THE SEARCH-TIME OF A GRAPH

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Dedicated to Pavol Hell on his 60th birthday.

Abstract. We consider the game of Cops and Robber played on finite and countably infinite connected graphs. The length of games is considered on cop-win graphs, leading to the new parameter called the search-time of the graph. While the search-time is bounded above by the number of vertices, we prove an upper bound of half the number of vertices for a large class of graphs including chordal graphs. Examples are given of cop-win graphs which have unique corners and have search-time within a small additive constant of the number of vertices. We consider the ratio of the search-time to the number of vertices, and extend this notion of search-time density to infinite graphs. For the infinite random graph, the search-time density can be any real number in $[0, 1]$. We also consider the search-time when more than one cop is required to win. We show that for the fixed number of cops, the search-time can be calculated by polynomial algorithm, but it is NP-complete to decide, whether k cops can capture the robber in no more than t moves for every fixed t.

1. Introduction

Cops and Robber is a vertex pursuit game played on a graph $G = (V, E)$ (we also use $V(G)$ and $E(G)$). All graphs we consider are simple, undirected, and countable (although usually finite) and we assume they are connected since it is enough to consider connected components, mutatis mutandis. The closed neighborhood of a vertex $v \in V$ is denoted $N[v]$; it consists of v together with the vertices joined to v. There are two players, a set of k *cops* (or *searchers*) $\mathcal{C}$, where $k > 0$ is a fixed cardinal, and the *robber* $\mathcal{R}$. The cops begin the game by occupying a set of k vertices, and the cops and the robber move alternately, the cop beginning. More than one cop is allowed to occupy

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a vertex, and the players may *pass*; that is, remain on their current vertex (this is known as the *passive* version of the game; in an *active* version each player *must* use an edge on a move). A *move* in a given round for either player consists of a pass or moving to an adjacent vertex initial positioning is thought of as move 0). The players know each others' current locations. The cops win and the game ends if at least one of the cops occupies the same vertex as the robber; otherwise, $\mathcal{R}$ wins. As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*, written $c(G)$, the minimum cardinality of a set of cops that have a winning strategy on G. A *strategy* is simply a mapping $\sigma_p : V^k \times V \longrightarrow V$ which tells the player (either the robber, or the set of cops) p what the next move is, based on the players' current positions. Obviously, a strategy is *winning* if it allows the player to win, no matter what the opponent's moves are.

The graphs with cop number 1, the *cop-win* graphs, were first characterised in [17] and [18]. Before proceeding, it is useful to introduce some notation. Consider a graph G whose vertex set is $\{v_1, v_2, \dots, v_n\}$. For $i = 1, \dots, n$, define G_i to be the subgraph induced by $\{v_i, \dots, v_n\}$. Similarly, for a vertex v_j, define $N_i(v_j) = N(v_j) \cap \{v_i, \dots, v_n\}$. The finite cop-win graphs are exactly those graphs with a *dismantling ordering*; that is, a linear ordering $(x_j : 1 \leq j \leq n)$ of the vertices so that for each $1 \leq i \leq n$, there is a $i < j \leq n$ such that $N_i[x_i] \subseteq N_i[x_j]$. The characterisation comes from the observation that the robber's last move must be from a vertex whose neighbourhood is contained in that of the cop's current vertex, together with the fact that a retract of a cop-win graph is cop-win. There is no known structural characterisation of graphs with cop number 2 or higher. For a survey of results on the cop number and related search parameters for graphs, see [2].

If $c(G) = k$, then how many moves does it take for the k cops to win? To be more precise, the *length* of a game is the number of rounds it takes (not including the initial or 0th round) to capture the robber (it is infinite if the robber can indefinitely evade capture). We say that a play of the game with $c(G)$ cops is *optimal* if its length is the minimum over all possible games played by the cops, assuming the robber is trying to evade capture for as long as possible. There may be many optimal plays possible (for example, on P_4, the cop may start on either vertex of the centre), but the length of an optimal game is an invariant of G. We denote this invariant $st(G)$, which we call the *search-time* of G. The search-time of G may be viewed as the temporal counterpart of the cop number. The search-time is in part motivated by the fact that in real-world networks with limited resources, not only the number of cops, but the time it takes to capture the robber on the network is of practical importance.

