# Mutually embeddable graphs and the Tree Alternative conjecture

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**Abstract**

We prove that if a rayless tree $T$ is mutually embeddable and non-isomorphic with another rayless tree, then $T$ is mutually embeddable and non-isomorphic with infinitely many rayless trees. The proof relies on a fixed element theorem of Halin, which states that every rayless tree has either a vertex or an edge that is fixed by every self-embedding. We state a conjecture that proposes an extension of our result to all trees.

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## 1 Introduction

A graph $G$ *embeds* in a graph $H$ if $G$ is isomorphic to an induced subgraph of $H$. If $G$ and $H$ are graphs, then we write $G \leq H$ if $G$ embeds in $H$. We write $G \sim H$ if $G \leq H$ and $H \leq G$, and we say that $G$ and $H$ are *mutually embeddable*.

Mutually embeddable finite graphs are necessarily isomorphic, but this is no longer the case for infinite graphs. For example, if the graph $G$ is a disjoint union of cliques, one of each finite cardinality, then $G$ is mutually embeddable with the graph consisting of a disjoint union of cliques with every even cardinality. In [1], we give many examples of mutually embeddable non-isomorphic graphs satisfying strong structural properties. On the other hand, the infinite two-way path is not mutually embeddable with any graph not isomorphic to it.

Define $ME(G)$ to be the set of isomorphism types of graphs $H$ so that $G \sim H$. Define the cardinal $m(G) = |ME(G)|$. Note that $|ME(G)| \leq 2^{|V(G)|}$, so that $m(G)$ is well-defined. For instance, with $|V(G)| = \aleph_0$ (that is, the cardinality of the set of natural numbers), there are examples of graphs where $m(G)$ is one of 1, $\aleph_0$, or $2^{\aleph_0}$. See Figure 1. As first stated in [1], we do not know of

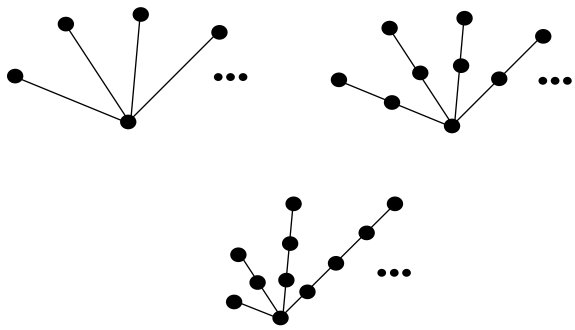

Fig. 1. Examples of countably infinite trees $T$ with $m(T) = 1$, $\aleph_0$, and $2^{\aleph_0}$, respectively.

any example with $m(G)$ finite but larger than 1. The structure of such graphs may prove to be intriguing if they exist.

If $G$ and $H$ are mutually embeddable, then composing an embedding of $G$ into $H$ with an embedding of $H$ into $G$ gives a self-embedding of $G$. Thus, the structure of the monoid of self-embeddings of $G$ may help us to determine the value of $m(G)$. A tree is *rayless* if it does not embed an infinite path. For example, each tree in Figure 1 is rayless. Self-embeddings, automorphisms, and various fixed element properties of rayless trees have been well-studied; for example, see [2–4,6]. Using such properties we are able to prove the following result for rayless trees, and we in fact conjecture an extension to all trees.

**Theorem 1** *If $T$ is a rayless tree, then $m(T)$ is 1 or infinite.*

**Tree Alternative Conjecture.** If $T$ is a tree, then $m(T)$ is 1 or infinite.

The rest of the paper is organized as follows. In Section 2, we prove a version of the Tree Alternative Conjecture for rooted rayless trees; see Theorem 2. In the final section we use a fixed element theorem of Halin's to derive Theorem 1 from Theorem 2. This suggests that if for all graphs $G$ we have that $m(G) = 1$ or $m(G) \geq \aleph_0$, then a proof may use interesting fixed element properties of graphs.

All the graphs we consider are undirected and simple. If graphs $G$ and $H$ are isomorphic, then we write $G \cong H$. We use the notation of [5] for graph theory. We work within ZFC; no additional set-theoretic axioms will be assumed. The set of natural numbers, considered as an ordinal, will be written as $\omega$.

