \to \mathbf{A}$ extending f.*

Definition 17 *A variety V* **has enough injectives (EI)** *iff every member of V can be embedded in some V-injective.*

For example, Day[4] showed that every variety generated by a primal algebra[5] has enough injectives; Bruns and Lakser[6] have shown that the variety of all semilattices has enough injectives.

Definition 18 *A variety V has the* **amalgamation property (AP)** *iff whenever f and g are embeddings between members of V with the same domain, there exist embeddings i and j (again in V) such that $i \circ f = j \circ g$.*

For example, groups, lattices, and semilattices all have the AP.

The following theorem, first proved by Banachewski, describes the interrelationship of all of these concepts. For a proof, see Taylor.

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[4]See p. 45 of [26].

[5]See Definition 7.2 of [3].

[6]See p. 397 of [8].

Theorem 11 (Banachewski) *A variety V has EI iff it has AP, CEP, and is residually small.*

We include examples to show the concepts in Theorem 11 are independent.

Example 6

1. *The variety of pseudocomplemented distributive lattices is residually large (proved by Lakser) but has the AP and CEP [26].*
2. *The variety V of commutative rings with unit satisfying $x^{22} \approx x$ is residually small (as proved by Banachewski), but does not have enough injectives [26].*

2.4 A Compendium of Residually Small Varieties

In this brief section, we include a list of some common varieties, and state whether they are residually small. Our motivation is to increase our stock of examples, and to lead the reader to the conclusion that residual smallness is somewhat of a rare property for varieties to possess (more will be said in this vein in the next chapter). Most of the results stated in the chart are well-known; the reader will find the relevant references to prove these results in the comprehensive bibliography of [13].

variety	*residually small*
unary algebras	yes
abelian groups	yes
semigroups	no
commutative semigroups	no
bands	no
semilattices	yes
monoids	no
commutative monoids	no
quasi-groups	no
loops	no
groups	no
rings	no
commutative rings satisfying $x^m \approx x$	yes
k-algebras	no
Lie algebras (over a field)	no
near-rings	no
lattices	no
modular lattices	no
distributive lattices	yes
Stone algebras	yes
Heyting algebras	no
Boolean algebras	yes
cylindric algebras	no
pseudocomplemented distributive lattices	no

Chapter 3

The Quackenbush and RS Conjectures

Our final chapter discusses the influence and impact of the Quackenbush and RS conjectures on the study of both residually small varieties and universal algebra in general. The key player here is McKenzie who has written numerous articles on these conjectures, and in whose book ([11]) the RS conjecture first appears. As we mentioned earlier, owing to early success in well-known varieties, McKenzie (mistakenly) believed that the conjecture would be proven true for all varieties. We examine the work of both McKenzie and other mathematicians on these conjectures.

3.1 Statement and Survey

The Quackenbush conjectures are in Chapter 1. We now state the RS[1] conjecture.

[1]Sometimes called the SI conjecture.

Conjecture 2 (Hobby-McKenzie) *Let* $\mathbf{A}$ *be a finite algebra. Then:*

$$\kappa(A) < \infty \rightarrow \kappa(A) < \omega.$$

The reader will note that the RS conjecture is stronger than the unrestricted Quackenbush conjecture. Also note that the above statement is no longer a conjecture, as McKenzie has refuted it in [21]. Nevertheless, it is true in many varieties; for example, it holds in congruence modular varieties (which include the varieties of groups, rings, and Heyting algebras); the variety of all semigroups; varieties of K-algebras (where K is any commutative ring with unit); and most generally, for any locally finite variety that is congruence P, for some nontrivial lattice equation P (that is, an equation false in at least one lattice). In the coming sections we will examine the proofs of the RS conjecture for each of the above mentioned classes of algebras.

3.2 An Early Result: The Variety of Groups

A. Ju. Ol'shankskii's paper of 1969 contains the following result:

Theorem 12 *Let* V *be variety of groups. Then* $\kappa(V) \leq \omega$ *iff* $V = V(\mathbf{G})$, *for a finite group* $\mathbf{G}$, *all of whose Sylow subgroups are abelian.*

This theorem is the first we are aware of that gives a "structural" condition to describe the property of being residually finite[2]. Theorem 12 predates Quackenbush's conjecture, and for that matter the notion of residual smallness. The original proof

[2]Ol'shanskii worked with "finitely approximable groups", a notion equivalent to residual finiteness.

involves fairly specialized results from group theory so we omit it. In hindsight, Theorem 12 may be viewed as a special case of the results of [7]. For this reason, we defer a proof based on those results to Section 3.5.

3.3 Taylor's Result: Congruence Permutability-Regularity

The following results of Taylor, published in 1979, were significant findings dealing with the Quackenbush conjecture. They prove that conjecture true in a variety satisfying special conditions that we describe now.

Definition 19 *Let V be a variety.*

1. *V is* **congruence permutable** *iff for any $\mathbf{A} \in V$, and for any θ_1, θ_2 in $Con(A)$, $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$.*
2. *V is* **congruence regular** *iff every congruence in every $\mathbf{A} \in V$ is determined by any one of its congruence blocks.*

There exist term conditions for a variety to have either of the properties in Definition 19. A variety V is congruence permutable[3] iff there is a ternary term p(x,y,z) (in the language of the type of V) such that

$$V \models p(x,x,y) \approx y$$

and

$$V \models p(x,y,y) \approx x.$$

[3]See Theorem 12.2 of [3].

V is congruence regular[4] iff there exist $n \in \omega$, and terms (in the language of the type of V) $p_i(x,y,z)$ (for $1 \leq i \leq 2n$) and $g_i(x,y,z,w)$ (for $1 \leq i \leq n$) such that V satisfies

$$\begin{aligned} z &\approx p_i(x,x,z) \qquad 1 \leq i \leq 2n, \\ x &\approx g_1(x,y,z,p_1(x,y,z)), \\ g_1(x,y,z,p_2(x,y,z)) &\approx g_2(x,y,z,p_3(x,y,z)), \\ g_2(x,y,z,p_4(x,y,z)) &\approx g_3(x,y,z,p_5(x,y,z)), \\ &\vdots \\ y &\approx g_n(x,y,z,p_{2n}(x,y,z)). \end{aligned}$$

Examples of congruence permutable-regular varieties include groups, Boolean algebras, and rings.

We prove the following result:

Theorem 13 (Taylor) *If* $\mathbf{A}$ *is a finite algebra with* $V(\mathbf{A})$ *both congruence regular and permutable, and if* V_{si} *contains arbitrarily large finite members, then* V_{si} *contains an infinite member.*

Hence, no algebra that is congruence regular-permutable would supply a counterexample to Quackenbush's conjecture. The proof hinges on the following technical Lemma:

Lemma 4 (Taylor) *If* $\mathbf{A}$ *is a finite algebra with* $V(\mathbf{A})$ *both congruence regular and permutable, then there exists a sequence* $\{\phi_i\}_{i \in I}$ *of congruence formulas such that the elements of every finite member* $\mathbf{B}$ *of* V_{si} *can be arranged in a sequence* $b_0, b_1, b_2, \ldots$ *such that* $\mathbf{B} \models \phi_j(b_i, b_j, b_0, b_1)$ *when* $i < j$.