As noted first in [3],

$$st(G) \leq |V(G)|^{c(G)+1}.$$

This upper bound is far from sharp in general; for example, it is not hard to see that the search-time of a tree is at most $\lfloor \frac{n}{2} \rfloor$ moves. The goal of the present article is to present some results and examples on the search-time number of a graph. We focus primarily on the search-time of cop-win graphs as they are better understood. We prove, in Theorem 1, that $st(G) \leq |V(G)| - 3$ if $|V(G)| \geq 5$. In Theorem 2, we prove that the upper bound of $\left\lfloor \frac{|V(G)|}{2} \right\rfloor$ applies to a large class of graphs including connected chordal graphs. Perhaps surprisingly, there are cop-win graphs of order n whose search-time is within an additive constant of n. These graphs exhibit the interesting structural property of possessing a unique corner (that is, a vertex whose closed neighbourhood is contained in the closed neighbourhood of some other vertex), see Theorem 3. Using the results of [13], we show in the end of Section 2 that if the number of cops k is fixed, then it is polynomial time computable to determine if $st(G) \leq t$, for a given t. We consider the search-time density for the infinite random graph; see Theorem 5. In the final section, we consider the case when more than one cop is allowed. We introduce the parameter $c_t(G)$ which is smallest number of cops needed for capture of robber in no more than t steps. We prove in Theorem 6 that it is NP-complete to determine if $c_t(G) \leq k$ if t is fixed, even if G is planar or chordal.

2. Cop-win graphs

The parameter $st(G)$ was first considered in [12] in an analysis of lengths of games for chordal graphs. For more general cop-win graphs, we have the following upper bound.

Theorem 1. *If G is cop-win and $|V(G)| \geq 5$, then $st(G) \leq |V(G)| - 3$.*

Proof. As $st(G)$ is finite, the upper bound is trivial in the case when G is infinite. For G finite, the proof is by induction on $|V(G)|$, with the case $|V(G)| = 5$ following by a direct check of all cop-win graphs of order 5. Assume that the theorem holds for graphs with $n \geq 5$ vertices and consider a cop-win graph G with $n+1$ vertices. Hence, G contains a corner, that is, a vertex u such that there is a vertex $v \in V(G)\backslash\{u\}$ with the property that $N[u] \subseteq N[v]$. Since $G - u$ is a retract of G, it also is cop-win (see [17]).

There is an optimal game of length $n-3$ on $G-u$. The cop plays this optimal game in $G-u$, and whenever $\mathcal{R}$ is on u, then $\mathcal{C}$ plays as if he were on v. After at most $n-3$ moves, either the robber is caught on $G-u$, or $\mathcal{R}$ is on u and $\mathcal{C}$ is on v. But then $\mathcal{C}$ can win in one more move, and so this strategy uses at most $n-2=(n+1)-3$ moves. As $st(G)$ is the length of an optimal game, we have that $st(G) \leq (n+1)-3$. $\square$

This improves the result of [9] (see Theorem 1.2.3) that if G is cop-win, then $st(G) \leq |V(G)|-1$. For many graphs (such as trees) the bound in Theorem 1 is not sharp. We introduce a new graph class where the bound for trees applies. If $N[x] \subseteq N[y]$, then we say x is *dominated* by y. Two corners a and b are *separate* if they are dominated by vertices not equal to either a or b. A graph G is 2-*dismantlable* if G is cop-win with a dismantling sequence $(x_1, x_2, \ldots, x_n)$ such that

(1) the graph G has at least two separate corners;
(2) for all $3 \leq i \leq n-2$, the graph G_i induced in G by $\{x_i, \ldots, x_n\}$) contains at least two separate corners.

Thus, deleting two corners from a 2-dismantlable graph with at least four vertices leaves another 2-dismantlable graph. Each connected chordal graph is 2-dismantlable (as chordal graphs contain at least two simplicial vertices). However, the 4-wheel (C_4 plus a universal vertex) is 2-dismantlable but not chordal.

