Solutions

**ACSC/STAT 3703 - Winter 2017 - Assignment 5**

**Due: Thursday, March 24, at beginning of class**

All problems from textbook.

8.5
8.10
8.11
8.13
8.15
8.17
8.19
8.21
8.23
8.25
8.26
8.27
8.28
8.33

(Out of 140)

10 points per question

problem 8.17

```
> (3.5e5/3)*(1-(6/9.5)^3)-(3e5*(1-(6/9.5)^2)-3.5e5*(6/9.5)^3)/3
[1] 56555.86
```

problem 8.19

```
> e=exp(1)
> -100^2*e^(-.1)-200000*e^(-.1)+2000000*(1-e^(-.1))+10000*e^(-.1)
[1] 9357.68

> 2*1000^2-9357.68-200*1000+200*1000*(1-e^(-.1))
[1] 1809675
```

problem 8.23

```
u=10
s=1

> exp(u+s^2/2)*pnorm((log(100000)-u-s^2)/s)+100000*(1-pnorm(log(100000)-u))
  -((exp(u+s^2/2)*pnorm((log(50000)-u-s^2)/s))+50000*(1-pnorm(log(50000)-u)))
  +(50000*(1-pnorm((log(50000)-u)/s)))

[1] 16229.65
```

problem 8.28

```
.8*( (95-95^3/30000 ) - (20-20^3/30000 )) / (1-(20/100)^2)
[1] 38.90625
```

Numerical

Calculations

8.5

with deductible $d$

$$E(Y^P) = \frac{E(X) - E(X \wedge d)}{S_X(d)}$$

before the deductible, $E Y^P = E(X)$

$= E[X \wedge \infty] = 20000$

and $P(X > 20000) = 1 - F(X \le 15000) = .3$

with 10000 deductible

$$E Y^P = \frac{20000 - 6000}{1 - .6} = \frac{14000}{.4} = 35000$$

and $P(Y^P > 10000) = 1 - .6 = .4$

want to raise deductible such that

$P(Y^P > d) = .2 \Rightarrow d = 22500$

when $d = 22500$ $\quad E Y^P = \dfrac{20000 - 9500}{.2}$

$= 52,500$

percent change is $\quad 100 \dfrac{(52500 - 35000)}{35000}$

$= 50\%$

8.10 $f(x) = \frac{1}{1000} e^{-x/1{,}000}$

$d = 500$

$$LER = \frac{E\, X \wedge 500}{E\, X} = \frac{1000\left(1 - e^{-500/1{,}000}\right)}{1000}$$

$$= 1 - e^{-.5}$$

for deductible $d$ $\quad LER_d = 1 - e^{-d/1000}$

for what $d$ is

$$1 - e^{-d/1000} = 2(1 - e^{-.5})$$

$$e^{-d/1000} = 1 - 2(1 - e^{-.5})$$

$$= 2e^{-.5} - 1$$

$$d = -1000 \ln(2e^{-.5} - 1)$$

$$= 1546.175$$

with no inflation

8.11 $$E\,Y^P = \frac{E(X) - E(X \wedge d)}{S_X(d)}$$

with $d = 1{,}000$

$$E(Y^P) = \frac{20000 - 7700}{1 - .7} = \frac{12300}{.3} = 41000$$

with 50% inflation

$$E\,Y^P = 1.5\,\frac{\left(E\,X - E\left(X \wedge \frac{d}{1.5}\right)\right)}{1 - F_X\left(\frac{d}{1.5}\right)}$$

with $d = 15000$ & 15% inflation

$$E\,Y^P = 1.5\,\frac{(20000 - E(X \wedge 1000))}{1 - F_X(10000)}$$

$$= 1.5\,\frac{(20000 - 6000)}{1 - .6} = \frac{1.5(14000)}{.4}$$

$$= 52500$$

8.13

$E(X) = 2000$

$d = 1000$

$LER = .3$

$LER = \frac{E(X \wedge d)}{E(X)} \Rightarrow E(X \wedge 1000) = .3(2000) = 600$

$P(X > 1000) = .4 \Rightarrow P(X \le 1000) = .6$

$$E(X \mid X \le 1000) = \frac{\int_{x=0}^{1000} x f_X(x)\,dx}{P(X \le 1000)} \quad ①$$

$$E(X \wedge 1000) - 1000\,P(X > 1000) = \int_{x=0}^{1000} x f_X(x)\,dx \quad ②$$