[4]See p.196 of [27].

We will not prove this Lemma here, but use it in the proof of Theorem 13.

PROOF of Theorem 13 ([27]). Let $\mathbf{B}_i$ be a finite member of V_{si}. Then there exist b_{ij} such that $B_i = \{b_{i0}, b_{i1}, \ldots\}$, as in Lemma 4. Let $\mathbf{B}$ be a nonprincipal ultraproduct of $\{\mathbf{B}_i : i \in I\}$ (that is, of all finite members of V_{si}, up to isomorphism). Define $b_j \in B$ so that for each j, the ith coordinate of b_j is b_{ij}, for almost all i. By Los' Theorem[5], $\mathbf{B} \models \phi_j(b_i, b_j, b_0, b_1)$ whenever $i < j$, where the ϕ_i are as in Lemma 4. If $< b_0, b_1 > \notin \theta$ for $\theta \in Con(B)$, then the b_j are all distinct in $\mathbf{B}/\theta$. Hence, $\mathbf{B}/\theta$ is infinite. If we let θ be a maximal congruence separating b_0 and b_1, we are done. □

Taylor's result was strengthened by Freese and McKenzie. We defer any discussion of this until Section 3.5.

3.4 The Variety of Semigroups

In 1979, Golubov and Sapir classified all residually finite varieties of semigroups. They proved that if a variety of semigroups has residual character $\leq \omega$, then the variety is finitely generated and has residual character $< \omega$. This proves the Quackenbush conjecture for semigroups. In 1981, McKenzie classified (independently of [9]) residually small varieties of semigroups modulo some questions concerning varieties of groups. He also showed that the Quackenbush conjecture is true for semigroups (a result weaker than that of [9]). In a sequel to his 1981 work, McKenzie classifies residually finite semigroup varieties, and partially addresses the group questions mentioned above. As a corollary of these results the RS conjecture holds true for semigroups.

[5]See Theorem 4.1.9 of[4].

We examine these three works and summarize their findings. The proofs of the results of this section require somewhat specialized results from semigroup theory, and so proofs are omitted. We show, however, how the RS conjecture follows from the results [18][6].

We make some introductory definitions.

Definition 20 *1. Let* $\mathbf{2} \equiv < \{0,1\}, \cdot >$, *where "·" is integer multiplication.*

2. *A* **zero semigroup** *is one with all products equal.*

3. *Let* $\mathbf{2}_0$ *be the two element zero semigroup.*

4. *A* **left (right) zero semigroup** *is one satisfying* $xy \approx x$ $(xy \approx y)$.

5. *Let* $\mathbf{L}_0$ $(\mathbf{R}_0)$ *represent the two element left zero (right zero) semigroup.*

6. *Let* $\mathbf{S}$ *be any semigroup. Define the semigroup* $\mathbf{S}^{(0)}$ *to have universe* $S \cup 0$ *(for* $0 \notin S$*), and* $xy = 0$ *if either* x *or* y *is* 0.

7. *Let* $\mathbf{G}$ *be a group,* U *a nonempty set, and* $\alpha : \mathbf{G} \to Perm(U)$ *a group morphism (*α *is a* **representation***). Define a semigroup* $\mathbf{R}(\mathbf{G}, U, \alpha) \equiv < G \cup U \cup \{0\}, \cdot >$, *where "·" is defined as follows:*

$$x \cdot y = \begin{cases} \alpha_x(y) & \text{if } x \in G \text{ and } y \in U \\ \text{the product in } \mathbf{G} & \text{if } x, y \in G \\ 0 & \text{otherwise} \end{cases}$$

8. $L(\mathbf{G}, U, \alpha)$ *is defined as follows. Again the universe of the algebra is* $G \cup U \cup \{0\}$. α *is defined to be a representation of* $\mathbf{G}^{op}$ *into* U*; that is,* $\alpha_{g \cdot h}(u) =$

[6] We note that the RS conjecture had not yet been posed at the time of the writing of [18].

$\alpha_h(\alpha_g(u)))$, *for* $g, h \in G$, *and* $u \in U$. *The operation of* $L(\mathbf{G}, U, \alpha)$ *is defined:*

$$y \cdot x = \begin{cases} \alpha_x(y) & \text{if } x \in G \text{ and } y \in U \\ \text{the product in } \mathbf{G} & \text{if } x, y \in G \\ 0 & \text{otherwise} \end{cases}$$

Definition 21 *Let* $n > 1$. *A semigroup* **S** *is a* **group of exponent n** *iff*

$$\mathbf{S} \models x^{n+1} \approx x, x^n \approx y^n.$$

Note that if a semigroup **S** is a group of exponent n, **S** becomes a group if we define $x^{-1} = x^{n-1}$.

Definition 22 *1. For* $n > 1$, *define* $V_1{}^n \equiv \{\mathbf{S} : \mathbf{S} \models ((xy)^{n+1} \approx xy) \wedge (x^n yz \approx x^n yx^n z) \wedge (xyz^n \approx xz^n yz^n)\}$.

2. For $n > 1$, *define* $V_2{}^n \equiv \{\mathbf{S} : \mathbf{S} \models ((x)^{n+1} y \approx xy) \wedge (x^n y^n z \approx y^n x^n z)\}$.

3. For $n > 1$, *define* $V_3{}^n \equiv \{\mathbf{S} : \mathbf{S} \models (xy^{n+1} \approx xy) \wedge (xy^n z^n \approx xz^n y^n)\}$.

Note that for each $n > 1$ the variety $V_3{}^n$ is the dual of the variety $V_2{}^n$.

3.4.1 Results of Golubov-Sapir

From [9] we include the following classification of all residually finite semigroups varieties:

Theorem 14 (Golubov-Sapir) *Let* **I** *be a two element semilattice*[7]; **P** *a semigroup with universe* $\{a, b, 0\}$ *with* $a^2 = a, ab = b$, *and all other products equaling* 0;

[7] $\mathbf{I} = \mathbf{2}_0$.

$\mathbf{Q}$ *the dual semigroup of* $\mathbf{P}$. *Then a variety of semigroups,* V, *is residually finite iff it is generated by one of the following semigroups:*

$\mathbf{L}_0 \times \mathbf{C} \times \mathbf{Q}$	$\mathbf{C} \times \mathbf{P} \times \mathbf{R}_0$	$\mathbf{L}_0 \times \mathbf{G} \times \mathbf{I} \times \mathbf{R}_0$	$\mathbf{G} \times \mathbf{I}$
$\mathbf{G} \times \mathbf{2}_0$	$\mathbf{L}_0 \times \mathbf{G} \times \mathbf{R}_0$	$\mathbf{L}_0 \times \mathbf{G} \times \mathbf{I}$	$\mathbf{C} \times \mathbf{Q}$
$\mathbf{L}_0 \times \mathbf{G} \times \mathbf{2}_0$	$\mathbf{G} \times \mathbf{2}_0 \times \mathbf{R}_0$	$\mathbf{G} \times \mathbf{2}_0 \times \mathbf{I}$	$\mathbf{G} \times \mathbf{R}_0$
$\mathbf{L}_0 \times \mathbf{G} \times \mathbf{2}_0 \times \mathbf{I}$	$\mathbf{G} \times \mathbf{2}_0 \times \mathbf{I} \times \mathbf{R}_0$	$\mathbf{L}_0 \times \mathbf{G} \times \mathbf{2}_0 \times \mathbf{R}_0$	$\mathbf{C} \times \mathbf{P}$
$\mathbf{L}_0 \times \mathbf{G} \times \mathbf{2}_0 \times \mathbf{I} \times \mathbf{R}_0$	$\mathbf{L}_0 \times \mathbf{G}$	$\mathbf{G} \times \mathbf{I} \times \mathbf{R}_0$	$\mathbf{G}$,

where $\mathbf{G}$ *is a finite group all of whose Sylow subgroups are abelian; and* $\mathbf{C}$ *is a finite cyclic group of prime power order.*

Golubov and Sapir show that any residually finite semigroup variety must be a subvariety of V_i^n for some $i \in \{1, 2, 3\}$. They then classify the subdirect irreducibles of each such subvariety.