Theorem 2. *If G is a finite 2-dismantlable graph, then* $st(G) \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$.

Proof. The proof is by induction on $n=|V(G)|$. The cases where $n \leq 3$ follow immediately. Suppose that G has $n+1 \geq 4$ vertices and is 2 -dismantlable. Let a, b be two corners of G, and let H be the 2 -dismantlable graph $G-\{a, b\}$. Suppose that x is dominated by x', where x is either a or b, and x' is not equal to a nor b.

There is an optimal game on H of length at most $\left\lfloor \frac{n-2}{2} \right\rfloor$. The cop plays this optimal game in H, and whenever $\mathcal{R}$ is on x, then $\mathcal{C}$ plays as if he were on x'. After at most $\left\lfloor \frac{n-2}{2} \right\rfloor$ moves, either the robber is caught on H, or $\mathcal{R}$ is on x and $\mathcal{C}$ is on x'. But then $\mathcal{C}$ can win in one more move, and so this strategy uses at most $\left\lfloor \frac{n-2}{2} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor$ moves. As $st(G)$ is the length of an optimal game, we have that $st(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$. $\square$

Not every cop-win graph has two corners. For an integer $n \geq 4$, define $G(n)$ by adjoining two vertices x and y joined to each vertex of a path P with n vertices. Add a vertex z that is joined to y and the endpoints of P. Then $G(n)$ is cop-win but z is the unique corner of $G(n)$. See Figure 1.

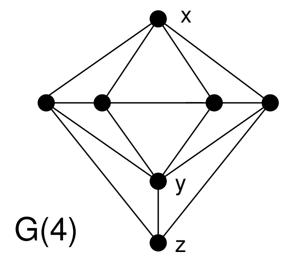

FIGURE 1. The graph $G(4)$.

Using the graph $G(4)$ as a template, we construct an infinite family of graphs whose search-time is within an additive constant of the upper bound $|V(G)|$ of Theorem 1. For $n \geq 7$, the graph $H(n)$ has vertices $1, \ldots, n$, where $1, 2, 3, 4, 5, 6, 7$ induce $G(4)$ (so that $x = 5, y = 3, z = 7$, and the remaining vertices on the path joined to x and y are (from left to right) $6, 2, 1, 4$.) For $i > 7$, the vertex i is joined to $j < i$ if j equals one of $i - 4$, $i - 3$, and $i - 1$. We name the vertices $7, 8, \ldots, n$ *special*. See Figure 2 for $H(11)$.

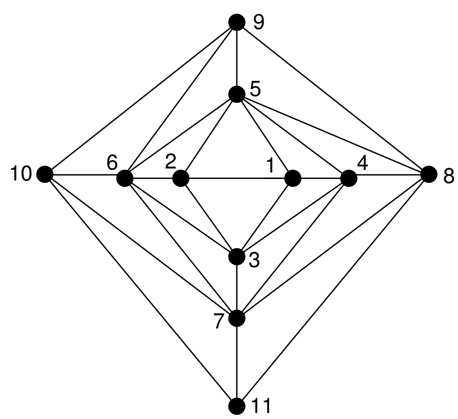

FIGURE 2. The graph $H(11)$.

Theorem 3. *For a fixed integer $n \geq 7$, the graphs $H(n)$ have the following properties.*

(1) *The graph $H(n)$ is planar.*
(2) *The graph $H(n)$ is cop-win and has a unique corner.*
(3) $st(H(n)) = n - 4$.

Proof. To see item (1), we describe a planar drawing of $H(n)$. The rough idea is to spiral the special vertices around $G(4)$ in an anti-clockwise fashion. To be more precise, first draw $G(4)$ as in Figure 1. Embed this drawing of $G(4)$ in the unit square in any fixed way, so that vertex 4 has coordinates $(1, 0)$, 5 has coordinates $(1, 0)$, 6 has coordinates $(-1, 0)$, and 7 has coordinates $(0, -1)$.

