On 2-e.c. line-critical graphs*

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Abstract

We continue the study of graphs defined by a certain adjacency property by investigating the n-existentially closed line-critical graphs. We classify the 1-e.c. line-critical graphs and give examples of 2-e.c. line-critical graphs for all orders ≥ 9.

1 Introduction

For a fixed integer $n \geq 1$, a graph G is called *n-existentially closed* or *n-e.c.* if for for every n-element subset S of the vertices, and for every subset T of S, there is a vertex not in S which is joined to every vertex in T and to no vertex in $S \setminus T$. N-e.c. graphs were investigated by Caccetta, Erdős, and Vijayan [4]; they referred to n-e.c. graphs as graphs with property $P(n)$. Although almost all finite graphs are n-e.c. for a given n (as labelled structures; see Fagin [6] and Blass and Harary [2]), very few explicit examples of n-e.c. graphs are known, especially for $n > 2$, with the exception being large Paley graphs (see Ananchuen and Caccetta [1]).

Induction is a potent tool when proving results about finite graphs. Graphs which are critical or minimal with respect to a given property play an important role in such investigations. In [3] we investigated the n-e.c. *point*-critical graphs: n-e.c. graphs which when a single vertex is deleted are

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no longer n-e.c.. In particular, we proved that there is a unique 2-e.c. minimal graph, and we found 2-e.c. criticals of all orders ≥ 9. One of the main tools of [3] was the operation of replicating an edge (see Definition 4 below). Replicating an edge preserves 2-e.c. and in some situations preserves 2-e.c. point-criticality.

[3] did not investigate the *n-e.c. line-critical* or *n-e.c.l.c.* graphs, which in this article we abbreviate as *n-l.c.* graphs: n-e.c. graphs with the property that when any edge is deleted the remaining graph is not n-e.c.. An easy exercise is that each n-e.c. graph has a spanning subgraph that is n-l.c.. Therefore, one way to find examples of n-l.c. graphs is to strategically delete edges in known n-e.c. graphs. In Section 2 we present a complete classification of the 1-l.c. graphs. In Section 3 we provide explicit examples of 2-l.c. graphs of all possible orders. Replication again proves valuable, and in Theorem 5 we find sufficient conditions for replication to preserve 2-l.c..

The countably infinite random graph R is the unique countable graph that is n-e.c. for all $n \geq 1$. As described in [5], R satisfies a first-order sentence φ in the language of graphs if and only if almost all finite graphs satisfy φ. Further, for any vertex x and edge e, $R - x$ and $R - e$ are isomorphic to R, so that for all $n \geq 1$, R is neither n-e.c. point- or line-critical. We leave it as an exercise to verify that the properties of being n-e.c. point- and line-critical are first-order definable. Thus, almost no finite graphs are n-e.c. point- or line-critical.

Throughout, all graphs are finite and simple. For a graph G, $V(G)$ will denote the vertex-set of G and $E(G)$ will denote its edge-set. (G may be dropped if it is clear from context.) The order of G is $|V(G)|$. We denote an edge by xy, or sometimes $(x)(y)$ to avoid confusion. We recall the following definition from [3].

Definition 1 *Let G be a graph, and let $n \geq 1$ be fixed.*

1. *An n-**e.c. problem** in G is a $2 \times n$ matrix $\begin{pmatrix} x_1 & \dots & x_n \\ i_1 & \dots & i_n \end{pmatrix}$, where $\{x_1, \dots, x_n\}$ is an n-element subset of $V(G)$, and for $1 \leq j \leq n$, $i_j \in \{0, 1\}$.*

2. *A **solution** to an n-e.c. problem $\begin{pmatrix} x_1 & \dots & x_n \\ i_1 & \dots & i_n \end{pmatrix}$ is a vertex $y \in V(G)$ so that if $i_j = 1$ then $yx_j \in E(G)$ and if $i_j = 0$ then $yx_j \notin E(G)$ and $y \neq x_j$.*

Observe that a graph G is n-e.c. if and only if each n-e.c. problem in G has a solution.

2 The 1-l.c. graphs

As was mentioned in [3], the 1-e.c. minimal graphs are $2K_2$, C_4, and P_4; observe that $2K_2$ is the only 1-e.c. minimal that is 1-l.c.. Recall that a graph is a *star* if it is one of the graphs $K_{1,n}$, for some $n \geq 1$. The following theorem completely classifies the 1-l.c. graphs, and reveals that they have a relatively simple structure.