Theorem 15 (Golubov-Sapir) *1. If* $V \subseteq V_1{}^n$, *and* V *is residually finite then the only possible subdirectly irreducible semigroups in* V *are (up to isomorphism):*

$$\mathbf{L}_2, \mathbf{L}_2{}^{(0)}, \mathbf{R}_2, \mathbf{R}_2{}^{(0)}, \mathbf{2}_0, \mathbf{I}, \mathbf{G}_i \mathbf{G}_i{}^0,$$

where $\mathbf{G}_i$ *is a subdirectly irreducible finite group all of whose Sylow subgroups are abelian, and* $i \in \{1, \cdots, m\}$, *where* m *is the number of subdirectly irreducible groups in* V.

2. If $V \subseteq V_2{}^n$, *and* V *is residually finite then the only possible subdirectly irreducible semigroups in* V *are (up to isomorphism):*

$$\mathbf{R}_2, \mathbf{R}_2{}^{(0)}, \mathbf{2}_0, \mathbf{I}, \mathbf{P}, \mathbf{C}_i, \mathbf{C}_i{}^{(0)}, \mathbf{C}_i \bigtriangledown \mathbf{P}^{(0)},$$

where $\mathbf{C}_i$ *is a subdirectly irreducible cyclic group,* $i \in \{1, \cdots, m\}$ *(*m *is defined as in 1), and* $S \bigtriangledown T^{(0)}$ *is the* **Cartesian 0-product** *of the semigroups* S *and*

$T^{(0)}$; that is, the Rees quotient semigroup of $S \times T$ modulo the congruence consisting of all pairs with second entry zero.

3. *If $V \subseteq {V_2}^n$, and V is residually finite then the only possible subdirectly irreducible semigroups in V are (up to isomorphism):*

$$\mathbf{L}_2, {\mathbf{L}_2}^{(0)}, \mathbf{2}_0, \mathbf{I}, \mathbf{Q}, \mathbf{C}_i, {\mathbf{C}_i}^{(0)}, \mathbf{C}_i \bigtriangledown \mathbf{Q}^{(0)},$$

where $\mathbf{C}_i$ is a subdirectly irreducible cyclic group, $i \in \{1, \cdots, m\}$ (m is as in 1), and $S \bigtriangledown T^{(0)}$ is as in 2.

As there are (up to isomorphism) only finitely any subdirect irreducible semigroups within any of the three possible families of residually finite semigroups. This proves the Quackenbush conjecture for semigroups[8].

3.4.2 Results of McKenzie - Part I

The following theorem gives a necessary condition of all residually small varieties of semigroups.

Theorem 16 (McKenzie) *Let V be a residually small variety of semigroups. Then V must be a subvariety of ${V_j}^n$, for some $j \in \{1, 2, 3\}$.*

McKenzie then classifies, within each of V_i^n for $i \in \{1, 2, 3\}$, the subclass of subdirectly irreducible semigroups. We list (after McKenzie) the subdirectly irreducible members of each of three families of varieties described in Theorem 16.

We first state a Proposition of McKenzie's.

[8] See Propositions 1 through 5 in [9].

Proposition 1 (McKenzie) *If* $\mathbf{G}$ *is a group of order at least two, then* $\mathbf{R}(\mathbf{G}, U, \alpha)$ *is subdirectly irreducible iff it satisfies the following:*

1. $g \neq h$ *implies that* $\alpha_g \neq \alpha_h$ *(that is,* α *is* **faithful***).*

2. $\alpha(\mathbf{G})$ *acts transitively on* U*.*

3. *For some* $u \in U$, $\{g \in G : \alpha_g(u) = u\}$ *is a completely meet irreducible member of the lattice of subgroups of* $\mathbf{G}$*.*

Theorem 17 (McKenzie) *Let* ${G^{(n)}}_i$ *be the subvariety of* ${V_i}^n$ *consisting of all groups of exponent* n*.*

1. *The subdirectly irreducible members of* ${V_1}^n$ *are* $\mathbf{G}$ *and* $\mathbf{G}^{(0)}$*, where* $\mathbf{G} \in {G^{(n)}}_1$ *is subdirectly irreducible,* $\mathbf{2}$, $\mathbf{2}_0$, $\mathbf{L}_0$, $\mathbf{R}_0$, ${\mathbf{L}_0}^{(0)}$*, and* ${\mathbf{R}_0}^{(0)}$*.*

2. *The subdirectly irreducible members of* ${V_2}^n$ *are* $\mathbf{G}$ *and* $\mathbf{G}^{(0)}$*, where* $\mathbf{G} \in {G^{(n)}}_2$ *is subdirectly irreducible,* $\mathbf{R}(\mathbf{G}, U, \alpha)$*, where* $\mathbf{G} \in {G^{(n)}}_2$ *and the conditions of Proposition 1 hold,* $\mathbf{2}$, $\mathbf{2}_0$, $\mathbf{R}_0$*, and* ${\mathbf{R}_0}^{(0)}$*.*

3. *The subdirectly irreducible members of* ${V_3}^n$ *are* $\mathbf{G}$ *and* $\mathbf{G}^{(0)}$*, where* $\mathbf{G} \in {G^{(n)}}_3$ *is subdirectly irreducible,* $\mathbf{L}(\mathbf{G}, U, \alpha)$*, where* $\mathbf{G} \in {G^{(n)}}_3$ *and the conditions of Proposition 1 hold,* $\mathbf{2}$, $\mathbf{2}_0$, $\mathbf{L}_0$*, and* ${\mathbf{L}_0}^{(0)}$*.*

Using model theoretical arguments, McKenzie proves the following theorem :

Theorem 18 (McKenzie) *If* $V(\mathbf{A})$ *is residually finite where* $\mathbf{A}$ *is a finite semigroup, then* $V(\mathbf{A})$ *has only finitely many non-isomorphic subdirectly irreducible semigroups.*

As alluded to previously, Theorem 18 proves the Quackenbush conjecture for semigroups.

3.4.3 Results of McKenzie - Part II

In [18], McKenzie partially answered some of the questions left over from [16]. His results are sufficient to both characterize all residually finite varieties of semigroups, and prove the RS conjecture true for semigroups. We prove the latter result.