## 2 Mutually embeddability of rooted rayless trees

The class of *rooted rayless trees* consists of all pairs $(T, r)$, where $T$ is a rayless tree and $r$ is some fixed vertex of $T$ called the *root* of $T$. An *embedding of rooted trees* $f : (T, r) \to (T', r')$ is an embedding of $T$ into $T'$ so that $f(r) = r'$; we write $(T, r) \leq (T', r')$. An *isomorphism of rooted trees* is a bijective embedding of rooted trees. If there is an isomorphism of rooted trees $(T, r)$ and $(T', r')$, then we write $(T, r) \cong (T', r')$. The cardinal $m(T, r)$ is defined in the obvious way. The main goal of this section is to prove the following theorem.

**Theorem 2** *If $(T, r)$ is a rayless rooted tree, then $m(T, r)$ is either 1 or is infinite.*

Before we give a proof of Theorem 2, we first introduce the following notation that will simplify matters. Let $\{(T_i, r_i) : i \in I\}$ be a family of rayless rooted trees, and let $r$ be a vertex not in $V(T_i)$, for all $i \in I$. Define

$$\sum_{i \in I} (T_i, r_i)$$

to be the rooted tree $(T, r)$ which has as its root the vertex $r$, so that $r$ is joined to each root $r_i$ of $T_i$, for all $i \in I$. We say that $(T, r)$ is the *sum* of the $(T_i, r_i)$, and each $(T_i, r_i)$ is a *summand* of $(T, r)$.

Note that if $(T,r)$ is a rooted tree, then

$$(T,r)=\sum_{i\in I}(T_i,r_i),$$

where the summands $T_i$ are the connected components of $T-r$, and $r_i$ is the unique vertex of $T_i$ joined to $r$. Further, this representation of $(T,r)$ is unique, up to a permutation of the summands. Clearly, $(T,r)$ is rayless if and only if each summand of $(T,r)$ is rayless.

If

$$f:\sum_{i\in I}(T_i,r_i)\to\sum_{j\in J}(T_j,r_j)$$

is an embedding, then $f$ induces an injection from $I$ into $J$, written $\hat{f}$, defined so that if $i\in I$, $\hat{f}(i)$ is the unique $j\in J$ such that $f(T_i,r_i)\leq(T_j,r_j)$. If $f$ is an isomorphism, then $\hat{f}$ is a bijection.

We next prove two lemmas about rooted trees that will be used in the proof of Theorem 2.

**Lemma 1** *Let*

$$(T,r)=\sum_{i\in I}(T_i,r_i)$$

*be a rooted tree such that for some* $k\in I$, $m(T_k,r_k)=\alpha\geq\aleph_0$. *Then* $m(T,r)\geq\alpha$.

**PROOF.** Let

$$ME(T_k,r_k)=\{(T_{k,n},r_{k,n}):n\in\alpha\}.$$

Define the rayless rooted tree

$$(T_n,r)=\sum_{i\in I}(X_i,x_i),$$

where

$$(X_i,x_i)=\begin{cases}(T_i,r_i) & \text{if } (T_i,r_i)\nsim(T_k,r_k);\\ (T_{k,n},r_{k,n}) & \text{if } (T_i,r_i)\sim(T_k,r_k).\end{cases}$$

Note that for all $n\in\alpha$, we have $(T_n,r)\sim(T,r)$ since $(X_i,x_i)\sim(T_i,r_i)$ for all $i\in I$. For $m\neq n$, $(T_m,r)\ncong(T_n,r)$, since $(T_n,r)$ contains summands of the form $(T_{k,n},r_{k,n})$ and $(T_m,r)$ does not. Hence, the family $\{(T_n,r):n\in\alpha\}$ is a witness to $m(T,r)\geq\alpha$. □

**Lemma 2** *Let*

$$(T,r)=\sum_{i\in I}(T_i,r_i)$$

*be a rooted tree such that $m(T_i, r_i) = 1$, for every $i \in I$. Then $m(T, r) = 1$ or $m(T, r) \geq \aleph_0$.*

**PROOF.** If $m(T, r) = 1$, then there is nothing to prove. Suppose for a contradiction that $1 < m(T, r) < \aleph_0$, and let

$$(T', r') = \sum_{j \in J} (T'_j, r'_j)$$

be a rooted tree not isomorphic to $(T, r)$ such that $(T', r') \sim (T, r)$. Fix embeddings $f : (T, r) \to (T', r')$, $g : (T', r') \to (T, r)$, and consider the injections $\hat{f} : I \to J$, $\hat{g} : J \to I$. We first prove the following claim.