① + ② $\Rightarrow$

$$E(X \mid X \le 1000) = \frac{E(X \wedge 1000) - 1000\,P(X > 1000)}{P(X \le 1000)}$$

$$= \frac{600 - 1000(.4)}{.6} = \frac{200}{.6} \simeq 333$$

8.15 $\quad X \sim \text{Pareto}(\alpha = 2, \theta = 100)$

$$e_X(d) = \frac{\int_{x=d}^{\infty} \left(\frac{\theta}{x+\theta}\right)^{\alpha} dx}{\left(\frac{\theta}{d+\theta}\right)^{\alpha}} = \frac{\int_{x=d}^{\infty} \left(\frac{100}{100+x}\right)^{2} dx}{\left(\frac{100}{d+100}\right)^{2}}$$

$$= (d+100)^2 \frac{1}{100+d} = d + 100$$

3

if $Y = 1.1X$, $\quad P(Y > y) = P(X > \frac{y}{1.1}) = \left(\frac{\theta}{\frac{y}{1.1} + \theta}\right)^{\alpha} = \left(\frac{1.1\theta}{y + 1.1\theta}\right)^{\alpha}$

$\Rightarrow$ $Y$ is Pareto$(\alpha = 2, \theta = 1.1(100))$

$\Rightarrow$ $e_Y(d) = d + 110$

$$\Rightarrow \frac{e_Y(d)}{e_X(d)} = \frac{d+110}{d+100}$$

this is a decreasing function of $d$, as its derivative is $-10/(d+100)^2$, which decreases from $1.1$ when $d=0$ to $1$, when $d \to \infty$.

4

With a policy limit of $u$ and a deductible of $d$, using theorem 8.7,

$$E(Z)=E(Y^P) = \frac{E(X\wedge u)-E(X\wedge d)}{S_X(d)}$$

and when $X \sim$ Pareto$(\alpha=2, \theta=100)$ and $u=500$

this equals

$$\frac{100\left(1-\frac{100}{600}\right)-100\left(1-\frac{100}{d+100}\right)}{\left(\frac{100}{d+100}\right)^2}$$

$$=\left(\frac{1}{d+100}-\frac{1}{600}\right)(d+100)^2 = d+100-\frac{(d+100)^2}{600}$$

$$=-\frac{d^2}{600}+d\left(\frac{2}{3}\right)+\frac{500}{6}$$

3

This is a quadratic in $d$, which has roots at $-100$ and $500$, is positive on $(0,500)$, takes its maximum value of $150$ at $d=200$, and is decreasing when $d>200$, with value $0$ at $d=500$. Hence its range is $[0,150]$.

8.17

$R = \frac{L}{P}$ , $B = P\left(\frac{.7 - R}{3}\right)$

$B > 0$ requires $.7 - R > 0 \Rightarrow L < .7P$

with $P = 5\times10^5$ , $B = 0$ if $L > 3.5\times10^5$

$$E(B) = \int_{l=0}^{3.5\times10^5} \left(\frac{3.5\times10^5 - l}{3}\right) f_L(l)\, dl$$

$$(*) \quad = \frac{3.5\times10^5}{3} F_L(3.5\times10^5) - \frac{1}{3}\int_{l=0}^{3.5\times10^5} l f_L(l)\, dl$$

using $E(X \wedge u) = \int_{-\infty}^{u} x f(x)\,dx + u S_X(u)$

$$\int_{l=0}^{3.5\times10^5} l f_L(l)\,dl = E(L \wedge 3.5\times10^5) - 3.5\times10^5\, S_L(3.5\times10^5)$$

when $L \sim \text{Pareto}(\alpha = 3,\ \theta = 6\times10^5)$

$$E(L \wedge 3.5\times10^5) = \frac{6\times10^5}{2}\left[1 - \left(\frac{6}{9.5}\right)^2\right]$$

and $S_L(3.5\times10^5) = \left(\frac{6}{9.5}\right)^3$

and $F_L(3.5\times10^5) = 1 - \left(\frac{6}{9.5}\right)^3$

combining, $(*) = 56555.86$ (see R calculation)