Proposition 2 *A graph G is 1-l.c. iff each component of G is a star and G has at least two components.*

PROOF. Sufficiency is easy, so we prove necessity only.

Claim 1: G is 1-l.c. iff G is 1-e.c. and for all $e = xy \in G$, one of x, y is isolated in $G - e$.

We prove the forward direction of the claim; the reverse direction is trivial. Fix $e = xy \in E(G)$. Then $G - e$ is not 1-e.c. so there is a 1-e.c. problem that cannot be solved in $G - e$, and this 1-e.c. problem must be $\begin{pmatrix} z \\ 1 \end{pmatrix}$ for some $z \in V(G)$. But then z must be one of x, y and so the deletion of e isolates one of x, y.

Fix a connected component, say C, of G.

Claim 2: If $|C| \geq 3$ then C has exactly one vertex of degree ≥ 2.

If each vertex of C had degree 1, then $C = K_2$, contrary to assumption. Now assume that both x, y have degree ≥ 2. We claim that $xy \in E(G)$. If not there is a path with length ≥ 2 and endpoints x, y, so that x is joined to some vertex $x_1 \neq y$ on the path, and x_1 is joined to y or to some vertex $x_2 \neq y$. As $\deg(x) \geq 2$, there is some x_0 joined to x distinct from x_1. If we delete xx_1 then neither x nor x_1 is isolated in $G - e$: x is joined to x_0 and x_1 is joined to x_2 or y. This contradicts Claim 1. Hence, $xy \in E(G)$. Since $\deg(x), \deg(y) \geq 2$, we can find $x' \neq y$ joined to x and $y' \neq x$ joined to y, where y' may not be x'. Deleting xy leaves neither x nor y isolated in $G - e$, and so Claim 2 follows.

From the claim, each component is a star; thus, for the graph to be 1-e.c. it must have at least two components. $\square$

3 2-l.c. graphs of all orders

We do not have a complete classification of the 2-l.c. graphs. However, we have found examples of 2-l.c. graphs of all possible orders. Before we present these examples we recall some results from [3].

1. The Cartesian product of K_3 with itself, written $K_3 \Box K_3$, is the unique 2-e.c. minimal graph. See Figure 1.

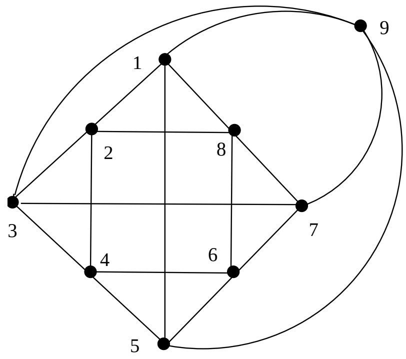

Figure 1: $K_3 \Box K_3$.

2. Define a graph $G = G^*(k)$ where k is even and $k \geq 6$ as follows (arithmetic is *mod* $2k$): $V(G) = \{1, \ldots, 2k+1\}$. Each even vertex i is joined to all other even vertices except $i+k$, and is joined to $i-1$ and $i+1$. Each odd vertex $\neq 2k+1$ is joined to $i-1$, $i+1$, $2k+1$, and $i+k$. $2k+1$ is joined to all of the odd vertices. Each graph $G^*(k)$ is 2-e.c. and 2-*e.c. critical* (when a vertex is deleted, the remaining graph is not 2-e.c.). We include a table, from [3], which will be useful later, which proves that $G^*(k)$ is 2-e.c.. By symmetry we may omit the first two rows of the "2nd only" column.

vertices	joined to both	neither	1st only	2nd only
i,j odd $\neq 2k+1$	$2k+1$	odd$\notin \{i,j,i+k, j+k\}$	$i-1$ if $j \neq i-2$ $i+1$ else	
i,j even	even$\notin \{i,j,i+k, j+k\}$	$2k+1$	$i-1$ if $j \neq i-2$ $i+1$ else	
i even, j odd $\neq 2k+1$	$j-1$ if $i \neq j-1+k$ $j+1$ else	odd$\notin \{j, j+k, i-1, i+1, 2k+1\}$	even$\notin \{i, i+k, j-1, j+1\}$	$2k+1$
i odd $\neq 2k+1$, $2k+1$	$i+k$	even$\notin \{i-1, i+1\}$	$i+1$	odd$\notin \{i, 2k+1, i+k\}$
i even, $2k+1$	$i-1$	$i+k$	even$\neq i+k$	$i+3$

$K_3 \square K_3$ is 2-l.c. as it is the unique 2-e.c. minimal graph. We claim that the following graphs H and J are 2-l.c. of orders 10 and 13, respectively (see Figures 2 and 3; note that J is a spanning subgraph of $G^*(6)$.)