**Claim** If for some $j^* \in J$, the summand $(T'_{j^*}, r'_{j^*})$ of $(T', r')$ is not isomorphic as a rooted tree to any summand of $(T, r)$, then $m(T, r) \geq \aleph_0$.

*Proof of Claim.* Let $L = \{l_i : i \in \omega\}$ be a fixed set disjoint from $I$. For an integer $n \geq 1$, define $L_n = \{l_1, \dots, l_n\}$, and let $L_0 = \emptyset$. Define for $n \in \omega$

$$(S_n, r) = \sum_{i \in (I \cup L_n)} (X_i, x_i),$$

where $(X_i, x_i) = (T_i, r_i)$ if $i \in I$, and $(X_i, x_i) = (T'_{j^*}, r'_{j^*})$ if $i \in L_n$. Then $(S_0, r) = (T, r)$ and $(S_n, r) \leq (S_{n+1}, r)$ for all $n \geq 0$. We show that $(S_{n+1}, r) \leq (S_n, r)$.

Let $I' = \{i_k : k \in \omega\}$ be the subset of $I$ defined by $i_0 = \hat{g}(j^*)$, and $i_m = \hat{g}\hat{f}(i_{m-1})$ for $m \geq 1$. If for some $m > 0$ we have that $i_m = i_0$, then $g$ and $f(gf)^{m-1}$ induce mutual embeddings between the non-isomorphic rooted trees $(T'_{j^*}, r'_{j^*})$ and $(T_{\hat{g}(j^*)}, r_{\hat{g}(j^*)})$, contradicting the hypothesis that $m(T_{\hat{g}(j^*)}, r_{\hat{g}(j^*)}) = 1$. Thus, $i_m \neq i_0$ for all $m > 0$, and since $\hat{g}\hat{f} : I \to I$ is injective, we have that $i_m \neq i_{m'}$ for all $m \neq m'$. Therefore, we can combine the restriction of $gf$ to

$$\sum_{i \in I'} (T_i, r_i)$$

with the restriction of $g$ to $(X_{l_{n+1}}, x_{l_{n+1}}) = (T'_{j^*}, r'_{j^*})$, and the identity on the remainder of $(S_{n+1}, r)$ to obtain an embedding of $(S_{n+1}, r)$ in $(S_n, r)$. Thus, we have that $(S_n, r) \sim (S_0, r) = (T, r)$ for all $n \geq 0$. Since $(S_n, r)$ contains exactly $n$ summands isomorphic to $(T'_{j^*}, r'_{j^*})$, the rooted trees $(S_n, r)$, $n \in \omega$ are pairwise non-isomorphic. The proof of the claim follows. □

Consider the set $\{(X_k, x_k) : k \in K\}$ of isomorphism types of the rooted trees $(T_i, r_i)$, and let $p : I \to K$ be the surjection defined by $(T_i, r_i) \cong (X_{p(i)}, x_{p(i)})$. By the Claim, there is a map $q : J \to K$ such that $(T'_j, r'_j) \cong (X_{q(j)}, x_{q(j)})$, for all $j \in J$. Therefore, $m(T'_j, r'_j) = 1$, for all $j \in J$. If $q$ were not surjective, then reversing the role of $(T,r)$ and $(T',r')$, the Claim would give that

$$m(T,r) = m(T',r') = \aleph_0.$$

Thus, $q$ is surjective since $1 < m(T,r) < \aleph_0$ by assumption.

We have that

$$(T,r) = \sum_{i\in I}(X_{p(i)}, x_{p(i)})$$

and

$$(T',r') = \sum_{j\in J}(X_{q(j)}, x_{q(j)}).$$

Since these two rooted trees are not isomorphic, there exists some $k \in K$ such that $|p^{-1}(k)| \neq |q^{-1}(k)|$. Without loss of generality, we may assume that $|p^{-1}(k)| < |q^{-1}(k)|$. Define

$$(T'',r) = \sum_{i\in(I\setminus p^{-1}(k))}(T_i, r_i).$$

We will show that $(T'',r) \sim (T,r)$.