We state three theorems from [18].

Theorem 19 (McKenzie) *If V is a variety of groups, λ a cardinal, and*

1. *for every $\mathbf{G} \in V$, every strictly meet irreducible element $\mathbf{N}$ of the lattice of normal subgroups of $\mathbf{G}$ satisfies $[\mathbf{G} : \mathbf{N}] < \lambda$; and*
2. *for every $\mathbf{G} \in V$, every strictly meet irreducible element $\mathbf{H}$ of the lattice of subgroups of $\mathbf{G}$ satisfies $[\mathbf{G} : \mathbf{H}] < \lambda$.*

Then every finite group in V is abelian.

To state our next theorem, we need some terminology. Let $N = V\{\mathbf{2}, \mathbf{2}_0, \mathbf{L}_0, \mathbf{R}_0\}$. Then N has sixteen subvarieties, generated by the subsets of its generating set. After McKenzie, we denote these by N_i, for $i \in \{0, \cdots, 15\}$. We let $R = V\{\mathbf{R}_0, \mathbf{P}\}$, and $L = V\{\mathbf{L}_0, \mathbf{Q}\}$. Then R (L) has two subvarieties not in N; namely, $R_0 \equiv R$ and $R_1 \equiv V\{\mathbf{P}\}$ ($L_0 \equiv L$ and $L_1 \equiv V\{\mathbf{Q}\}$).

Theorem 20 (McKenzie) *A variety V of semigroups is residually small iff the groups in V constitute a subvariety W of $G^{(n)}$ for some n and one of the following holds:*

1. *$V = N_i \vee W$ for some $i < 16$ and W satisfies 1. of Theorem 19 for some λ.*

2. $V = R_i \vee W$ *or* $V = L_i \vee W$ *for some* $i < 2$ *and* W *satisfies both 1. and 2. of Theorem 19 for some* λ.

To describe the subdirect irreducibles in each of the twenty classes of Theorem 20, we need more notation. Let $\mathbf{H}$ be a subgroup of a group $\mathbf{G}$. Define a semigroup $R(\mathbf{G},\mathbf{H})$ to have universe the disjoint union of $\mathbf{G}$, $\{x \cdot H : x \in G\}$, and $\{0\}$, and with an operation defined so that $\mathbf{G}$ is a subsemigroup, $x\cdot(y\cdot H) = (xy)\cdot H$, and all other products equal 0. The semigroup $L(\mathbf{G},\mathbf{H})$ is defined to be the dual of $R(\mathbf{G},\mathbf{H})$. For K a class of groups, let $R(G)$ ($L(G)$) be the class of all subdirectly irreducible semigroups $R(\mathbf{G},\mathbf{H})$ ($L(\mathbf{G},\mathbf{H})$) with $\mathbf{G} \in K$.

The class $F \equiv \{\mathbf{2},\mathbf{2_0},\mathbf{L_0},\mathbf{R_0},\mathbf{P},\mathbf{Q},\mathbf{L_0}^{(0)},\mathbf{R_0}^{(0)}\}$ is (up to isomorphism) the class of subdirectly irreducible members of $N \cup R \cup L$[9].

Theorem 21 *Let* $V = G \vee W$*, where* W *is one of the classes* N, R, L*, and* G *is a variety of groups of finite exponent. Then*

$$V_{si} = \begin{cases} (W \cap F) \cup G_{si} \cup G_{si}^{(0)} & \textit{if } W = N \\ (W \cap F) \cup G_{si} \cup G_{si}^{(0)} \cup R(G) & \textit{if } W = R \\ (W \cap F) \cup G_{si} \cup G_{si}^{(0)} \cup L(G) & \textit{if } W = L \end{cases}$$

The classification of all residually small varieties of semigroups is still incomplete. As McKenzie states in [18], such a classification depends on both the classification of residually small varieties of groups of finite exponent, and the classification of varieties of groups possessing property 2. in Theorem 19.

From the above results, we can (with the help of some of the results of the next section) prove the RS conjecture for semigroups.

[9]See [18].

PROOF of the RS conjecture for semigroups. Let **S** be a finite semigroup. Assume that $V = V(\mathbf{S})$ has residual character $< \infty$. Then, by Theorem 20, the groups in V constitute a subvariety G of $G^{(n)}$ for some n, and either $V = N_i \vee G$, for some $i < 16$ and G satisfies 1. of Theorem 19 for some λ, or V equals one of $R_i \vee G$ or $L_i \vee G$ for some $i < 2$, and G satisfies both 1. and 2. of Theorem 19 for some λ.

Assume the first case holds. By Theorem 21, $V_{si} = (N \cap F) \cup G_{si} \cup {G_{si}}^{(0)}$. Now, $(N \cap F)$ contributes only finitely many non-isomorphic subdirectly irreducible semigroups. ${G_{si}}^{(0)}$ will contribute only finitely many non-isomorphic subdirectly irreducible semigroups if G_{si} does. We show that G has residual character $< \omega$. As G satisfies 1. of Theorem 19, G is residually small. But G is finitely generated[10]. By the results of [7] (to be discussed in the next section), G has residual character $< \omega$.

Assume the second case holds. Then $V_{si} = (W \cap F) \cup G_{si} \cup {G_{si}}^{(0)} \cup W(G)$, where $W = R$ or $W = L$. We prove V is residually finite in the case $W = R$ (the case $W = L$ follows similarly). As G satisfies now both 1. and 2. of Theorem 19, every finite group in G is abelian. As in the first case, G has residual character $< \omega$. The question is now whether $R(G)$ is finite. As every member of G_{si} is abelian, so is every group in G. If $\mathbf{H} \subseteq \mathbf{G}$ is a pair of groups, then the semigroup $R(\mathbf{G},\mathbf{H})$ is subdirectly irreducible iff **H** is a strictly meet irreducible element in the lattice of subgroups of **G**, which contains no nontrivial normal subgroup of **G**[11] But then $R(G)$ is a finite set of semigroups of the form $R(\mathbf{C}, \{1\})$, where **C** is a cyclic group of prime power order.

In either case, $\kappa(\mathbf{S}) < \omega$. □

[10]See Lemma 28 of [16].

[11]See p.145 of [18].

3.5 Congruence Modular Varieties

In 1981, Ralph Freese strengthened Taylor's result by dropping the condition of congruence permutability[12]. Freese and McKenzie strengthened this result, in that the variety need only be congruence modular. The purpose of this section is to explore the validity of the RS conjecture in the congruence modular case.

The following is the main result of [7], which we state without proof:

Theorem 22 *Let* $\mathbf{A} \in V$*, where* V *is congruence modular, with* $|A| = m < \omega$*. Then the following are equivalent:*

1. $\kappa(\mathbf{A}) < \infty$.

2. *V(A) is residually* $\leq (l+1)!m$*, where* $l = m^{m^{m+1}}$.

3. *For any* $\alpha, \beta \in Con(B)$*, where* $\mathbf{B} \subseteq \mathbf{A}$*,* $\alpha \leq [\beta, \beta] \rightarrow \alpha = [\beta, \alpha]$.