For each vertex i, $i \geq 8$, write i as $r4 + s$, where $r \geq 2$ and s is one of $0, 1, 2, 3$. Place $r4 + s$ at coordinate $(r, 0)$ if $s = 0$, at $(0, r)$ if $s = 1$, at $(-r, 0)$ if $s = 2$, and $(0, -r)$ if $s = 3$. Hence, we place the special vertices u around $G(4)$, so u is on the positive or negative arms of the x- or y-axes depending on the residue of x $(mod\ 4)$. It is easy to see that this is a planar drawing of $H(n)$.

For item (2), note that the graph $H(n)$ is cop-win, since we may dismantle the special vertices in reverse order $n, n-1, \ldots, 2, 1$. By a straightforward (and so omitted) induction argument, n is the unique corner of $H(n)$.

For item (3), we present a strategy $\mathcal{S}$ for the cop to win which always results in a game of length at most $n - 4$. First note that each vertex $5 \leq x \leq n-4$ has neighbours exactly $\{x-4, x-3, x-1, x+1, x+3, x+4\}$. So the cop and robber may move to vertices with index $1, 3$ or 4 more or less than their current index.

The strategy $\mathcal{S}$ has three parts, with the third part repeated until the robber is captured (which we will demonstrate eventually happens).

($S1$) In the 0th round, place the cop on vertex 1.

($S2$) After robber places himself on $i > 1$ in the 0th round, in the first round move the cop to $j \in V(G(4))$ with $j \in \{2, 3, 4, 5\}$ so that such that $i \equiv j$ $(mod\ 4)$.

($S3$) Repeat the following steps until the robber is eventually caught.
- (a) If robber moves from i to $i + k$, then the cop moves from j to $j + k$, where $k = 1, 3$, or 4.
- (b) If robber moves from i to $i - 1$, then cop moves from j to $j + 3$.
- (c) If robber moves from i to $i - 3$, then cop moves from j to $j + 1$.
- (d) If robber moves from i to $i - 4$, then cop moves from j to $j + 4$.

Let the cop play with $\mathcal{S}$, and let $cop(t)$ and $robber(t)$ be the positions of the cop and robber at round t in this game. Note that for all $t \geq 0$, $cop(t + 1) > cop(t)$. We prove by induction that for all $t \geq 1$,

$$cop(t) \equiv robber(t - 1)\ (mod\ 4). \tag{2.1}$$

The base case of (2.1) follows by ($S1$) and ($S2$). Suppose (2.1) holds for a fixed $t \geq 1$. Suppose that $cop(t) = j$, with $robber(t-1) = i$. At time t, the robber moves to $i + m$, where $m \in \{-4, -3, -1, 1, 3, 4\}$.

Then the cop moves at round $t+1$ to $j+m'$ for some m' as instructed by ($S3$). It is straightforward to check that $i + m \equiv j + m'$ $(mod\ 4)$ holds for all possible moves of the robber. Hence, the induction step follows.

It follows that the difference of the indices of the cop and robber is kept 0 $(mod\ 4)$, and when the robber goes to a higher or lower index, the difference is monotonically decreasing. Over time the cop gets closer and closer to the robber. Note that for all rounds except for the last one where the robber is captured, $cop(t) < robber(t)$. Eventually $robber(t-1) = i$ and $cop(t) = j$ so that $i = j + 4$. In that case, the robber can only evade capture by moving to $i + 1$ (which is not joined to j). The cop then moves to $j + 5$ as instructed by ($S3a$). The index of the robber will then eventually increase to n, with the robber in the unique corner n of $H(n)$, with the cop in $n-4$ (by repeated applications of ($S3a$)). The cop will then capture the robber in the next move.

To prove that $st(H(n)) = n - 4$, we need to show that the robber can survive this many steps against any strategy of the cop (including $\mathcal{S}$). To see this, note that the robber may use the lowest indexed vertex that is available. In round 0, the robber may be placed on a vertex $1, \ldots, 6$, since no vertex of $H(n)$ dominates all six of these vertices. In round $t \geq 0$, if $i = robber(t-1) \in \{1, 2, 3, 4\}$, then in such a case the robber can always move to a vertex of label at most 7 (we leave the verification to the reader). If $5 \leq i \leq n-1$, then the robber's moves are instructed by the following table.