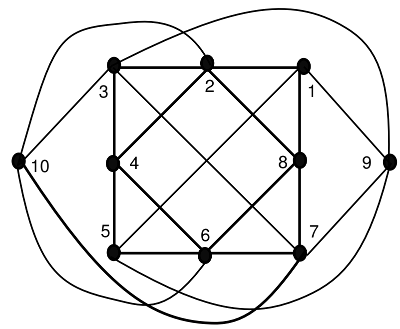

Figure 2: The graph H.

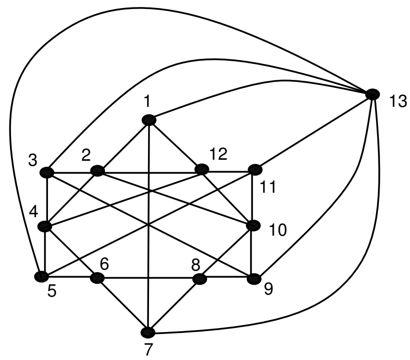

Figure 3: The graph J.

We leave it as an exercise to show that H is 2-e.c.. For the line-criticality of H, we supply the following table that lists a problem that cannot be solved if the given edge is deleted. Symmetry covers the remaining cases as $12 \sim 56$, $18 \sim 45$, $19 \sim 59$, $23 \sim 67$, $24 \sim 68$, $28 \sim 64$, $(2)(10) \sim (6)(10)$, $34 \sim 78$, $39 \sim 79$ and $(3)(10) \sim (7)(10)$, where $e \sim f$ means there is an automorphism of the graph which maps the ends of e onto the ends of f.

edge deleted	12	18	15	19	23
cannot solve	$\begin{pmatrix} 1 & 8 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 9 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix}$
edge deleted	24	28	(2)(10)	34	37
cannot solve	$\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 10 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 9 \\ 1 & 1 \end{pmatrix}$
edge deleted	39	(3)(10)			
cannot solve	$\begin{pmatrix} 9 & 7 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 10 & 2 \\ 1 & 1 \end{pmatrix}$			

We verify that J is 2–l.c.; we leave it as an exercise to show that J is 2-e.c.. For line-criticality of J, we supply the following table.

xy deleted	$(i)(i+6)$, i odd$\neq 13$	$(i)(13)$, i odd$\neq 13$	$(i)(i+1)$, i odd$\neq 13$
cannot solve	$\begin{pmatrix} i & 13 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} i+6 & 13 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} i & i-1 \\ 1 & 1 \end{pmatrix}$
xy deleted	$(i)(i-1)$, i odd$\neq 13$	(2)(10) ((4)(12) is similar)	$(i)(i+2)$, i even
cannot solve	$\begin{pmatrix} i & i+1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 8 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} i & i+1 \\ 1 & 1 \end{pmatrix}$

3.1 Orders ≥ 17, $\equiv 1$ $(mod\ 4)$

For each even $k \geq 8$, define a graph $G^{**}(k)$ which is a spanning subgraph of $G^{*}(k)$. All the edges are the same except between even vertices. In $G^{**}(k)$, an even i is joined to $i \pm 2$, and to $i + 4l \neq i + k$ with $l \geq 1$ $(mod\ 2k)$.

Theorem 3 *$G^{**}(k)$ is 2-l.c. for each $k \geq 8$.*

Proof. There are two cases. In the first case, $k \equiv 0$ $(mod\ 4)$; here, i, even, is joined to $i + k \pm 4$ but is not joined to $i + k \pm 2$. In the second case, $k \equiv 2$ $(mod\ 4)$; here, i, even, is not joined to $i + k \pm 4$ and is joined to $i + k \pm 2$. We give the proof for the first case; the second case is similar.

We first show that $G^{**}(k)$ is 2-e.c.. We consider 2-e.c. problems of the form $\begin{pmatrix} x & y \\ p & q \end{pmatrix}$, where $p, q \in \{0, 1\}$. $G^{**}(k)$ is obtained from $G^{*}(k)$ by deleting some edges between even vertices. So for 2-e.c. problems where 1) x, y are both odd, 2) p, q are both 0, or 3) the solution z in $G^{*}(k)$ is odd, a solution in $G^{*}(k)$ is a solution in $G^{**}(k)$.