To prove this result we require the commutator theory of modular varieties. We do not develop this theory here, and refer the reader to [7]. The congruence modular varieties are a broad class of algebras, containing, for example, the varieties of classical algebra (groups, rings, lattices, etc.). Accepting Theorem 22 and repressing one's historical hindsight, it is easy to see how the RS conjecture could be believed to be true.

We prove a special case of Ol'shanksii's theorem (for finitely generated varieties) in the form of two theorems.

Let **G** be a finite group.

[12]The result was unpublished.

Theorem 23 *If every Sylow subgroup of* $\mathbf{G}$ *is abelian, then* $\kappa(\mathbf{G}) < \omega$.

PROOF. We follow a proof as in [7]. Define a finite algebra $\mathbf{A}$ to be **critical** iff $\mathbf{A} \notin HSP((HS)*(\mathbf{A}))$, where $(HS)*(\mathbf{A})$ is $\{\mathbf{B} \in HS(\mathbf{A}) : |B| < |A|\}$. Let $(*)$ be the property of having no nonabelian Sylow subgroups.

We show that if $\mathbf{G}$ is a finite group satisfying $(*)$, then $V(\mathbf{G})$ has only finitely many finite subdirectly irreducibles (up to isomorphism); the result will then follow from Quackenbush's theorem. It is not hard to check that if $\mathbf{G}$ satisfies $(*)$, then so will any finite group in $V(\mathbf{G})$ (it is enough to show that $(*)$ is preserved by H, S, and finite products). Kovács and Newman[13] showed that any finite subdirectly irreducible group satisfying $(*)$ is critical. Oates and Powell[14] proved that $V(\mathbf{G})$ has only finitely many critical groups, which establishes the result. □

Theorem 24 *If some Sylow subgroup of G is nonabelian, then* $\kappa(\mathbf{G}) = \infty$.

PROOF. The proof here is inspired by the general case in [7].

Step 1 Let $\mathbf{H} \in HS(\mathbf{G})$, be chosen so that H has a nonabelian Sylow subgroup and $|H|$ is minimum. Then $\mathbf{H}$ is subdirectly irreducible. To see this, note that since $|H|$ is finite, $\mathbf{H}$ is isomorphic to a finite subdirect product of finite subdirectly irreducible algebras $\mathbf{S}_i$, each of which are homomorphic images of $\mathbf{H}$[15]. If $\mathbf{H}$ is not subdirectly irreducible, each of the $\mathbf{S}_i$ must have cardinality strictly less $|H|$; hence, they all satisfy $(*)$ by the minimality of $|H|$. As alluded to in the proof of Theorem 23, groups satisfying $(*)$ form a local subvariety of $V(\mathbf{G})$, which gives us our contradiction.

[13]See [14].

[14]See [23].

[15]See Corollary 8.7 of [3].

Let **N** be the monolith of **H**. If the commutator subgroup, **H**$'$, was a trivial subgroup, then **H** would be abelian. Hence, each of the subgroups of **H** would be abelian, contradicting our hypothesis. We next show that **H** is a p-group; therefore, the center of **H**, $Z(\mathbf{H})$, is nontrivial. As **H** is finite, $|H| = {p_1}^1 \cdots {p_n}^n$, for primes $p_1, \ldots, p_n$. By hypothesis, **H** has a nonabelian Sylow p-subgroup, say **S**. As $|S| = {p_i}^i$ for some $i \in \{1, \ldots, n\}$, it follows that **S** will have a nonabelian p_i subgroup; namely, itself. By the minimality of $|H|$, it follows that **H**=**S**, and we are done.

As $Z(\mathbf{H})$ and **H**$'$ are normal subgroups of **H**, they contain **N**.

Step 2 Let K be the normal subgroup of **H**×**H** generated by $\{< x, x^{-1} >: x \in \mathbf{H}\}$. We show that $\mathbf{N}\times\{1\} \subseteq K$:

As $\mathbf{N}\subseteq\mathbf{H}'$, it suffices to show that $< [x,y], 1 > \in K$ for all $x, y \in H$. But this follows since: $< [x,y], 1 >=< x^{-1}, x > (< y, x >^{-1}< x, x^{-1} >< y, x >)$.

Step 3 Let λ be an infinite cardinal. Let

$$\mathbf{B} \equiv \{f \in H^\lambda : f(i) = 1 \text{ for all but finitely many } i < \lambda\}.$$

It is easy to show that **B** is a subgroup of $\mathbf{H}^\lambda$.

For each $i < j < \lambda$ define

$$K_{ij} \equiv \{f \in B :< f(i), f(j) > \in K \text{ and } f(k) = 1 \text{ for all } k \neq i, j\}.$$

Further define

$$\mathbf{M} \equiv \{f \in B : range(f) \subseteq N \text{ and } \prod_{i<\lambda} f(i) = 1\}.$$

Then $K_{ij} \lhd \mathbf{B}$, for all $i < j$, and $\mathbf{M} \lhd \mathbf{B}$ (because **N** is normal and is in the center of **H**).

For each $c \in H$ and $i < j$ define $f_i{}^c \in B$ by:

$$f_i{}^c(j) = \begin{cases} c & \text{if } j = i \\ 1 & \text{otherwise.} \end{cases}$$

Fix $a \in N - \{i\}$. The following facts will hold:

Fact 1 From Step 2 and by the definition of K_{ij}, it follows that $f_i{}^a \in K_{ij}$, for all $i < j < \lambda$.

Fact 2 By the definition of M, $f_i{}^a \equiv f_j{}^a(\text{mod } \mathbf{M})$, for all $i < j < \lambda$.

Fact 3 Therefore, $f_0{}^a \in M \vee K_{ij}$, for all $i < j < \lambda$.

Fact 4 Again by the definition of M, $f_0{}^a \notin M$.

Step 4 Let **S** be any normal subgroup of **B** maximal with respect to the property that $\mathbf{M} \subseteq \mathbf{S}$, and $f_0{}^a \notin S$. Hence, $\mathbf{B}/\mathbf{S}$ is subdirectly irreducible.

We show that $|B/S|$ can be arbitrarily large. To do this I show that for any infinite cardinal κ, if we chose $\lambda = (2^\kappa)^+$, then $|B/S| \geq \kappa^+$.

Fix $i < j < \lambda$. K_{ij} is the smallest normal subgroup (of **B**) containing the set

$$\{f_i{}^x(f_j{}^x)^{-1} : x \in H\}$$

(From the definitions, one can check that K_{ij} is the normal closure in **B** of $\{f_i{}^x(f_j{}^x)^{-1} : x \in H\}$.) By Fact 3, observe that $\mathbf{M} \subseteq \mathbf{S}$, and $f_0{}^a \notin S$ imply that $K_{ij} \nsubseteq \mathbf{S}$. Hence,

$$\{f_i{}^x(f_j{}^x)^{-1} : x \in H\} \nsubseteq \mathbf{S}$$

Therefore, for all $i < j < \lambda$, there exists $x \in H$ such that $f_i{}^x(f_j{}^x)^{-1} \notin S$. By the Erdös-Rado Theorem, there exists $X \subseteq \lambda$, such that $|X| = \kappa^+$, and a $\mathbf{c} \in H$, such that

$$(3.1) \qquad (i, j \in X \text{ and } i < j) \to f_i{}^c(f_j{}^c)^{-1} \notin S$$

By 3.1 the elements f_i^c are pairwise inequivalent (mod $\mathbf{S}$). Hence, $|B/S| \geq |X| \geq \kappa^+$. As κ was arbitrary, it follows that $V(\mathbf{G})$ is residually large. □

3.6 Varieties of K-algebras

In 1982 McKenzie proved the analogue of the RS conjecture for locally finite varieties of K-algebras. He also gave a syntactic condition equivalent to residual smallness for all varieties of K-algebras. In this section, we describe these results.

