A2 Solutions

1.

(a) $$P\left(\frac{1}{X} \le y\right) = P\left(X \ge \frac{1}{y}\right)$$

$$= \left(\frac{y\theta}{1 + y\theta}\right)^{\alpha}$$

b) $$P\left(X^{1/\tau} \ge y\right) = P\left(X \ge y^{\tau}\right)$$

$$= \left(\frac{\theta}{y^{\tau} + \theta}\right)^{\alpha}$$

c) With $\tau < 0$, $\tau \neq -1$ let $\tau^* = -\tau > 0$

$$P\left(X^{\frac{1}{\tau}} \ge y\right) = P\left(X^{-\frac{1}{\tau^*}} \ge y\right)$$

$$= P\left(\frac{1}{X^{\tau^*}} \ge y\right) = P\left(X \le \frac{1}{y^{\frac{1}{\tau^*}}}\right)$$

$$= 1 - \left(\frac{\theta}{\theta + \frac{1}{y^{1/\tau^*}}}\right)^{\alpha} = 1 - \left(\frac{\theta}{\theta + y^{1/\tau}}\right)^{\alpha}$$

2.
(a)

$$P(-\log X \leq y)$$

$$= P(X \geq e^{-y})$$

$$= e^{-e^{-y}} \quad -\infty < y < \infty$$

b) $$P(-\sigma \log X + \mu \leq y)$$

$$= P\left(X \geq e^{-\frac{y-\mu}{\sigma}}\right)$$

$$= \exp\left(-\exp\left(-\left(\frac{y-\mu}{\sigma}\right)\right)\right)$$

$$-\infty < y < \infty$$

3.

$$p(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$p(\lambda) = \theta e^{-\lambda \theta}$$

$$p(x, \lambda) = \frac{\theta \lambda^x e^{-\lambda(\theta+1)}}{x!}$$

$$p(x) = \int_{\lambda=0}^{\infty} \theta \frac{\lambda^x e^{-\lambda(\theta+1)}}{x!} \, \partial \lambda$$

$$= \frac{\Gamma(x+1)}{x!} \frac{\theta}{(\theta+1)^{x+1}}$$

$$= \binom{x+1}{x} \left(\frac{\theta}{\theta+1}\right) \left(\frac{1}{\theta+1}\right)^x,$$

$$x = 0, 1, \ldots$$

4.

let $N = 0$ wp $.6$, $= 1$ w.p. $.4$)

a) $P(X=1) =$

$$P(X=1 \mid N=0)\,P(N=0)$$

$$+ P(X=1 \mid N=1)\,P(N=1)$$

$$= e^{-1}(.6) + 2e^{-2}(.4)$$

b) $E\,X \mid N=0 = 1$

$E\,X \mid N=1 = 2$

$\Rightarrow E(X) = 1(.6) + 2(.4) = 1.4$

c) $V(X \mid N=0) = 1$ $\quad V(X \mid N=1) = 2$

$\Rightarrow E\,V(X \mid N) = .6(1) + .4(2)$

$V(E(X \mid N)) = .6\,(1-1.4)^2 + .4\,(2-1.4)^2$

$= .6(.16) + .4(.36)$

$= 1.104$

5.

$$p(x) = \frac{\lambda^{x} e^{-\lambda}}{x\,!}$$

$$= \frac{\frac{1}{x!}\, e^{x \log \lambda}}{e^{\lambda}}$$

$$p(x) = \frac{1}{x\,!}$$

$$r(\lambda) = \log \lambda$$

$$q(\lambda) = e^{\lambda}$$

6.

$$\lim_{\substack{\alpha \to \infty \\ \theta \to \infty}} \left(\frac{\theta}{x + \theta} \right)^{\alpha}$$

$\frac{\alpha}{\theta} = c$

$$= \lim \left(1 + \frac{x}{\theta} \right)^{-\alpha}$$

$$= \lim_{\alpha \to \infty} \left(1 + \frac{c x}{\alpha} \right)^{-\alpha}$$

$$= e^{-cx} \quad , x > 0$$

3.

$P(X = x)$

$$= \int_{\lambda} P(X = x \mid \lambda) f_{\lambda}(\lambda)\, d\lambda$$

4. $P(X=1) = P(X=1 \mid N=0)P(N=0) + P(X=1 \mid N=1)P(N=1)$

$\cancel{N}$ $\quad P(N=0) = .6 \quad P(N=1) = .4$

$f_1(x) = 1, \quad 0 < x < 1$

$f_2(x) = \lambda e^{-\lambda(x-1)}, \quad x > 1$

$$f(x) = a_1 f_1(x) + a_2 f_2(x)$$

$$a_1 + a_2 = 1$$

$$a_1 f_1(1) = a_2 f_2(1)$$

$$a_1 = a_2 \lambda$$

$$a_2 (1 + \lambda) = 1$$

$$a_2 = \frac{1}{1+\lambda} \qquad a_1 = 1 - \frac{1}{1+\lambda} = \frac{\lambda}{\lambda+1}$$

$$f_2(1) = \lambda$$

$$a_2 f_2(1) = \frac{\lambda}{1+\lambda}$$