8.19

$d = 100$ $\qquad$ $X \sim$ exponential with mean 1000

$$E(Y^L) = E(X) - E(X \wedge 100)$$

$$= 1000 - 1000(1 - e^{-100/1000})$$

$$= 904.8$$

$$E[(Y^L)^2] =$$

$$E X^2 - E[(X \wedge 100)^2] - 200\, E X + 200\, E(X \wedge 100)$$

$$= 2(1000^2) - 9357.68 - 200(1000) + 200(1000)(1 - e^{-\frac{100}{1000}})$$

$$= 1809675$$ (see R calculations)

where we used $E[(X \wedge 100)^2]$

$$= \int_{x=0}^{100} x^2 f_X(x)\,dx + 100^2 \int_{x=100}^{\infty} f_X(x)\,dx$$

$$= \int_{x=0}^{100} x^2 \frac{e^{-\frac{x}{1000}}}{1000}\,dx + 100^2 e^{-\frac{100}{1000}}$$

$$= -x^2 e^{-\frac{x}{1000}}\Big|_0^{100} + \int_{x=0}^{100} 2x e^{-x/1000}\,dx + 100^2 e^{-\frac{100}{1000}}$$

$$= -100^2 e^{-100/1000} - 2000 x e^{-\frac{x}{1000}}\Big|_0^{100} + 2000 \int_0^{100} e^{-x/1000}\,dx + 100^2 e^{-\frac{100}{1000}}$$

$$= -100^2 e^{-.1} - 200000\, e^{-.1} + 2000000\, (1 - e^{-.1})$$

$$+ 10000\, e^{-.1}$$

$$= 9357.68$$

(see R calculations)

8.21 $E\left\{\sum_{j=1}^{n} x_j\right\} = 10^7$

$x_j \sim \text{Pareto}(\alpha = 2, \theta = 2000)$

$E(x_j) = \dfrac{\theta}{\alpha - 1} = 2000$

$\Rightarrow$ # policies $n = \dfrac{10^7}{2\times10^3} = 5\times10^3$

year a.

reinsurer pays $(x-3000)_+$

$E\left[(x-3000)_+\right] = E(x) - E(x \wedge 3000)$

$= 2000 - 2000\left(1 - \frac{2}{5}\right) = \frac{2}{5}(2000) = 800$

reinsurer is paid a premium of 110% of expected covered losses.

per loss this is $800(1.1) = 880$

for 5000 policies this is $880(5000)$
$= 4400 \times 10^3$

year b. losses include 5% inflation

$E(Y^L) = (1+r)\left[E(x) - E\left(x \wedge \frac{3000}{1+r}\right)\right]$

$= 1.05\,(2000 - E(x \wedge 2857.143))$

$= 1.05\left(2000 - 2000\left(1 - \frac{2000}{4857.143}\right)\right) = 864.71$

the reinsurer is paid a premium

of 110% of expected loss, so

$$1.1(864.71) = 951.18$$

the ratio of premiums is

$$\frac{951.8}{880} \cong 1.081$$

8.33

$$Y^L = \begin{cases} 0 & , \; x < 50000 \\ x & , \; 50000 < x < 100000 \\ 100000 & , \; x > 100000 \end{cases}$$

$$E(Y^L) = \int_{x=50000}^{10000} x f(x)\,dx + 100000\, S_x(100000)$$

$$= \int_{x=0}^{100000} x f(x)\,dx - \int_{x=0}^{50000} x f(x)\,dx + 100000\, S_x(100000)$$

$$= E(X \wedge 100000) - 100000\, S_X(100000)$$

$$- \left(E(X \wedge 50000) - 50000\, S_x(50000)\right)$$

$$+ 100000\, S_x(100000)$$

$$= E(X \wedge 100000) - E(X \wedge 50000) + 50000\, S_x(50000)$$

$$= \exp(10.5)\, \Phi(\ln(100000) - 10.5) + 100000\left(1 - \Phi(\ln(100000) - 10)\right)$$

$$- \exp(10.5)\, \Phi(\ln(50000) - 10.5) - 50000\left(1 - \Phi(\ln(50000) - 10)\right)$$

$$+ 50000\left(1 - \Phi(\ln(50000) - 10)\right)$$

$$= 16229.65 \quad \text{(see R code)}$$

8.25

$E(X \wedge 350)^2$

$$f_X(x) = \begin{cases} .3 \frac{1}{50} & 0 < x < 50 \\ .36 \frac{1}{50} & 50 < x < 100 \\ .18 \frac{1}{100} & 100 < x < 200 \\ .16 \frac{1}{200} & 200 < x < 400 \end{cases}$$

$$E(X \wedge 350)^2 = \int_0^{50} \frac{.3}{50} x^2 \, dx + \frac{.36}{50} \int_{50}^{100} x^2 \, dx$$

$$+ \frac{.18}{100} \int_{100}^{200} x^2 \, dx + \frac{.16}{200} \int_{200}^{350} x^2 \, dx$$

$$+ \frac{.16}{200} 350^2 \int_{x=350}^{400} dx$$

$$= \frac{.3}{50} \frac{50^3}{3} + \frac{.36}{50} \frac{100^3 - 50^3}{3}$$

$$+ \frac{.18}{100} \frac{200^3 - 100^3}{3} + \frac{.16}{200} \frac{350^3 - 200^3}{3}$$

