Solutions

ACSC/STAT 3703 - Winter 2016 - Midterm 1, Feb 12

NAME: ____________________ SIGNATURE: ____________________

THIS EXAM HAS 3 PAGES.

SHOW YOUR WORK IN THE SPACE PROVIDED.

(Out of 55 points.)

1. Let X have density function given by

$$f_X(x) = \begin{cases} (x+1)e^{\theta x}\frac{\theta^2}{2\theta e^\theta - e^\theta - \theta + 1} & , 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

where $\theta > 0$.

(5) (a) Show that the distribution of X is from the linear exponential family, and calculate the functions $p(x)$, $q(\theta)$, and $r(\theta)$.

$p(x) = x+1$

$r(\theta) = \theta$

$$q(\theta) = \frac{2\theta e^\theta - e^\theta - \theta + 1}{\theta^2}$$

(5) (b) Calculate the mean of X (which will be a function of θ).

$$E(X) = \frac{q'(\theta)}{q(\theta)r'(\theta)} \qquad r'(\theta) = 1$$

$$q'(\theta) = \frac{\left(\theta^2(2e^\theta + 2\theta e^\theta - e^\theta - 1) - 2\theta(2\theta e^\theta - e^\theta - \theta + 1)\right)}{\theta^4}$$

$$E(X) = \frac{q'(\theta)}{q(\theta)r'(\theta)} = \frac{q'(\theta)}{q(\theta)}$$

2. X is a mixture of two exponential random variables with density function

$$f_X(x) = .25e^{-x} + .75(2)e^{-2x}, x > 0$$

(5) (a) Find the mean of X.

$$.25\int_0^\infty x e^{-x}\,dx + .75\int_0^\infty x\,2e^{-2x}\,dx$$

$$= .25 + .75\,\tfrac{1}{2} = \tfrac{1}{4} + \tfrac{3}{8} = \tfrac{5}{8} \approx .625$$

(5) (b) Find the variance of X.

$$EX^2 = .25\left(\frac{2}{1^2}\right) + .75\,\frac{2}{2^2} = \frac{1}{2} + \frac{3}{4}\cdot\frac{1}{2} = \frac{7}{8}$$

$$V(X) = EX^2 - (EX)^2 = \frac{7}{8} - \frac{25}{64} = \frac{56-25}{64} = \frac{31}{64} \approx .484$$

(5) (c) Find the survivor function of X.

$$S(x) = .25\int_x^\infty e^{-s}\,ds + .75\int_x^\infty 2e^{-2s}\,dx$$

$$= .25e^{-x} + .75e^{-2x}$$

(5) (d) Where $\tau > 0$, let $Y = X^{\frac{1}{\tau}}$. Find the density function of Y.

$x = y^\tau$

$\frac{dx}{dy} = \tau y^{\tau-1}$

using change of variables

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$$= (.25e^{-y^\tau} + .75(2)e^{-2y^\tau})\,\tau y^{\tau-1}$$

OR.

$$P(Y > y) = P(X^{1/\tau} > y) = P(X > y^\tau)$$

$$= .25e^{-y^\tau} + .75e^{-2y^\tau}$$

$$f_Y(y) = -\frac{d}{dy}P(Y > y)$$

$$= .25\,\tau y^{\tau-1}e^{-y^\tau} + .75(2)\tau y^{\tau-1}e^{-2y^\tau}$$

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3. X is a continuous random variable which has density function given by

$$f_X(x) = \begin{cases} \frac{1}{2} & , 0 < x < 1 \\ \frac{1}{2x^2} & , x \geq 1 \end{cases}$$

(5) (a) Integrate the density to find the distribution function $F_X(x)$ of X. (Your answer should be such that $S_X(x) = 1/(2x)$ when $x > 1$.)

$$F(x) = \begin{cases} x/2 & 0 < x < 1 \\ \frac{1}{2} + \int_1^x \frac{1}{2s^2}\,ds & x \geq 1 \end{cases}$$

$$\int_1^x \frac{1}{2s^2}\,ds = -\frac{1}{2s}\Big|_1^x = \frac{1}{2} - \frac{1}{2x}$$

$$= \begin{cases} x/2 & 0 < x < 1 \\ \cancel{1} - \frac{1}{2x} & x \geq 1 \end{cases}$$

(5) (b) Find the hazard function of X.

$$h_X(x) = \frac{f_X(x)}{S_X(x)} = \begin{cases} \frac{1}{2}\Big/\left(1 - \frac{x}{2}\right) & 0 < x < 1 \\ 1/x & x \geq 1 \end{cases}$$

$$\frac{1}{2x^2}\Big/\frac{1}{2x} = \frac{1}{x}$$

(5) (c) Based on the hazard function, would you say that the distribution has a heavier or lighter right hand tail than an exponential distribution? Why?

heavier tailed

exponential has constant hazard function so hazard function of X is smaller than for exponential at large X

(5) (d) Find the value at risk at the 99% level, $VaR_{.99}(X) = \pi_{.99}$.

$F(x) = .99$

$1 - \frac{1}{2x} = .99 \qquad \frac{1}{2x} = .01 \qquad x = \frac{1}{.02} = 50$

(5) (e) Find the tail value at risk of X at the 99% security level, $TVaR_{.99}(X)$.

$$TVaR_{.99}(x) = 50 + \frac{\int_{50}^{\infty} \frac{1}{2x}\,dx}{1 - .99} = \infty$$