Define

$$J_0 = \{j \in J : q(j) = k \text{ and } p(\hat{g}(j)) \neq k\}.$$

Since $|p^{-1}(k)| < |q^{-1}(k)|$, we have that $J_0 \neq \emptyset$. We define the sets $I_0 = \hat{g}(J_0) \subseteq I$ and $I_m = \hat{g}\hat{f}(I_{m-1}) \subseteq I$ for $m \geq 1$. Let

$$I' = \bigcup_{i\in\omega} I_m.$$

By reasoning similar to that given in the proof of the Claim, we have that $I_m \cap I_{m'} = \emptyset$, whenever $m \neq m'$. Sequences of composition of the maps $f$ and $g$ demonstrate that $(X_k, x_k) \leq (T_i, r_i)$ whenever $i \in I'$. Moreover for some $m$, we have that

$$\left|\bigcup_{0\leq j\leq m-1} I_j\right| \geq |p^{-1}(k)|.$$

Indeed if $|p^{-1}(k)| < \aleph_0$ we can put $m = |p^{-1}(k)|$, and if $|p^{-1}(k)| \geq \aleph_0$, then since $|q^{-1}(k) \setminus J_0| \leq |p^{-1}(k)|$, we have $|J_0| = |q^{-1}(k)|$ whence

$$|I_0| = |J_0| = |q^{-1}(k)| > |p^{-1}(k)|.$$

For this integer $m$, define

$$I'' = \bigcup_{0\leq j\leq m-1} I_j$$

and fix an injection $\phi : p^{-1}(k) \to I''$. We may then combine embeddings $h_i : (T_i, r_i) \to (T_{\phi(i)}, r_{\phi(i)})$, where $i \in p^{-1}(k)$, with the restriction of $(gf)^m$ to

$$\sum_{i\in I'}(T_i, r_i)$$

and the identity on the remainder of $(T, r)$ to define an embedding of $(T, r)$ in $(T'', r)$. Since $(T'', r) \leq (T, r)$, we then have $(T'', r) \sim (T, r)$. However, since $(T, r)$ has summands isomorphic to $(X_k, x_k)$ and $(T'', r)$ does not, the Claim applied to $(T'', r)$ gives that $m(T, r) = m(T'', r) \geq \aleph_0$, which contradicts our assumption that $m(T, r) < \aleph_0$. $\square$

With Lemmas 1 and 2 in hand, we now turn to the proof of Theorem 2.

**PROOF OF THEOREM 2.** Suppose for a contradiction that there exists a rooted rayless tree $(T, r)$ such that $1 < m(T, r) < \aleph_0$. By Lemma 1 and Lemma 2, there is some summand $(T_1, r_1)$ of $(T, r)$ satisfying $m(T_1, r_1) \in (1, \aleph_0)$. By repeated application of Lemma 1 and Lemma 2, we may recursively choose a sequence $((T_i, r_i) : i \in \omega)$, with $(T_0, r_0) = (T, r)$, and where $(T_{i+1}, r_{i+1})$ is a summand of $(T_i, r_i)$ such that $m(T_{i+1}, r_{i+1}) \in (1, \aleph_0)$. But then the path in $T$ beginning with $r_0$ and whose remaining vertices are the $r_i$ constitutes a ray in $T$, which is a contradiction. $\square$

In fact, it is straightforward to modify the argument to prove that for every rooted tree $(T, r)$ (not necessarily rayless), we have $m(T, r) = 1$ or $m(T, r) \geq \aleph_0$. In the next section, the absence of rays is used more explicitly in the transition from rooted trees to general trees.

## 3 Mutually embeddability of rayless trees

Define a *fixed vertex* $u$ of a graph $G$ to be one with the property that for all self-embeddings $f$ of $G$, $f(u) = u$. Define a *fixed edge* $uv$ of $G$ to be one with the property that for all self-embeddings of $G$, $\{f(u), f(v)\} = \{u, v\}$. The following "fixed element" theorem was first proved by Halin [2], and will be used in the proof of Theorem 1.

**Theorem 3** *If $T$ is a rayless tree, then there is either a vertex or an edge fixed by every self-embedding of $T$.*

Note that the maps that we refer to as *self-embeddings* are referred to as *endomorphisms* in [2].