Definition 23 *Let K be a commutative associative ring with identity. A* **K- algebra, A**, *is an algebra $< A, +, \cdot, -, 0, t_k (k \in K) >$ so that $< A, +, \cdot, -, 0 >$ is an associative ring, and the map $k \mapsto t_k$ is a ring (with identity) homomorphism of K into End($< A, + >$) so that $t_k(x \cdot y) = x \cdot t_k(y) = t_k(x) \cdot y$, for all $x, y \in A$, and $k \in K$.*

For a fixed K, the class of all K-algebras is a variety, which we denote by A_K. A K-algebra has an identity iff it has a constant, 1, and satisfies $x \cdot 1 \approx x \approx 1 \cdot x$. The variety of all K-algebras with identity is denoted by ${A_K}^{(1)}$.

The following is McKenzie's main result:

Theorem 25 (McKenzie) *Let V be a variety so that either $V \subseteq A_K$ or $V \subseteq {A_K}^{(1)}$. Then the following are equivalent:*

1. *V is residually small.*
2. *$I_0 \cap [I_1, I_1] \subseteq [I_0, I_1]$ holds for all ideals in every algebra of V.*
3. *V satisfies an identity of the form $x \cdot y \approx f(x, y)$ whose monomials on the right side each have a total degree not less than 3.*

4. $V \models (x - x^n)(y - y^n) \approx [(x - x^n)(y - y^n)]^n$*, for some integer* $n > 1$*.*

As the proof is largely ring-theoretical, we omit it.

Corollary 7 *Every residually small variety* $V \subseteq {A_K}^{(1)}$ *is residually* $< n$ *for some* $n \in \omega$*. Every residually small and locally finite variety* $V \subseteq A_K$ *is residually* $< n$ *for some* $n \in \omega$*.*

3.7 Monotone Clones

We present a non-central result of McKenzie's that proves the RS conjecture true for a special class of varieties, which are "order-generated". McKenzie studied such varieties as part of his quest to prove the RS conjecture true for all varieties. In such varieties, we also prove that congruence distributivity is equivalent to residual smallness.

We first make some definitions.

Definition 24 1. *An algebra* $\mathbf{A} \equiv < P, F >$ *is* **monotone with respect to an ordered set** $\mathbf{P} \equiv < P, \leq >$ *iff each operation of* $\mathbf{A}$ *preserves the order of* $\mathbf{P}$*.*

2. $\mathbf{A}$ *is* **order primal** *with respect to* $\mathbf{P}$ *iff* $\mathbf{A}$ *is monotone with respect to* $\mathbf{P}$ *and the clone*[16] *of all monotone operations (over* $\mathbf{P}$*) equals the clone of term operations of* $\mathbf{A}$*.*

   Denote: $\mathbf{A}$ *is order primal over* $\mathbf{P}$ *by* $\mathbf{A}(\mathbf{P})$*.*

3. $V(\mathbf{A}(\mathbf{P}))$ *is called an* **order-primal variety**.

[16]A clone is a class of functions closed under composition, and containing all the projection functions; for example, the term operations of **A** form a clone.

McKenzie proved that the RS conjecture is true for bounded order-primal varieties ("bounded" in the sense that the underlying order has a least and greatest element). As congruence distributivity is equivalent to residual smallness in such varieties, the result follows as an application of Jónsson's Theorem.

McKenzie's main result is the following theorem.

Theorem 26 (McKenzie) *Let* $\mathbf{P}$ *be a bounded ordered set with 0 and 1 as least and largest elements, respectively, and let* $\mathbf{A}$ *be an algebra monotone over* $\mathbf{P}$ *which has ternary terms* $A(x,y,z)$ *and* $B(x,y,z)$*, such that* $\mathbf{A}$ *satisfies:*

$$A(0,0,x) \approx 0, \; A(0,x,x) \approx A(x,0,x) \approx A(x,x,x) \approx x,$$
$$B(1,1,x) \approx 1, \; B(1,x,x) \approx B(x,1,x) \approx B(x,x,x) \approx x.$$

If $V(\mathbf{A})$ *is residually small, then it is congruence distributive.*

An immediate Corollary is the following:

Corollary 8 (McKenzie) $V = V(\mathbf{A}(\mathbf{P}))$ *(where* $\mathbf{P}$ *is a finite bounded ordered set) is residually small iff it is congruence distributive.*

PROOF. We mentioned in our previous remarks why the reverse direction is true (if V is congruence distributive, then it is residually $< |A| + 1$). For the forward direction define the following term operations: let $A(x,y,z) = z$ unless $x = y = 0$, and define $A(0,0,z) = 0$. Define $B(x,y,z)$ dually. These operations are clearly monotone and satisfy the hypotheses of Theorem 26. □

To prove Theorem 26 we need a technical Lemma.

Lemma 5 (McKenzie) *Let* $\mathbf{A}$ *be a monotone algebra over an ordered set with zero. Assume* $\mathbf{A}$ *has a term operation* $A(x,y,z)$ *as in Theorem 26. If* $V(\mathbf{A})$ *is* *residually small, then* **A** *possesses terms* $d_0(x,y,z),\dots d_n(x,y,z)$ *such that* **A** *satisfies* $d_i(x,0,x) \approx x$ *(for all* $i \leq n$*) and which are Jónsson terms*[17] *except that* **A** *may not satisfy* $d_i(x,y,z) \approx x$.

PROOF ([19]). By hypothesis, there exists a cardinal λ greater than the cardinals of members of $V(\mathbf{A})_{si}$. Define $X \equiv A \times A \times \lambda$. We define elements of A^X. Define f_1, $f_2 \in A^X$ by $f_1(a,b,c) = a$, and $f_2(a,b,c) = b$. Define $f_{1,n}$ and $f_{2,n}$, for $n \in \lambda$, as follows:

$$f_{1,n}(a,b,c) = \begin{cases} a & \text{if } n \neq c \\ 0 & \text{otherwise} \end{cases}$$

$$f_{2,n}(a,b,c) = \begin{cases} b & \text{if } n \neq c \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{S} \leq A^X$ be the subalgebra generated by f_1, f_2, and the $f_{1,n}, f_{2,n}$, for each $n \in \lambda$. Let θ be the congruence in $\mathbf{S}$ generated by the ordered pairs of elements:

$$Z \equiv \{< f_{1,n}, f_{2,n} >: n \in \lambda\}.$$

Our aim is to show that: $< f_1, f_2 > \in \theta$. If this is false, then let ϕ be a congruence of $\mathbf{S}$, maximal with respect to the property of not containing $< f_1, f_2 >$; hence, $\mathbf{B} \equiv \mathbf{S}/\phi$ is subdirectly irreducible. We show, to obtain a contradiction, that $|B| \geq \lambda$. If $|B| < \lambda$, then there exist distinct $i, j \in \lambda$, such that $< f_{1,i}, f_{1,j} > \in \phi$; hence, $< f_{2,i}, f_{2,j} > \in \phi$. Let A stand for the term operation in $\mathbf{S}$ induced by $A(x,y,z)$ in $\mathbf{A}$. Then by the equations satisfied by $A(x,y,z)$, if $m \in \{1,2\}$, we have that: $A(f_{m,i}, f_{m,i}, f_m) = f_{m,i}$ and $A(f_{m,i}, f_{m,j}, f_m) = f_m$. Now, $< f_{m,i}, f_{m,j} > \in \phi$ implies that $< f_m, f_{m,j} > \in \phi$. Then $< f_{1,i}, f_{2,j} > \in \phi$ implies that $< f_1, f_2 > \in \phi$, which contradicts our assumption. Hence, $|B| \geq \lambda$, a contradiction, so that in turn, our assumption that $< f_1, f_2 > \notin \theta$ is erroneous.

[17] See Theorem 12.6 in [3].

We can now show the existence of the desired term operations.

As $< f_1, f_2 >\in \theta$, there exists a finite sequence of elements of **S**:

$$f_1 = g_0, g_1, \ldots, g_{m-1}, f_2 = g_m$$

such that for every $k < m$, there exists a binary polynomial operation $p_k(x, y)$ of **S**, and some $i_k \in \lambda$, such that

$$g_k = p_k(f_{1,i_k}, f_{2,i_k}) \text{ and } g_{k+1} = p_k(f_{2,i_k}, f_{1,i_k})$$

For each of the generators "f" of **S**, there is a corresponding term operation $B(x, y)$ of **A** so that $f(a, b, c) = B(a, b)$ unless $f = f_{1,c}$ or $f_{2,c}$ (take either $B(x, y) = x$ or $B(x, y) = y$.) I claim that for every $f \in$ **S**, there exists a binary term operation $B(x, y)$ and a finite subset F of λ, so that

$$B(a, b) = f(x) \text{ for all } x = (a, b, c) \in X \text{ such that } c \notin F.$$

We prove this by induction. As **Q**$= Sg^{\mathbf{A}^X}(Z) = \cup_{n\in\omega} E^n(Z)$ (our notation E^n is taken from Theorem 3.2 of [3]), $f \in \mathbf{Q}$ is in E^n for some $n \in \omega$. Then either $f \in E^{n-1}$ in which case we are done by induction, or $f = F^{\mathbf{A}^X}(g_1, \cdots, g_m)$, for some m-ary fundamental operation F of $\mathbf{A}^X$, and $g_i \in E^{n-1}$, for $i \in \{1, \cdots m\}$. By inductive hypothesis, for each g_i there is a binary term operation $B_i(x, y)$ and a finite set F_i such that for all $\in \{1, \cdots, m\}$

$$B_i(a, b) = g_i(x) \text{ for all } x = (a, b, u) \in X \text{ such that } u \notin F_i.$$

Let $B(x, y)$ be the binary term operation $F^{\mathbf{A}^X}(B_1(x, y), \cdots, B_m(x, y))$, and let F be the finite set $\bigcup_{1\leq i\leq m} F_i$. Take $(a, b, c) \in X$, such that $c \notin F$. Then $c \in \bigcap_{1\leq i\leq m} \lambda - F_i$. This means that for such a c, $g_i(a, b, c) = B_i(a, b)$ by inductive hypothesis. Hence, for $c \notin F$,

$$\begin{aligned} f(a,b,c) &= F^{\mathbf{A}^X}(g_1(a,b,c),\cdots,g_m(a,b,c)) \\ &= F^{\mathbf{A}^X}(B_1(a,b),\cdots,B_m(a,b)) \\ &= B(a,b), \end{aligned}$$

as desired.

For $k < m$ let the operation $B(x,y)$ corresponding to g_k be $B_k(x,y)$. Then $B_0(x,y) = x$, $B_m(x,y) = y$, and

$$B_s(a,b) = p_s(f_{1,i_s}(x), f_{2,i_s}(x))$$

and

$$B_{s+1}(a,b) = p_{s+1}(f_{2,i_s}(x), f_{1,i_s}(x))$$

for all $x = (a,b,u) \in X$ so that $u \notin L_s$, for some fixed finite subset L_s of λ.

Fix $k < m$. The polynomial operation $p_k(x,y)$ can be expressed as:

$$p_k(x,y) = K_k(x,y,f_{1,i_k},f_{2,i_k},f_{1,j_0},f_{2,j_0}\ldots,f_{1,j_{l-1}},f_{2,j_{l-1}},f_1,f_2),$$

for some $l \in \omega$, some $2l+6$-ary operation K_k of $\mathbf{A}$, and some distinct elements $j_0,\ldots j_{l-1} \in \lambda$, distinct from the i_k. By evaluating the above equations at any $x \in X$ such a that $x = (a,b,c)$ where $c \notin L_k \cup \{j_0,\cdots,j_{l-1}\}$, we have that

$$\begin{aligned} B_k(a,b) &= K_k(a,b,a,b,\ldots,a,b) \text{ and} \\ B_{k+1}(a,b) &= K_k(b,a,a,b,\ldots,a,b). \end{aligned}$$

Let $R_k(x,y,z,w) = K_k(x,y,z,w,z,w,\ldots,z,w)$.

Hence,

$$B_k(a,b) = R_k(a,b,a,b), \text{ and} \tag{3.2}$$

$$B_{k+1}(a,b) = R_k(b,a,a,b). \tag{3.3}$$

We next show that

$$(3.4) \qquad R_k(0,0,z,z) = R_k(0,z,z,z) = R_k(z,0,z,z) = R_k(z,z,z,z) = z$$

As every pair of generators of θ agree at $x = (a,a,b) \in X$, we have

$$g_k(x) = g_{k+1}(x) = g_0(x) = a.$$

Evaluating at $x = (a,a,i_k)$, and at $x = (a,a,c)$, for $c \notin \{i_k, j_0, \cdots, j_{k-1}\}$ we the find that

$$K_k(0,0,0,0,a,a,\ldots,a,a) = a = K_k(a,\ldots,a).$$

Equation 3.4 now follows from this and our definition of R_k, as K_k is monotone. Define:

$$(3.5) \qquad D_{2k+1}(x,y,z) = R_k(y,z,x,z),\ D_{2k+2}(x,y,z) = R_k(z,y,x,z).$$

As $k < m$ was arbitrary, we can define $D_i(x,y,z)$ in this way for $1 \le i \le 2m$. Equations 3.2 and 3.5 give us that

$$\begin{aligned} D_{2k+1}(x,z,z) &= D_{2k+2}(x,z,z) \qquad \text{for } 0 \le k < m, \text{ and} \\ D_{2k}(x,x,z) &= B_k(x,z) = D_{2k+1}(x,x,z) \text{ for } 1 \le k < m\ . \end{aligned}$$