$cop(t)$	$robber(t)$
$i-4$	$i+1$
$i-3$	$i-1$
$i-1$	$i-3$
$i+1$	$i-4$
$i+3$	$i-4$
$i+4$	$i-4$

Hence, $robber(t) \leq 6 + t$ for every $t \geq 1$, and so the robber cannot be caught in fewer than $n - 4$ moves of the cop. □

It is an open problem to find infinitely many cop-win graphs G with n vertices such that $st(G) = n-3$. Another open problem is to characterise graphs, called *k-cop-win*, on which $k \geq 2$ cops are necessary and sufficient to catch a robber. Almost nothing is known about directed graphs and cop-and-robber games. In an attempt to close these knowledge gaps, Hahn and MacGillivray [13] provide an algorithmic characterisation whose byproduct is a polynomial (in the number of vertices) time algorithm for determining if $st(G) \leq k$ for a given (di)graph G and a given k. It is based (in retrospect) on the characterisation of cop-win graphs given in [17] that also applies to infinite graphs. The idea is to assign an ordinal to each pair of vertices (u, v), indicating how many rounds the cop at u needs to win if the robber is at v.

Let G be a graph (finite or infinite). Note that in what follows, "$<$" means "$\lneqq$". Define a sequence of order relations $<_\alpha$, $\alpha \leq |V(G)|$, recursively as follows.

- $u <_0 u$ for all $u \in V(G)$;
- $u <_\alpha v$ if for each $x \in N[u]$ there is a $y \in N[v]$ and a $\beta < \alpha$ such that $x <_\beta y$.

Since $<_\beta \subseteq <_\alpha$ whenever $\beta \leq \alpha$, and, clearly, $<_{|V(G)|+1} = <_{|V(G)|}$, there is a least α such that $<_\gamma = <_\alpha$ for all $\gamma \geq \alpha$. Set $\preceq = <_\alpha$.

Theorem 4 ([17]). *A graph G is cop-win if and only if the relation $\preceq$ is trivial, that is, $u \preceq v$ for all $u, v \in V(G)$*

In [13], an algorithm based on this idea is developed that decides, in time polynomial in the number of vertices of the (obviously finite) input graph G, whether or not the graph is k-cop-win (k fixed). From the labelling (really by the appropriate β) of the *vertices* one can read off the length of an optimal game. The problem of deciding whether the search-time of a graph is at most t is, therefore, polynomial in the number of vertices of the graph, provided the number k of cops is fixed.

One absence that motivates the present paper is that of a good (that is, achievable) bound on the search-time in terms of some known graph parameters. For example, the diameter might seem a likely candidate, but it is shown in [12] that for every natural number k there is a chordal, diameter 2 finite cop-win graph with search-time k (and, by compactness, that there are chordal, diameter 2 infinite graphs which are not cop-win). Other parameters do not bound the search-time, such as the length of a longest path (consider the complete graph), or the length of a longest chordless path (consider a graph obtained from a path by adding a new universal vertex). One might hope to get a bound in terms of the least l such that $v_l, \ldots, v_n$ induce a complete graph in a finite cop-win graph G with an enumeration of its vertices as guaranteed by the structural characterisation theorem, but the last-mentioned counterexample works here as well. The question is as interesting - and as open - for infinite graphs. Let S be the graph obtained from $(\mathbb{N}, \{0i : 0 \neq i \in \mathbb{N}\})$ by replacing each edge $0i$ by a path $0v_i^1 v_i^2 \dots v_i^i$ of length i; this is a star with 0 as its centre and paths of length i (one for each $i > 0$) as rays. Clearly the robber determines the length of the game by his choice of his starting vertex. Let SP be the graph obtained from the ray (one-way infinite path) $v_0v_1v_2\dots$ by the addition of a new vertex v adjacent to all the vertices of the ray (call SDP the graph obtained from the double ray by the same method). The two graphs SP and SDP provide counterexamples to hypotheses as to the bounds on the number of moves in terms of the diameter or of the length of a longest chordless path.