**PROOF OF THEOREM 1.** Suppose that $m(T) \geq 2$. By Theorem 3, there exists a fixed vertex $u$ or a fixed edge $e = uv$ of $T$. Consider the rooted tree $(T, u)$. We will use Theorem 2 and Theorem 3 to prove that in both cases we have that:

(1) $m(T, u) \geq \aleph_0$.

(2) If $\{(T_i, u_i) : i \in \omega\}$ is a family of pairwise non-isomorphic rooted trees mutually embeddable with $(T, u)$, then $\{T_i : i \in \omega\}$ is a family of rayless trees mutually embeddable with $T$, with the additional property that for all $i \in \omega$, there is *at most* one $j \in \omega$ such that $T_i \cong T_j$.

Once items (1) and (2) are proven, it will follow that $m(T)$ is infinite, and our proof of Theorem 1 will be concluded.

To prove item (1), we argue as follows. As $m(t) \geq 2$, let $T'$ be a rayless tree that is non-isomorphic and mutually embeddable with $T$. Then there exists embeddings $f : T \to T'$ and $g : T' \to T$. If $gf(u) = u$, then $f$ and $g$ act as mutual embeddings between the non-isomorphic rooted trees $(T, u)$ and $(T', f(u))$. Hence, $m(T, u) \geq \aleph_0$ by Theorem 2.

Otherwise, since $gf$ is a self-embedding of $T$ and $gf(u) \neq u$, we are dealing with the case where $uv$ is an edge fixed by all self-embeddings of $T$, where $gf(u) = v$ and $gf(v) = u$. Therefore, $f$ and $gfg$ act as mutual embeddings between the two rooted trees $(T, u)$ and $(T', f(u))$, which again implies that $m(T) \geq \aleph_0$.

We prove item (2) by contradiction, assuming that there are distinct $i, j, k \in \omega$ such that there exist isomorphisms $h_{ij} : T_i \to T_j$ and $h_{ik} : T_i \to T_k$. Since $(T_i, u_i)$, $(T_j, u_j)$, and $(T_k, u_k)$ are mutually embeddable with $(T, u)$, there exist embeddings $f_i : T \to T_i$, $g_j : T_j \to T$, and $g_k : T_k \to T$ such that

$$f_i(u) = u_i, \;\; g_j(u_j) = u, \;\; g_k(u_k) = u. \tag{1}$$

See Figure 2.

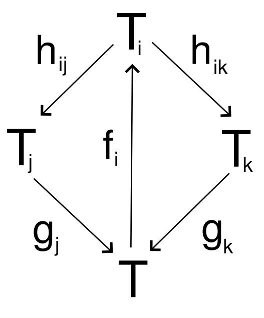


Fig. 2. Maps in the proof of Theorem 1.

Since $(T_i, u_i)$, $(T_j, u_j)$, and $(T_k, u_k)$ are pairwise non-isomorphic as rooted trees, we have that $h_{ij}(u_i) \neq u_j$ and $h_{ik}(u_i) \neq u_k$. This implies by (1) that $g_j h_{ij} f_i(u) \neq u$, and that $g_k h_{ik} f_i(u) \neq u$. Therefore, we are in the case when $uv$ is a fixed edge of $T$, and both self-embeddings $g_j h_{ij} f_i$ and $g_k h_{ik} f_i$ interchange

$u$ and $v$. Hence,

$$g_j h_{ij} f_i(v) = u, \;\; g_k h_{ik} f_i(v) = u. \tag{2}$$

Equations (1) and (2) imply that

$$h_{ij}(f_i(v)) = {g_j}^{-1}(u) = u_j, \;\; h_{ik}(f_i(v)) = {g_k}^{-1}(u) = u_k. \tag{3}$$

Equations (1), (2), and (3) together imply that

$$h_{ik}{h_{ij}}^{-1}(u_j) = u_k.$$

Therefore, $h_{ik}{h_{ij}}^{-1}$ is an isomorphism from $T_j$ to $T_k$ which maps $u_j$ to $u_k$, contradicting the fact that $(T_j, u_j)$ and $(T_k, u_k)$ are non-isomorphic as rooted trees. □


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