ACSC/STAT 3703 - Winter 2026 - Assignment 3 solutions

1. For any discrete random variable X with moment generating function $M_X(z)$ and probability generating function $P_X(z)$

   (a) Show that $P_X(z) = M_X(log_e(z))$.
   $P_X(z) = E[z^X] = E[e^{log(z^x)}] = E[e^{xlog(z)}] = M_X(log_e(z))$.

   (b) Using the fact that $E[X^k] = M_X^{(k)}(0)$, show that $P^{'}(1) = E[X]$ and $P^{''}(1) = E[X(X-1)]$.

$$P'(z) = E[Xz^{X-1}] \to P'(1) = E[X]$$

$$P''(z) = E[X(X-1)z^{X-2}] \to P''(1) = E[X(X-1)]$$

2. (a) where X has pmf $p_k = (\frac{\beta}{1+\beta})^k \frac{1}{1+\beta}$, $k = 0, 1, 2, \ldots$, show that $P_X(z) = (1 - \beta(z-1))^{-1}$

   (b) Suppose that $X_1, X_2, \ldots, X_r$ are independent, identically distributed with the pmf given in part (a). Derive the probability generating function of $N = \sum_{j=1}^{r} X_j$.

$$P_N(z) = E[Z^N] = E[Z^{\sum_{j=1}^{r} X_j}] = E[\prod_{j-1}^{r} Z^{x_j}] = \prod_{j-1}^{r} E[Z^{x_j}] = \prod_{j-1}^{r} P_{x_j}(z) = \prod_{j-1}^{r}(1-\beta(z-1))^{-1} = (1-\beta(z-1))^{-r}$$

   (c) Using the fact that $E(N) = P_N^{'}(1)$ and $E(N(N-1)) = P_N^{''}(1)$, derive the mean and variance of N.

$$P_N(z) = (1-\beta(z-1))^{-(r+1)}(r\beta) \to P'(1) = r\beta$$

$$P''(z) = (1-\beta(z-1))^{-(r+2)}(r(r+1)\beta^2) \to P''(1) = (r+1)r\beta^2$$

   The use $V(N) = E[N(N-1)] + E[N] - E[N]^2$.

3. For the Poisson distribution with mean λ, derive the values of a and b for the (a,b,0) recursion by examining $\frac{p_k}{p_{k-1}}$.

$$\frac{p_k}{p_{k-1}} = \frac{\lambda^k e^{-\lambda}/k!}{\lambda^{k-1}e^{-\lambda}/(k-1)!} = \frac{\lambda}{k}$$

   which means $a = 0$, $b = \lambda$.

4. Using the values of a and b which you just calculated, letting $\lambda = 1$, and approximating $p_0 = e^{-1}$ as .368,

   (a) use the (a,b,0) recursion to calculate p_1, p_2, p_3.

$$p_1 = p_0\frac{b}{1} = .368(1) = .368$$

$$p_2 = p_1\frac{b}{2} = .368(1/2) = .184$$

$$p_3 = p_2\frac{b}{3} = .184(1/3) \approx .061$$

   (b) now consider the associate 0-modified Poisson with $p_0^M = 0.10$. Calculate the alues of p_1^M, p_2^M, p_3^M
   multiply the vales above by $\frac{1-p_0^M}{1-p_0} = (1-.9)/(1-.368)$.

5. Problem 6.4. Show that for the ETNB with $\beta > 0$, $r > -1$ but $r \neq 0$, the values of p_k given by the (a,b,1) recursion are positive and $\sum_{k=1}^{\infty} p_k < \infty$.

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k} = \frac{\beta}{\beta+1} + (r-1)\frac{1}{k}\frac{\beta}{\beta+1}$$

and so $lim_{k\to\infty}\frac{p_k}{p_{k-1}} = \frac{\beta}{\beta+1} < 1$. Then the ratio test for convergence of infinite series says that $\sum_{k=1}^{\infty}|p_k| < \infty$.

Need also to show that $p_k > 0$ for all k. Suppose that $p_{k-1} > 0$.

Then by the recursion, $p_k = p_{k-1}(\frac{\beta}{\beta+1} + (r-1)\frac{1}{k}\frac{\beta}{\beta+1})$ whenever $\beta > 0$. The first term in brackets is > 0, so the bracketed term is greater than $\frac{\beta}{\beta+1}(1+\frac{r-1}{k})$ which is greater than or equal to 0 provided $k \geq 2$.

Provided that $p_1 > 0$, this means that $p_k > 0$ for $k \geq 2$, by induction.

6. Suppose you have a sample of size n from a discrete distribution on $0, 1, 2, \ldots$, and in the sample there are n_0 values equal to 0, n_1 equal to 1, and so on. You make a plot of $k\frac{n_k}{n_{k-1}}$ vs k and observe that the slope of the plot is close to 1. Does this suggest that the underlying distribution is binomial, negative binomial or Poisson?

   Slope is close to 1,which suggests that a is positive, which is compatible with a negative binomial distribution. (See table 6.1 in book)