3. Search-time density and infinite graphs

In this section we introduce a new parameter which measures limits of the ratio of the cop time to the number of vertices over chains of induced subgraphs. A similar approach was given in [5] for the cop number.

As proven by Erdős and Rényi [10], with probability 1, a countably infinite random graph has a unique isomorphism type written R. The (deterministic) graph R is called the *infinite random graph* or the *Rado* graph. The graph R is the unique isomorphism type of countable graph satisfying the *e.c. property*: for all finite disjoint sets of vertices A, B, there is a vertex $z \notin (A \cup B)$ such that z is joined to each vertex of A and to no vertex of B. From the results of [5], $c(R) = \aleph_0$; that is, R is *infinite-cop-win*. By [10], it follows that almost all countably infinite graphs are infinite-cop-win.

Similar to [5] we consider the density of the search-time parameter, relative to the number of vertices. A *chain of graphs* is a sequence $(G_n : n \in \mathbb{N})$, each G_n is an induced subgraph of G_{n+1}, for all $n \in \mathbb{N}$. Given a chain $(G_n : n \in \mathbb{N})$ of induced subgraphs of G, we write $G = \lim_{n\to\infty} G_n$ if $V(G) = \bigcup_{n\in\mathbb{N}} V(G_n)$ and $E(G) = \bigcup_{n\in\mathbb{N}} E(G_n)$. Note that every countable graph G is the limit of a chain of finite graphs, and there are infinitely many distinct chains with limit G. Suppose that $G = \lim_{n\to\infty} G_n$, where $C = (G_n : n \in \mathbb{N})$ is a fixed chain of induced subgraphs of G. We say that C is a *full chain for* G. For G a finite graph, define the *search-time density* by

$$D_{st}(G) = \frac{st(G)}{|V(G)|}.$$

By Theorem 1, $D_{st}(G)$ is a rational number in $[0, 1)$. We may extend the definition of D_{st} as follows. For a full chain $C = (G_n : n \in \mathbb{N})$ in

G, define

$$D_{st}(G, C) = \lim_{n\to\infty} D_{st}(G_n).$$

We refer to this as the *search-time density of G relative to C*. For simplicity, we will always consider G and C where this limit exists. Note $D_{st}(G, C)$ is a real number $[0, 1]$.

For example, let C be the chain with G_n isomorphic to P_n, and so the limit graph G is the infinite one-way path. In this case, $D_{st}(G, C) = \frac{1}{2}$. If we let our chain consist of the graphs $H(n)$, for $n \geq 7$ (where $H(n)$ is embedded in obvious way in $H(n+1)$), then the search-time density of the limit graph relative to this chain is 1.

For the infinite random graph, we obtain the surprising result that the search-time density can be any real number in $[0, 1]$.

Theorem 5. *For all $r \in [0, 1]$, there is a full chain C in R such that $D_{st}(G, R) = r$.*

Proof. Let $(p_n : n \in \mathbb{N})$ be a sequence of rationals such that $p_n \in [0, 1)$ if $n \geq 1$, $p_0 = 0$, and $\lim_{n\to\infty} p_n = r$. We construct a chain $C = (G_n : n \in \mathbb{N})$ in $G = R$ such that $G = \lim_{n\to\infty} G_n$, and with the property that $D_{st}(G) = p_n$, and each G_n is cop-win. Enumerate $V(G)$ as $\{x_n : n \in \mathbb{N}\}$.

We proceed inductively on n. For $n = 0$, let G_0 be the subgraph induced by x_0. Then $D_{st}(G_0) = 0 = p_0$.

Fix $n \geq 1$, suppose the induction hypothesis holds for all $k \leq n$, and let $p_{n+1} = \frac{a}{b}$, where a, b are positive integers. Further suppose for an inductive hypothesis that $\{x_0, \ldots, x_n\} \subseteq V(G_n)$. Without loss of generality, we may assume that $a < b$ and that $\gcd(a, b) = 1$.