Equations 3.4 and 3.5 give us that $D_i(x,0,x) = x$ for $1 \le i \le 2m$. Further, $D_1(x,x,z) = B_0(x,z) = x$ and $D_{2m}(x,x,z) = B_m(x,z) = z$. Letting $D_0(x,y,z) = x$ and $D_{2m+1}(x,y,z) = z$, we see that the operations

$$D_0, \ldots D_{2m+1}$$

are the required term operations. □

PROOF of Theorem 26. We construct Jónsson terms for $V(\mathbf{A})$. By hypothesis we have terms $A_i(x,y,z)$ as in Lemma 5. As our hypotheses are self-dual, we will obtain a dual set of term operations $B_i(x,y,z)$ satisfying a dual set of term conditions. Define

$$C_{i,j}(x,y,z) = A_i(x, B_j(x,y,z), z).$$

As $B_i(a,1,a) = a$ implies that $B_i(a,b,a) \leq a$, $A_i(a,0,a) = A_i(a,a,a) = a$, and A_i is monotone, we have that $C_{i,j}(a,b,a) = a$. Further,

$$\begin{aligned} C_{1,0}(x,y,z) &= x, \\ C_{k,2j}(x,x,z) &= C_{k,2j+1}(x,x,z), \\ C_{k,2j+1}(x,z,z) &= C_{k,2j+2}(x,x,z), \\ C_{2i+1,m}(x,y,z) &= C_{2i+2,m}(x,y,z), \\ C_{2i,0}(x,y,z) &= C_{2i+1,0}(x,y,z). \end{aligned}$$

Thus, the sequence:

$$C_{2,0}, \ldots, C_{1,m} = C_{2,m}, \ldots, C_{2,0} = C_{3,0}, C_{3,1}, \ldots, C_{3,m}, \cdots C_{n,0}$$

will yield a set of Jónsson terms. □

McKenzie also shows that residual smallness in a bounded order- primal variety is equivalent to congruence modularity. We will not pursue this here.

3.8 Results from Tame Congruence Theory

The RS conjecture first appeared in the book "The Structure of Finite Algebras", written by David Hobby and Ralph McKenzie. In this book, the authors present tame congruence theory – a powerful tool in the study of finite algebras – for the first time. Among the many results in that book is the following:

Theorem 27 (Hobby-McKenzie) *Every locally finite, residually small variety that omits types 1 and 5 is congruence modular.*

Hence, based on the results of [7], the RS conjecture is true for locally finite varieties that omit type 1 and 5. It follows that the RS conjecture is true for every locally finite variety whose congruence lattices obey a non-trivial lattice equation[18]. This result proves the RS conjecture for the broadest class of algebras we have come across, and was (not surprisingly) the strongest evidence of the truth of the RS conjecture. As a Corollary to this result, we see that any finite algebra that generates a non-congruence modular variety that omits types 1 and 5 is residually large. Here we again have more evidence that residual smallness is an uncommon property of varieties. On this matter, McKenzie states in [20]: '[The results from [11]] show in a striking fashion just how restrictive the hypothesis of residual smallness is.'

The proof of Theorem 27 is beyond the scope of this paper.

3.9 Open Problems

The last twenty years has witnessed a large amount of work on residually small varieties. We feel that there are still unanswered questions concerning such varieties. In this final section we sample a few open problems and conjectures which may point the way for future research.

First, we have noted that the restricted Quackenbush conjecture is still open. This problem appears much more difficult than the unrestricted question. As we mentioned earlier, a resolution of the restricted conjecture may require new methods or techniques.

[18]See Theorem 9.19 of [7].

We restate a subset of the open problems as stated in [21].

1. Does there exist a finite algebra **A** with $\omega \leq \kappa(\mathbf{A}) < \infty$ so that

   i) $|A| = 3$?

   ii) $V(\mathbf{A})$ omits type 5?

2. Does there exist a locally finite variety V of finite type with $\kappa(V) = \omega$?

3. Does there exist a non-finitely based[19] finite algebra **A**, of finite type, with $\kappa(\mathbf{A}) < \omega$?

4. Does there exist a congruence distributive variety V of finite type with $\kappa(V) = \omega$?

5. Suppose that $\kappa(V) = \omega$ and every subvariety of V is generated by a finite, congruence distributive algebra. Is V congruence distributive?

The number "3" is relevant in Problem 1 i). McKenzie's counterexample in [21] has cardinality 4, and (as McKenzie states in [21]) Taylor proved all two element algebras satisfy the RS conjecture.

By our previous discussion of results from tame congruence theory, type 1 and 5 algebras would be possible culprits to violate the RS conjecture. In [21], the counterexample admits type 5. Therefore, an algebra answering Problem 1 ii) affirmatively would admit type 1.

McKenzie expects that the answer to Problem 1 ii) is no. McKenzie believes that the answer for Problem 3 is yes.

We present a problem that in some sense generalizes the RS conjecture.

[19] See Definition 4.1 of [3].

Problem 1 *For which $\eta \geq \omega$ does it follow that:*

$$\mathcal{K}(\mathbf{A}) \geq \eta \rightarrow \mathcal{K}(\mathbf{A}) = \infty \tag{3.6}$$

for an algebra $\mathbf{A}$ *of cardinality* $< \eta$*?*

McKenzie has shown that such a cardinal cannot be ω. If we weaken the problem by insisting that $\mathbf{A}$ be infinite, then we have shown in Chapter 1 that the strongly inaccessible cardinals would satisfy condition 3.6 in Problem 1.

Another problem is the classification of all residually small varieties of groups[20]. Apart from the commutator condition of [7], we know of no other intrinsic characterization of such varieties.

[20]This problem was mentioned in [20].

Conclusion

A property of mathematical structures may be considered useful if it meets any one of the following conditions[21]:

1. The property is nontrivial; that is to say, it does not hold in every structure.
2. Whether the property holds in a structure can be determined by some syntactic or semantic criteria.
3. Consideration of the property may lead to new and unexpected results, or even to new mathematical paradigms.

We claim that residual smallness meets the first two criteria, and is therefore, a noteworthy property of varieties. Further, it may prove to satisfy the third.

Most varieties are not residually small; hence, residual smallness meets the first of our criteria. On the basis of our previous exposition, it could be argued that residual smallness is a rare property of varieties.

Throughout our discussion, we have noted many structural conditions for a variety to be residual small: the condition for the variety of groups determined by Ol'shanksii; Taylor's eleven equivalent conditions for residual smallness; Freese and

[21]We make no claim that this list is exhaustive.

McKenzie's commutator condition for congruence modular varieties; McKenzie's classification of residually small semigroups, and so on.

In answer to the last of our criteria – namely, whether consideration of residual smallness has lead to the unexpected – the results are pending. We include a remark of McKenzie from 1990 which looks hopefully toward the future.

> *... our interest in discovering whether the [RS] conjecture is valid for* all *algebras remains strong. It seems to be a question that had the potential of forcing us to dream of new things yet unseen. We should not like to believe that all interesting varieties and phenomena of varieties have already been discovered.* [[20] p. 188]

Now, after the refutation of the RS conjecture, this remark may yet find justification. Our hope is that the final chapter on residually small varieties is still unwritten, and echoing McKenzie, that our understanding of varieties will continue to evolve in new and unforeseen directions.

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