The remaining problems are of the form i) $\begin{pmatrix} even & even \\ 1 & 1 \end{pmatrix}$,

ii) $\begin{pmatrix} even & odd \neq 2k+1 \\ 1 & 1 \end{pmatrix}$, iii) $\begin{pmatrix} even & odd \neq 2k+1 \\ 1 & 0 \end{pmatrix}$, and iv) $\begin{pmatrix} even & 2k+1 \\ 1 & 0 \end{pmatrix}$.

iv) $\begin{pmatrix} x & 2k+1 \\ 1 & 0 \end{pmatrix}$, for x even, is solved by $x+2$ or any other even vertex that x is joined to.

iii) $\begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}$, where x is even and y is odd, $y \neq 2k+1$, is solved by any even vertex that x is joined to other than $y-1$ and $y+1$.

i) Consider $\begin{pmatrix} x & y \\ 1 & 1 \end{pmatrix}$ where x, y are both even. Without loss of generality, we can assume that $x = 2$ and that $y \in \{4, 6, \ldots, k+2\}$ (the remaining cases follow by symmetry). We have: $\begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ is solved by 3, $\begin{pmatrix} 2 & 6 \\ 1 & 1 \end{pmatrix}$ is solved by 4. If $l \geq 1$, $6+4l \neq k+2$, then $\begin{pmatrix} 2 & 6+4l \\ 1 & 1 \end{pmatrix}$ is solved by $6+4(l-1)$. If $l \geq 0$, $6+4l+2 \neq k+2$ then $\begin{pmatrix} 2 & 6+4l+2 \\ 1 & 1 \end{pmatrix}$ is solved by $6+4l$. $\begin{pmatrix} 2 & k+2 \\ 1 & 1 \end{pmatrix}$ is solved by $k-2$.

ii) Consider $\begin{pmatrix} x & y \\ 1 & 1 \end{pmatrix}$ where x is even, y odd$\neq 2k+1$.

Without loss of generality, we may assume that $x = 2$ and $y \in \{3, \ldots, k+1\}$. For $3 \leq y \leq k-1$, $\begin{pmatrix} 2 & y \\ 1 & 1 \end{pmatrix}$ is solved by either $y-1$ or $y+1$ (depending on the position of y). $\begin{pmatrix} 2 & k+1 \\ 1 & 1 \end{pmatrix}$ is solved by 1.

We next show line-criticality of $G^{**}(k)$. The majority of cases are handled in the following table.

xy deleted	$(i)(i+k)$, i odd$\neq 2k+1$	$(i)(2k+1)$, i odd $\neq 2k+1$	$(i)(i+1)$, i odd
cannot solve	$\begin{pmatrix} i & 2k+1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} i & i+k \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} i & i-1 \\ 1 & 1 \end{pmatrix}$
xy deleted	$(i)(i-1)$, i odd$\neq 2k+1$	$(i)(i+2)$, i even	
cannot solve	$\begin{pmatrix} i & i+1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} i & i+1 \\ 1 & 1 \end{pmatrix}$	

The last case is when i, j are both even and $i-j \not\equiv \pm 2$ $(mod\ 2k)$. As before, by symmetry we may assume $i = 2$ and $j \in \{6, \ldots, k-2\}$. If we delete 26 then we cannot solve $\begin{pmatrix} 2 & 7 \\ 1 & 1 \end{pmatrix}$ (since $k \geq 8$, $7 < 1+k$, so 1 and 3 cannot solve the problem). If we delete $(2)(6+4l)$, where $l \geq 1$ and $6+4l < 2+k$ then we cannot solve $\begin{pmatrix} 2 & 6+4l-1 \\ 1 & 1 \end{pmatrix}$. $\Box$

3.2 Orders $\equiv 0, 2, 3$ $(mod\ 4)$

We can realize the rest of the odd spectrum of 2-l.c. graphs with the aid of the following definition that played a crucial role in [3].

Definition 4 *Let G be a graph and let $e = ab \in E(G)$. The **replicate**, $R = R(G, e)$, is the graph with vertices $V(G) \cup \{a', b'\}$ and edges $E(G) \cup \{a'b'\} \cup \{a'c : ac \in E(G)$ and $c \neq b\} \cup \{b'c : bc \in E(G)$ and $c \neq a\}$ (in other words, add new nodes a' and b' and edge $a'b'$ to G, join a' to $N(a) - \{b\}$ and do the analogous for b').*

As was shown in [3], if G is 2-e.c. then for any $e \in E(G)$, $R(G, e)$ is 2-e.c.. We now present conditions for replication to "preserve 2-l.c.".