$$+ \frac{.16}{200} 350^2 (50)$$

$$= 20750$$

10 points ~~5 points for details of solution~~

8.36

$$Y^L = \begin{cases} 0 & 0 < x < 1000 \\ .8(x-1000) & 1000 < x < 6000 \\ 4000 & 6000 < x < 14000 \\ 4000 + .9(x-14000) & x > 14000 \end{cases}$$

$$E(Y^L) = .8 \int_{1000}^{6000} x f(x)\,dx - 800\big(F(6000) - F(1000)\big)$$

$$+ .9 \int_{14000}^{\infty} x f(x)\,dx - .9(14000)\big(1 - F(14000)\big)$$

$$+ 4000 \underbrace{P(X > 6000)}_{1-F(6000)}$$

integrating by parts

$$\int_a^b x f(x)\,dx = \int_a^b x \frac{\alpha \theta^\alpha}{(x+\theta)^{\alpha+1}}\,dx$$

$$= \frac{a\theta^\alpha}{(a+\theta)^\alpha} - \frac{b\theta^\alpha}{(b+\theta)^\alpha} + \frac{\theta^\alpha}{(\alpha-1)(a+\theta)^{\alpha-1}} - \frac{\theta^\alpha}{(\alpha-1)(b+\theta)^{\alpha-1}}$$

and $F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha$

with $\alpha = 2$, $\theta = 5000$

$E Y^L = 2699.36$ (see R output)

can also argue that

$$E(Y^L) = -.8\,E(X\wedge 1000) + .8\,E(X\wedge 6000) - .9\,E(X\wedge 14000) + .9\,E(X\wedge 14000)$$

8.27 $X \sim \text{Poisson}(\lambda = 3)$ $EX = 3$

contract 1

$d = 2$ $EY^L = E(X) - E(X \wedge d)$

$= 3 - 1.75 = 1.2489.$

because.

$$E\, X \wedge d = \sum_{x=0}^{1} x\, p(x) + 2 \sum_{x=2}^{\infty} p(x).$$

$$= p(1) + 2 \sum_{x=2}^{\infty} p(x)$$

$$= p(1) + 2(1 - p(1) - p(0))$$

$$= 2 - p(1) - 2p(0) = 2 - 3e^{-3} - 2e^{-3}$$

$$= 2 - 5e^{-3} \approx 1.75$$

contract 2. no deductible
co-insurance rate $\alpha$

$EY^L = \alpha E(X) = 3\alpha$

equating expected values

$3\alpha = 1.2489 \Rightarrow \alpha \approx .416$

8.28 $\quad F(x) = \left(\frac{x}{100}\right)^2 \quad 0 \le x \le 100$

$$f(x) = \frac{1}{50}\,\frac{x}{500} = \frac{x}{5000}, \quad 0 \le x \le 100$$

$d = 20 \qquad \alpha = .8$

maximum payment is 60

$$\Rightarrow \alpha(u-d) = 60$$
$$\Rightarrow .8(u-20) = 60$$
$$\Rightarrow u = 20 + 60\left(\frac{5}{4}\right) = 95$$

$$E[Y^P] = \frac{.8\left(E(X \wedge 95) - E(X \wedge 20)\right)}{1 - F(20)}$$

$$E(X \wedge c) = \int_{x=0}^{c} \frac{x^2}{5000}\,dx + c\int_{x=c}^{100} \frac{x}{5000}\,dx$$

$$= \frac{c^3}{3 \cdot 5000} + \frac{c}{5000}\frac{(100^2 - c^2)}{2} = \frac{-c^3}{30000} + \frac{100^2 c}{10000}$$

substituting

$$= c - c^3/30000$$

$E[Y^P] \doteq 38.91$ (see R code)

8.33

$N_L \sim NB(r=3, \beta=5)$

$X \sim$ Weibull$(\tau=.5, \theta=1000)$

with a deductible of 200,

the probability that a loss results

in a payment is

$$P(X>200) = e^{-\left(\frac{200}{1000}\right)^{.5}} \doteq .54$$

expected # payments is

$$E(N_L)\,P(X \geq 200) = 3(5)(.54) \doteq 8.09$$

this is because

$$N_p = \sum_{j=1}^{N_L} I_j \quad \text{with } I_j = \begin{cases} 1, & x_j > 200 \\ 0, & x_j \leq 200 \end{cases}$$

and $E(I_j) = P(X_j > 200)$

$$\Rightarrow E(N_p) = E(N_L)\,P(X>200)$$