ACSC/STAT 3703 Assignment 6 solutions

1. Let $X_j = I_j B_j$ and $S = \sum_{j=1}^{n} X_j$, where $I_1, I_2, \ldots, I_n, B_1, B_2, \ldots, B_n$ are independent.

   B_j has moment generating function $M_{B_j}(z)$.

   $I_j = 1$ with probability q_j, and $I_j = 0$ with probability $1 - q_j$.

   It follows that $E[e^{B_j I_j z} | I_j = 0] = E[e^0] = 1$ and $E[e^{B_j I_j z} | I_j = 1] = M_{B_j}(z)$.

   (a) Combine these to find $M_{X_j}(z) = M_{B_j I_j}(z) = E[e^{B_j I_j z}]$.

   $$M_{X_j}(z) = E[e^{B_j I_j z}] = P(I_j = 0)E[e^0] + P(I_j = 1)E[e^{B_j z}]$$
   $$= (1 - q_j) + q_j M_{B_j}(z)$$

   (b) Now find $M_S(z)$.

   Just form the product of the previous, from $j = 1$ to $j = n$.

2. (9.63)

   (a) $E[S] = E[N^L]E[Y^L]$. $E[N^L] = 2(1.5)$. $E[Y^L]$ is calculated using theorem 8.7 as

   $$E[Y^L] = E[X \wedge 175] - E[X \wedge 50]$$

   From the appendix,

   $$E[X \wedge c] = \alpha\theta\Gamma(\alpha + 1, c/\theta) + c(1 - \Gamma(\alpha, c/\theta))$$

   where $\Gamma(\alpha, x)$ can be calculated in R as pgamma(x,alpha,1).

   $$V[S] = E[V[S|N^L]] + V[E[S|N^L]] = E[N^L]V[Y^L] + (E[Y^L])^2 V[N^L]$$

   From theorem 8.8,

$$E[(Y^L)^2] = E[(X \wedge u)^2] - E[(X \wedge d)^2]) + 2 * d(E[X \wedge d] - E[X \wedge u])$$

From the appendix,

$$E[(X \wedge c)^2] = \alpha(\alpha+1)\theta^2\Gamma(\alpha+2, c/\theta) + c^2(1 - \Gamma(\alpha, c/\theta))$$

and $V[N^L] = 2(1.5)(2.5) = 7.5$.

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> d=50
> u=175
>
> Exc=function(c,alpha=2,theta=100){
+   return(alpha*theta*pgamma( c/theta,alpha+1)+c*(1-pgamma(c/theta,al
> Exc(u)
[1] 134.8348
> Exc(d)
[1] 48.36734
> EYL=Exc(u)-Exc(d)
>
> Exc2=function(c,alpha=2,theta=100){
+   return(alpha*(alpha+1)*theta^2*pgamma( c/theta,alpha+2)+c^2*(1-pga
>
> Exc2(u)
[1] 20683.65
> Exc2(d)
[1] 2379.587
> EYL2=Exc2(u)-Exc2(d)+2*d*(Exc(d)-Exc(u)); EYL2
[1] 9657.314
>
> VYL=EYL2-EYL^2
>
> VS=3*VYL + EYL^2 *7.5; VS
[1] 62616.72
>
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(b) negative binomial with $r = 2$ and $\beta = 1.5P(X > 50)$. $P(X > 50) = 1 - pgamma(50/100, 2) \approx .91$, giving $\beta \approx 1.365$.

(c) for $x > 50$, $P(X > x|X > 50) = P(X > x \cap X > 50)/P(X > 50) = P(X > x)/P(X > 50)$. The distribution function is 1 minus this. In R, $P(X < x)$ is given by pgamma(x,alpha,theta).

3. (9.69)

$S = \sum_{j=1}^{900} I_j B_j$, where I_j is Bernoulli with success probability $q_j = .03$ (.07) [.10] for j=1 to 400 (401 to 700) [701 to 900], and B_j is exponential with mean $\theta_j = 5, 3, 2$ respectively for the three groups. $E[I_j] = q_j$, $V(I_j) = q_j(1 - q_j)$, $E[B_j] = \theta_j$ and $V[B_j) = \theta_j^2$. Also $E[I_j B_j] = E[I_j]E[B_j]$ and

$V[I_j B_j] = V(E[I_j B_j|I_j]) + E(V[I_j B_j|I_j]) = V[I_j](E[B_j])^2 + E[I_j]V[B_j]$.

For the exponential distribution, the variance is the square of the expectation, and so

$$V[I_j B_j] = \theta_j^2(q_j + q_j(1 - q_j)) = \theta_j^2(2q_j - q_j^2)$$

Calculate mean $E[S]$ and $\sigma_S = \sqrt{V(S)}$ of S using these results, then find s so that $P(S > s) = .05$ using $P(Z > (s - E[S])/\sigma_S) = .05$, which gives $s = 1.645\sigma_S + E[S]$.

$$E(S) = 400(.03)(5) + 300(.07)3 + 200(.1)(2) = 163$$

$$V(S) = 400(25)(.06-.03^2)+300(9)(.14-.07^2)+200(4)(.2-.1^2) = 1107.07$$

The premium to be charged is $s = 1.645 * \sqrt{1107.07} + 163 = 217.73$.