Assignment 8 Solutions

1. Let X_1, X_2, X_3 and X_4 be independent random variables with probability mass function $P(X = 0) = 3/4$ and $P(X = 1) = 1/4$. Find the probability mass function of $S = X_1 + X_2 + X_3 + X_4$.

   This can be done as in problem 1 of assignment 5. Easier solution is just to note that by definition S has a binomial distribution with $n = 4$ and success probability $1/4$.

2. Suppose that X has a discrete distribution which is a finite mixture of a Poisson distribution with mean 2 and a Poisson distribution with mean 3, given by:

$$P(X = k) = .25\frac{2^k e^{-2}}{k!} + .75\frac{3^k e^{-3}}{k!} \qquad , k = 0, 1, 2, \ldots$$

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   (a) Calculate $P(X \geq 2)$.
   Let Z=0 with probability .25 and Z=1 with probability .75. $P(X \leq 1|Z = 0) = e^{-2}(1 + 2)$ and $P(X \leq 1|Z = 1) = e^{-3}(1 + 2)$. Hence

$$P(X \geq 2) = .25(1 - 3e^{-2}) + .75(1 - 3e^{-3})$$

   (b) Calculate the probability generating function of X, $P_X(z) = E(z^X)$.
   $E[z^X|Z = 0] = e^{2(z-1)}$ and $E[z^X|Z = 1] = e^{3(z-1)}$. Therefore $E[z^X] = .25e^{2(z-1)} + .75e^{3(z-1)}$.

   (c) Evaluate the mean of X as $P_X'(1)$. Just differentiate the above and evaluate at z=1.

   (d) Evaluate $E[X(X - 1)] = P_X''(1)$.
   Differentiate twice and evaluate at z=1.

   (e) Combine the results of the previous two parts to obtain the variance of X.

$$V[X] = E[X(X - 1)] + E[X] - (E[X])^2$$

3. Suppose that N is a Poisson random variable with mean 100, and that losses $X_1, X_2, \ldots$ are i.i.d. having a normal distribution with mean 100 and standard deviation 20.

   (a) What is the mean of the aggregate claim in the compound model, for which $S = \sum_{i=1}^{N} X_i$?

$$E[S] = E[N]E[X] = 100(100)$$

   (b) What is the variance of $S = \sum_{i=1}^{N} X_i$?

$$V[S] = E[V[S|N]] + V[E[S|N]]] = E[N \times 400] + V[N \times 100] = 400(100) + 100^2(100)$$

   (c) What is $P(S > 200|N = 2)$?
   When N=2, S is normally distributed with mean 200, so the probability is 1/2.

(d) Do you think that the distribution of S will be well approximated by a normal distribution? Why or why not?

No, because it is a mixture of normal distributions with different means and variances.

4. In the previous question, if the individual losses $X_1, X_2, \ldots$ instead had exponential distributions with common mean 100, what would be the mean of the aggregate claim $S = \sum_{i=1}^{N} X_i$?

Same as the mean in previous problem.

5. Suppose that X has an exponential distribution with mean 100, that there is an ordinary deductible of $d = 20$.

   This is problem 2 of assignment 5, so solution may be found there.

   (a) Out of 10 losses which are exponentially distributed with mean 100, how many would you expect to exceed the deductible of $d = 20$.?

   (b) In this situation (10 independent exponentially distributed losses with mean 100, a deductible of 20) what is the probability that exactly 9 of the losses exceed the deductible?

   (c) Where $E(X \wedge d)/E(X)$ is the loss elimination ratio, what would be the deductible required to have a loss elimination ratio of 1/2?

   (d) Suppose now that in addition to a deductible d=20, there is a policy limit of $u = 200$, and a uniform inflation rate of 10%.

      i. What is the mean of the per loss variable Y_L?

      ii. What is the expected cost per payment Y_P?

6. Consider the aggregate loss in the individual risk model $S = \sum_{i=1}^{200} X_i$, where the base losses $X_1, X_2, \ldots$ are exponential with mean 100. Use a normal approximation to approximate $P(S > 22772)$.

   This is problem 3 of assignment 5. See solution there.

7. Let X have an exponential distribution with mean 1.

   (a) Find the value at risk at the 99% level, $VaR_{.99}(X) = \pi_{.99}$.
   This is the 99'th percentile of the distribution, so solve
   $1 - e^{-\pi_p} = .99$, or $\pi_p = -log(.01)$.

   (b) Find the tail value at risk of X at the 99% security level, $TVaR_{.99}(X)$.

$$\pi_p + \frac{\int_{x=\pi_p}^{\infty} e^{-x} dx}{1 - p}$$

   with $p = .99$.

8. For holders of a certain nursing home insurance policy, the average length of a nursing home stay is 300 days, and 50% of the stays are terminated in the first 20 days. The terminations are uniformly distributed in that period. The policy pays 50 per day for the first 20 days, and 200 per day after that. What are the expected benefits payable for a single stay?

   Let L be length of stay. Given $E[L] = 300$, and $P(L \leq 20) = .5$. The conditional distribution of L given that $L \leq 20$ is given as uniform on [0,20]. Also, $P(L \leq 20) = .5$, so the density function of L on the interval $0 < x < 20$ is $f(x) = \frac{1}{20} \times .5 = 1/40$.
   The policy pays 50 per day for the first 20 days, and 200 per day after that.

   Therefore the payment $P = 1000 + 200(L - 20)$ if $L > 20$, and $P = 50L$ if $L \leq 20$.

$$E[P] = \int_{x=0}^{20} 50x f_L(x)dx + \int_{x=20}^{\infty} (1000 + 200(x-20)) f_L(x)dx$$

$$= 50 \int_{x=0}^{20} x f(x)dx - 3000 \int_{x=20}^{\infty} f(x)dx + 200 \int_{x=20}^{\infty} x f(x)dx$$

$$= -3000 P(X > 20) + 200 \int_{x=0}^{\infty} x f(x)dx - 150 \int x = 0^{20} x f(x)dx$$

$$= -3000(.5) + 200 E[L] - 150 \int_{x=0}^{20} x \frac{1}{40} dx$$

$$= -15000 + 200(300) - 150(20^2/80)$$

9. (a) Use probability generating functions to show that if $X_1, X_2, \ldots, X_n$ are independent Poisson random variables with mean λ, then $S = \sum_{i=1}^{n} X_i$ also has a Poisson distribution. What is the mean of that distribution?

   The sum is Poisson with mean n. See proof at beginning of Chapter 6.

   (b) Use probability generating functions to show that if $X_1, X_2, \ldots, X_N$ are independent Poisson random variables with mean λ, and are independent of N which has a Poisson distribution with mean θ, then $S = \sum_{i=1}^{N} X_i$ does NOT have a Poisson distribution.

   This was a problem on midterm 2. See solution there.