Theorem 5 *Let G be 2-l.c. and fix $e = ab \in E(G)$. Suppose G satisfies:*

1. *For edges f incident with e,*
   (a) *if $f = au$, there is a vertex c such that u is the unique solution to $\begin{pmatrix} a & c \\ 1 & 1 \end{pmatrix}$ in G;*
   (b) *if $f = bu$, there is a vertex c such that u is the unique solution to $\begin{pmatrix} b & c \\ 1 & 1 \end{pmatrix}$ in G;*
2. *For edges $f = uv$, where u, v are distinct from a, b, there exists a vertex c such that v is the unique solution to $\begin{pmatrix} u & c \\ 1 & 1 \end{pmatrix}$ in G or there exists a vertex d such that u is the unique solution to $\begin{pmatrix} v & d \\ 1 & 1 \end{pmatrix}$ in G.*

Then $R = R(G, e)$ is 2-l.c..

PROOF. Fix $f \in E(R)$. We consider cases based on the location of f.

(i) If $f = e$, then $\begin{pmatrix} a & a' \\ 1 & 0 \end{pmatrix}$ is uniquely solved by b in R, so this problem has no solution in $R - f$.

(ii) If $f = au$, there exists a vertex c such that u is the unique solution to $\begin{pmatrix} a & c \\ 1 & 1 \end{pmatrix}$. This problem cannot be solved in $R - f$ since neither a' nor b' is joined to a. Case (1b) is analogous.

(iii) Suppose $f = uv$ where u, v are distinct from a, b. Suppose Case (2) holds so there exists a vertex c such that v is the unique solution to $\begin{pmatrix} u & c \\ 1 & 1 \end{pmatrix}$ in G. Suppose this problem is solved by a' in R. By hypothesis $u \neq a$, and since $a'c \in E(R)$, $u, c \neq a$. Thus a solves $\begin{pmatrix} u & c \\ 1 & 1 \end{pmatrix}$ in G, a contradiction. Similarly, b' cannot solve $\begin{pmatrix} u & c \\ 1 & 1 \end{pmatrix}$. The other case of (2) is analogous.

(iv) Suppose $f \in E(R) - E(G)$.

a) If $f = a'b'$ then $\begin{pmatrix} a' & a \\ 1 & 0 \end{pmatrix}$ cannot be solved in $R - f$.

b) Suppose $f = a'u$ for some $u \in V(G) - \{a, b\}$. Then $au \in E(G)$. By (1a) there is a vertex $c \neq u$ such that u is the unique solution to $\begin{pmatrix} a & c \\ 1 & 1 \end{pmatrix}$ in G. We claim that $\begin{pmatrix} a' & c \\ 1 & 1 \end{pmatrix}$ has no solution in $R - f$. Otherwise, say d solves this problem in $R - f$, so that $d \neq u$. Then $a'd, cd \in E(R - f)$. Since $a'd \in E(R)$, $d \neq a, b$.

If $c = b$ then $d \neq a', b'$ so that $ad \in E(G)$. Hence, $\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$ is solved by $d \neq u$ in G, which is a contradiction.

We therefore assume that $c \neq b$. If $d = b'$ then $cb' \in E(R)$ so that $cb \in E(G)$ as $c \neq a, b$. But then $\begin{pmatrix} a & c \\ 1 & 1 \end{pmatrix}$ is solved by $b \neq u$ in G, which is a contradiction. Thus, $d \neq b'$ and so $ad \in E(G)$; therefore, $d \neq u$ solves $\begin{pmatrix} a & c \\ 1 & 1 \end{pmatrix}$ in G, which is a contradiction. $\square$

We leave it to the reader to check that the conditions of Theorem 5 are satisfied by $K_3 \square K_3$ when $e = 15$ (or, for any other edge, since $K_3 \square K_3$ is edge-transitive). For J, we claim the conditions of Theorem 5 hold when $e = 17$. The verification of this is similar to the case for $K_3 \square K_3$ except for the edges (2)(10) and (4)(12). (2)(10) is not incident with 17, so we show (2). But 10 is the unique solution of $\begin{pmatrix} 2 & 8 \\ 1 & 1 \end{pmatrix}$. (4)(12) is handled similarly.