We add vertices to G_n in several ways. First, if necessary, add x_{n+1} to G_n, to form the graph G''_{n+1}. Next, form G'_{n+1} by adding a universal vertex u (u exists by the e.c. property). Then $D_{st}(G'_{n+1}) = 1$, and say that $|V(G'_{n+1})| = b'$, where b' is a positive integer. If $\frac{1}{b'} = \frac{a}{b}$, then let $G_{n+1} = G'_{n+1}$. Otherwise, we add some new vertices to adjust the parameter $D_{st}(G'_{n+1})$.

We may add an arbitrary finite number y of endvertices to u. For some $x \geq 7$, by the e.c. property add a copy of $H(x)$ by identifying the vertex u with 1 and keeping all other vertices disjoint from the existing ones. This gives the graph G_{n+1}, with $x \geq 5$ and y to be determined. The graph G_{n+1} is cop-win, since $H(x)$ can be dismantled to $1 = u$, and all of G'_{n+1} can be dismantled to 1.

We claim that $st(G_{n+1}) = x - 4$. To see this place the cop first at 1. If the robber is in G'_{n+1}, then he is caught in the next round. If the robber is in $H(x)$, then play $\mathcal{S}$ as in the proof of Theorem 3. The optimal play for the robber is then also is as in the proof of Theorem 3. As the robber is always on a vertex $i \geq 6$, by properties of $\mathcal{S}$ he can never "escape" to 1 and G'_{n+1} without being captured in less than $x-4$ moves. Note that the y endvertices attached to 1 do not increase the search-time. In this way, the robber can play so that the game takes $x-4$ moves.

Thus,

$$D_{st}(G_{n+1}) = \frac{x-4}{b'+x+y} = \frac{a}{b} \tag{3.1}$$

where x, y are positive integers. We must find positive integer solutions (x, y) with $x \geq 5$ to the Diophantine equation

$$(b-a)x - ay = b'a + 4b.$$

As $\gcd(b-a, -a) = \gcd(a, b)$ and $b-a$ and $-a$ are opposite signs, we may find infinitely many of the desired (x, y). This completes the induction step in constructing G_{n+1}.

As $\{x_0, \ldots, x_n\} \subseteq V(G_n)$ for all $n \in \mathbb{N}$, we have that $C{=}(G_n : n \in \mathbb{N})$ is a full chain for G. Further,

$$D(G, C){=} \lim_{n \to \infty} p_n = r.$$

$\square$

4. More than one cop

The general situation for graphs with cop number greater than 1 appears to be more complex. For example if $n \geq 3$, then it can be shown that

$$st(C_n) = \begin{cases} \frac{n-i}{4} & \text{if } n \equiv i \pmod 4,\ i = 0, 1, 2; \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

To study the case for more than one cop, we introduce another parameter related to the lengths of games. If G is a graph, then let $c_t(G)$ smallest number of cops needed to capture of robber in no more than t rounds. Then $c_0(G) = |V(G)|$, and $c_1(G)$ is the domination number of G. For chordal (but not general) graphs, $c_t(G)$ is t-domination number.

The following result contrasts with the complexity results shown in Section 2 for the search-time. Consider the following problem.

BOUNDED-TIME COP NUMBER

INSTANCE: A graph G and a nonnegative integer k.

QUESTION: Is $c_t(G) \leq k$?

Theorem 6. *For each positive integer t, the* BOUNDED-TIME COP NUMBER *problem is NP-complete.*

Proof. The problem is in NP since for a given strategy of cops we can construct a directed rooted tree of all possible moves of the robber (where the first move is a choice of the initial position) and corresponding moves of the cops. For every vertex of this tree the current positions of the cops and robber are stored (only initial positions of cops are stored for the root). This tree is a rooted tree of restricted size (out-degrees of vertices are restricted by $|V(G)|$, and distance from the root to leaves is no more than t). Hence, given the tree it may be checked in polynomial time whether the strategy of cops is winning.

