ACSC/STAT 3703 - Assignment 5, solutions

1. Let X_1,X_2 and X_3 be independent random variables with probability mass function $P(X = 0) = 1/2$, $P(X = 1) = 1/4$, and $P(X = 3) = 1/4$.

   (a) Find the probability mass function of $S = X_1 + X_2$.
   Where p_0, p_1 and p_3 are the probabilities above

$$P(S = 0) = p_0^2$$

$$P(S = 1) = 2p_0p_1$$
$$P(S = 2) = p_1^2$$
$$P(S = 3) = 2p_0p_3$$
$$P(S = 4) = 2p_1p_3$$
$$P(S = 6) = p_3^2$$

   (b) Find the probability mass function of $S_3 = X_1 + X_2 + X_3$.
   S_3 has non-zero probability for S_3=0,1,2,3,4,5,6,7,9.
   For example, $P(S_3 = 1) = P(S_2 = 1)P(X_3 = 0) + P(S_2 = 0)P(X_3 = 1) = 2p_0p_1p_0 + p_0^2p_1$.

2. Suppose that X has an exponential distribution with mean 100, that there is an ordinary deductible of $d = 20$.

   (a) Out of 10 losses which are exponentially distributed with mean 100, how many would you expect to exceed the deductible of $d = 20$.? (Hint: first find $S_X(20)$. Then note that the number exceeding d is $\sum_{j=1}^{10} I[X_j > d]$ where $I[A]$ is the indicator of event A. Then use fact that mean of the sum is the sum of the means.)
   $P(X > 20) = e^{-20/100} = .819$, so the expected value is $10(.819) = 8.19$.

   (b) In this situation (10 independent exponentially distributed losses with mean 100, a deductible of 20) what is the probability that exactly 9 of the losses exceed the deductible? (Hint: the number exceeding d has a binomial distribution. Which binomial distribution?)
   The distribution os number exceeding the deuctible is binomial(10,.819), so P(N=9) is given by
   dbinom(9,10,.819)
   [1] 0.3000777

   (c) Where $E(X \wedge d)/E(X)$ is the loss elimination ratio, what would be the deductible required to have a loss elimination ratio of 1/2?
   $d = -100log(1/2) = 69.3$

   (d) Suppose now that in addition to a deductible d=20, there is a policy limit of $u = 200$, and a uniform inflation rate of 10%.

      i. What is the mean of the per loss variable Y_L?
      Using theorem 8.7, the mean is
      $1.1 * (E[X \wedge 200/1.1] - E[X \wedge 20/1.1]$, where $E[X \wedge c] = \theta(1 - e^{c/\theta})$

```
Ewedge=function(c,theta) return(theta*(1-exp(-c/theta)))
EYl= 1.1*(Ewedge(200/1.1, 100)-Ewedge(20/1.1,100))
EYl              [1] 73.85755
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ii. What is the expected cost per payment Y_P?

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EY1/exp(-20/100)
[1] 90.20982
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3. Consider the aggregate loss in the individual risk model $S = \sum_{i=1}^{200} X_i$, where the base losses $X_1, X_2, \ldots$ are exponential with mean 100. Use a normal approximation to approximate $P(S > 22772)$. You can leave the answer as $P(Z > c)$, where c is to be determined.

   The mean and standard deviation of the X_i's are both 100, so the mean and standard deviation of S are 20000 and $\sqrt{(200)20000}$. Standardize to get $z = (22772 - 20000)/\sqrt{200 * 100^2} \approx 1.96$

4. Suppose that $X_1, X_2, \ldots, X_N$ are independent Poisson random variables with mean λ, and are independent of N which has a Poisson distribution with mean θ. Find the probability generating function of $S = \sum_{i=1}^{N} X_i$?

$$P_S(z) = P_N(P_X(z)) = e^{\theta(e^{\lambda(z-1)}-1)}$$

5. Using the recursion for the (a,b,0) class given in section 6.5, and the values of p_0, a and b given in table 6.1, find p_1, p_2 and p_3 for the binomial distribution with parameters $m = 6$ and $q = .25$.

```
q=.25; m=6;  a=-q/(1-q); b= (m+1)*q/(1-q); p0=(1-q)^m
p=rep(0,4)
p[1]=p0
for (k in 2:7) p[k]=(a+b/(k-1))* p[k-1]
p
[1] 0.1779785156 0.3559570312 0.2966308594 0.1318359375 0.0329589844
[6] 0.0043945313 0.0002441406

#check
dbinom(0:6,6,.25)
[1] 0.1779785156 0.3559570312 0.2966308594 0.1318359375 0.0329589844
[6] 0.0043945312 0.0002441406
```

(a) find p_1^T, p_2^T, and p_3^T for the associated zero truncated random variable.

```
    pt=p[-1]/(1-p[1]); pt
[1] 0.4330264330 0.3608553609 0.1603801604 0.0400950401 0.0053460053
[6] 0.0002970003
```

(b) find p_1^M, p_2^M, and p_3^M for the associated zero modified random variable with $p_0^M = .5$.

```
p0M=.5
p0=p[1]
c=(1-p0M)/(1-p0)
pM=p*c
pM[1]=p0M
pM
[1] 0.5000000000 0.2165132165 0.1804276804 0.0801900802 0.0200475200
[6] 0.0026730027 0.0001485001
> sum(pM)
[1] 1
```

6. Let X_1, X_2, ... be independent gamma random variables with parameters α and β, independent of N which has a Poisson distribution with mean λ.

   Let $S = \sum_{i=1}^{N} X_i$. (The X's might represent individual insurance claims, and N the number of claims in a year, in which case S is the total of claims in a year.) Find the mean and variance of S.

$$E[S] = E[N]E[X] = \lambda\alpha\beta$$

$$V[S] = V[E[S|N]] + E[V[S|N]] = V[NE[X]] + E[NV[X]] = E[X]^2 V[N] + E[N]V[X]$$

$$= (\alpha\beta)^2\lambda + \lambda\alpha\beta^2$$