For $G^{**}(k)$ we let $e = (1)(1+k)$. We leave it to the reader to verify that the conditions of Theorem 5 hold for all edges $f = xy$ when one of x or y is odd. Hence, we verify the conditions of the Theorem only for x, y both even. We can assume $x = 2$, and $4 \leq y \leq 2 + k$. Note that 4 is the unique solution for $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ and 6 is the unique solution for $\begin{pmatrix} 2 & 7 \\ 1 & 1 \end{pmatrix}$ (using the fact that $k \geq 8$). We now consider edges of the from $(2, 6 + 4l)$, with $l \geq 1$, $6 + 4l \neq 2 + k$.

Case 1. $k \equiv 0$ $(mod\ 4)$.

We claim that $y = 6+4l$ is the unique solution to $\begin{pmatrix} 2 & 7+4l \\ 1 & 1 \end{pmatrix}$. Now $7+4l$ is joined to $6+4l, 8+4l, 2k+1$, and $7+4l+k$. Note that $7+4l+k$ is joined to 2 only if (all arithmetic *mod* $2k$) either *i*) $7+4l+k \equiv 1 \equiv 1+2k$, or *ii*) $7+4l+k \equiv 3 \equiv 3+2k$. For i) $6+4l \equiv k$. But 2 is not joined to k in Case 1. As 2 is not joined to $8+4l$ or $2k+1$ our claim follows.

For ii) $6+4l \equiv 2+k$, which is not joined to 2.

Case 2. $k \equiv 2$ $(mod\ 4)$.

We claim that $y = 6+4l$ is the unique solution to $\begin{pmatrix} 2 & 5+4l \\ 1 & 1 \end{pmatrix}$. The argument is similar to that of Case 1 and so is omitted. Hence, we have found 2-l.c. graphs of all odd orders ≥ 9.

For even orders, we rely on the following lemma. Fix G and $e = ab \in E(G)$. Define $R_1(G,e) = R(G,e)$, and $R_{n+1}(G,e) = R(R_n(G,e),e)$; the replicated edge in $R_{n+1}(G,e)$ is denoted $e_{n+1} = a_{n+1}b_{n+1}$.

Lemma 6 *If G is 2-l.c. and G and $e \in E(G)$ satisfy the conditions of Theorem 5, then $R_n(G,e)$ is 2-l.c. for each $n \geq 1$.*

Proof. We prove the lemma by induction on $n \geq 1$; the case for $n = 1$ follows by Theorem 5. Assume $R_n = R_n(G,e)$ is 2-l.c. and for each $1 \leq j \leq n$, a_j is the unique solution of $\begin{pmatrix} b_j & b \\ 1 & 0 \end{pmatrix}$ and b_j is the unique solution of $\begin{pmatrix} a_j & a \\ 1 & 0 \end{pmatrix}$.

Fix an edge f in $R_{n+1} = R_{n+1}(G,e)$; we show that $R - f$ is not 2-e.c.. The cases when f is one of the edges $\{e, e_1, \ldots, e_n\}$ follow by remarks at the end of the preceding paragraph. a_{n+1} is the unique solution of $\begin{pmatrix} b_{n+1} & b \\ 1 & 0 \end{pmatrix}$, and so $R - e_{n+1}$ is not 2-e.c..

Case i) $f \in E(G) - \{e\}$.

The argument in this case is similar to Cases ii) to iii) in the proof of Theorem 5, replacing the roles of a' and b' by a_j and b_j, respectively, and using the facts that each of a_j and b_j are not joined to a, b nor any of the a_k, b_k when $k \neq j$.

Case ii) $f \in E(R) - (E(G) \cup \{e_1, \ldots, e_{n+1}\})$.

The argument in this case is similar to that of Case iv) of Theorem 5, again using the fact that each a_j and b_j are not joined to a, b nor any of the a_k, b_k when $k \neq j$. $\square$

For examples of 2-l.c. graphs of even orders, we first note that from the above tables, H satisfies the conditions of Theorem 5 with $e = 15$. Now use Lemma 6 to replicate the edge 15 in H repeatedly.

Since the complement of an n-e.c. graph is n-e.c., the complements of our 2-l.c. graphs are 2-e.c. graphs that are critical the "other" way: adding an edge that is not already there results in a graph that is not 2-e.c.. We thank the anonymous referee for this and other useful remarks.

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