For the NP-completeness, we reduce the SATISFIABILITY problem (see [16]) to our problem. Let C be a boolean formula in conjunctive normal form with variables $x_1, x_2, \ldots, x_k$ and clauses $C_1, C_2, \ldots, C_m$. From the formula C we construct the graph $G(C)$ as follows. For every variable x_i, introduce vertices x_i and $\overline{x_i}$ which are connected by an edge. A path P_t with one endpoint u_i is added, and vertex u_i is joined by edges with x_i and $\overline{x_i}$. For every clause C_j add a path P_t with endpoints y_j and z_j. For every literal from C_j, join the vertex y_j to x_i if this literal contains a positive occurrence of this variable, and to $\overline{x_i}$ if this literal contains a negative occurrence.

Suppose that k cops can capture the robber in no more than t steps. Then for every $i = 1, 2, \ldots, k$, the cops must be placed on either x_i or $\overline{x_i}$, or on one of vertices of the path with endpoint u_i. If a cop occupies x_i, then let $x_i =$ true, and $x_i =$ false if a cop occupies $\overline{x_i}$. If a cop is placed on any other vertex, then the value of x_i is chosen arbitrarily. If some clause C_j is not satisfied by this truth assignment, then clearly the robber can occupy the vertex z_i and avoid capture in the first t rounds. Conversely, suppose that variables $x_1, x_2, \ldots, x_k$ have a truth assignment for which C has value true. If $x_i =$ true, then at the beginning of the game we place a cop on vertex x_i, otherwise a cop is placed on $\overline{x_i}$ for $i = 1, 2, \ldots, k$. Since each clause contains a positive literal, each y_i is joined to a vertex with a cop. The strategy for the cops to win is now clear and captures the robber in at most t rounds, no matter what his strategy is. □

The BOUNDED-TIME COP NUMBER remains NP-complete for planar graphs and for chordal graphs. The proof of the NP-complexity of the problem for chordal graphs follows since the t-dominating problem for chordal graphs is NP-complete (see [8]). For planar graphs, we need a bit more preliminary work.

Let C be a boolean formula in conjunctive normal form with variables $x_1, x_2, \ldots, x_k$. Define $H(C)$ to be the bipartite graph with vertices $x_1, x_2, \ldots, x_k$ and $C_1, C_2, \ldots, C_m$ such that x_i and C_j are joined if and only if clause C_j contains the literal x_i or $\overline{x_i}$. It is known (see [15]) that Satisfiability (and 3-Satisfiability) remains NP-complete if the graph $H(C)$ is planar and every variable occurs in no more than four clauses. We require the following lemma.

Lemma 7. Satisfiability *remains NP-complete even when $H(C)$ has a plane embedding such that if clauses C_r and $C_{r'}$ contain x_i, and C_s and $C_{s'}$ contain $\overline{x_i}$, then the edges x_iC_r and $x_iC_{r'}$ are edges in the boundary of one face.*

Proof. Consider a fixed planar embedding of $H(C)$. Suppose that the condition of the claim is not fulfilled for a fixed vertex x_i. We add a new variable x_i' and replace x_i by x_i' in $C_{r'}$ and $C_{s'}$. Then clauses $x_i \vee \overline{x'}$ and $\overline{x_i} \vee x_i'$ are added. Let C' be the resulting boolean formula. It can be easily seen that the given embedding of $H(C)$ can be replaced by a planar embedding of $H(C')$, for which the condition of the claim for x_i and x_i' is satisfied, without violating the condition for the other vertices. Since $(x_i \vee \overline{x'}) \wedge (\overline{x_i} \vee x_i') =$ true if and only if variables x_i and x_i' have same values, the formula C can be satisfied if and only if C' can be satisfied. □

It is not difficult to see that if $H(C)$ satisfies the conditions of Lemma 7, then $G(C)$ is planar. Thus, the bounded-time cop number problem remains NP-complete when $G(C)$ is planar.

Theorem 6 contrasts with the complexity of computation of the cop number (see [3, 11, 13]) and for search-time for both chordal and planar graphs. It is interesting to note that the value of $c_t(G)$ can be calculated in polynomial time for trees [14, 19], strongly chordal graphs [7], and interval graphs [1].

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