ACSC/STAT 3703 - Winter 2026 - Assignment 1 solutions

1. Let X have the gamma(α,θ) distribution with density function

$$f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\Gamma(\alpha)\theta^\alpha}, \; x > 0$$

where both α and θ are positive.

(a) Find the moment generating function of X.

$$M_X(t) = \frac{\int_{x=0}^{\infty} x^{\alpha-1}e^{-x(\frac{1}{\theta}-t)}dt}{\Gamma(\alpha)\theta^\alpha} = \frac{\Gamma(\alpha)\left(\frac{\theta}{1-t\theta}\right)^\alpha}{\Gamma(\alpha)\theta^\alpha} = (1-\theta t)^{-\alpha}$$

(b) By suitable manipulation of the moment generating function, find the first three moments of the gamma(α,θ) distribution.

i. $M'(t) = \alpha\theta(1-\theta t)^{-(\alpha+1)}$, so $\mu'_1 = M'(0) = \alpha\theta$.

ii. $M''(t) = \alpha(\alpha+1)\theta^2(1-\theta t)^{-(\alpha+2)}$, so $\mu''_2 = M''(0) = \alpha(\alpha+1)\theta^2$.

iii. $M'''(t) = \alpha(\alpha+1)(\alpha+2)\theta^3(1-\theta t)^{-(\alpha+3)}$, so $\mu'''_3 = M'''(0) = \alpha(\alpha+1)(\alpha+2)\theta^3$.

(c) Suitably manipulate these moments to find the skewness of the gamma(α,θ) distribution.
$\mu_3 = E[(X-\mu)^3] = E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3$. Substituting the non-central moments, this equals $2\alpha\theta^3$. Where the variance is $\sigma^2 = \alpha\theta^2$, the skewness is $\frac{2\alpha\theta^3}{(\alpha\theta^2)^{3/2}} = \frac{2}{\sqrt{\alpha}}$.

2. Let X have a two parameter Pareto distribution with survivor function $S(x) = \left(\frac{\theta}{x+\theta}\right)^\alpha$, where x, θ and α are all positive,

(a) Find the density of X.

$$f(x) = -S'(x) = -\alpha\left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\frac{-\theta}{(\theta+x)^2} = \frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$$

(b) Derive the hazard function of X.

$$h(x) = \frac{f(x)}{S(x})= \frac{\alpha}{\theta+x}$$

(c) Derive the mean excess loss function $e_X(d)$ of X.

$$\int_{x=d}^{\infty} S(x)dx = \left.\frac{\theta^\alpha(\theta+x)^{1-\alpha}}{1-\alpha}\right|_d^\infty = \frac{\theta^\alpha(\theta+d)^{1-\alpha}}{\alpha-1}$$

Dividing by $S(d)$, $e_X d = \frac{\theta+d}{\alpha-1}$. This requires $\alpha > 1$ as otherwise, the integral diverges.

3. Let X have density function

$$f(x) = \frac{2}{x^3}, \; x \geq 1$$

(a) Derive the distribution function of X.

$$F(x) = \int_1^x f(t)dt = \left.\frac{1}{t^2}\right|_{t=x}^1 = 1 - \frac{1}{x^2}$$

(b) Find the mean of X

$$E[X] = \int_1^\infty xf(x)dx = \int_1^\infty \frac{2}{x^2}dx = \left.\frac{2}{x}\right|_\infty^1 = 2$$

(c) Find the median of X

Median solves $1 - \frac{1}{x^2} = .5$, so $\frac{1}{x^2} = .5$, or $x = \sqrt{2}$.

(d) Find the mode of X

The density function is monotone decreasing, so has its maximum at 1, which is therefor the mode.

4. Let X have the Weibull density function with survivor function

$$S(x) = e^{-\left(\frac{x}{\theta}\right)^{\tau}}$$

where τ, θ and x are all positive.

(a) Derive the density function of X.

Differentiating, $f(x) = e^{-\left(\frac{x}{\theta}\right)^{\tau}} \tau(x/\theta)^{\tau-1}\frac{1}{\theta}$,

(b) Derive the formula for π_p, the p'th percentile of X. That is, solve $p = F(\pi_p)$.

$\pi_p = \theta(-log(1-p))^{1/\tau})$

(c) If the 25'th percentile is 1000 and the 75'th percentile is 10000, solve for τ.

$1000 = \theta(-log(.75))^{1/\tau}$ and $10000 = \theta(-log(.25))^{1/\tau}$, so

$10 = \left(\frac{-log(.25)}{-log(.75)}\right)^{1/\tau}$, solving for τ gives $\tau \approx .683$.

(d) Calculate the hazard function $h(x)$.

Using the density and survivor functions as above, the hazards function is $h(x) = \tau(x/\theta)^{\tau-1}\frac{1}{\theta}$.

(e) On the basis of the hazard function, would you say that the inverse Weibull is heavy tailed or light tailed?

Heavy tailed if $\tau > 1$ in which case $h(x)$ is increasing in x, and light tailed if $\tau < 1$ in which case $h(x)$ is decreasing in x. If $\tau = 1$ hazard function is constant, which is neither heavy tailed nor light tailed.

5. If $X_1, X_2, \ldots X_{100}$ are i.i.d. gamma distributed with mean 5000 and variance 25×10^6.

(a) Find the parameters α and θ of the gamma distribution.pp

Mean is $\alpha\theta$ and variance is $\alpha\theta^2$, so θ =25 $\times 10^6/5000 = 5000$.

The $\alpha = mean/\theta = 1$.

(b) Use the central limit theorem to approximate the probability that $\bar{X} = \frac{\sum_{i=1}^{100} X_i}{100}$ is greater than 5250. (You can leave the answer as $P(Z > c)$ for the appropriate value of c.)

$$P(\bar{X} > 5250) \approx P(Z > 10(5250 - 5000)/\sqrt{25 \times 10^6}) \approx P(Z > 0.5)$$

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6. Let X have an exponential distribution with density function

$$f(x) = \lambda e^{-\lambda x}$$

where x and λ are both positive.

(a) Find the hazard function $h(x)$ of X.

$S(x) = e^{-\lambda x}$, so $h(x) = \lambda$.

(b) Verify that

$$S(x) = e^{-\int_{y=0}^{x} h(y)dy}$$

$\int_{y=0}^{x} \lambda dy = \lambda x$, so $e^{-\int_{y=0}^{x} h(y)dy} = e^{-\lambda x}$

(c) Show that for any $x > 0$ and $c > 0$ that $P(X > x + c|X > c) = P(X > x)$. (This is

$$P(X > x + c|X > c) = \frac{P(X > x + c \bigcap X > c)}{P(X > c} = \frac{P(X > x + c)}{P(X > c} = \frac{e^{-\lambda(x+c)}}{e^{-\lambda(c)}} = e^{-\lambda x}$$

referred to as the memoryless property of the exponential distribution.)

(d) Suppose that $\lambda = 1$.

i. Find $VaR_{.999}(X)$, the Value at Risk of X at the 99.9% level.
This is just the 99.9'th percentile, so solves $1 - e^{-x} = .999$, or $x = -log(.001)$.

ii. Find $TVaR_{.999}(X)$, the Tail Value at Risk of X at the 99.9% security level.
This is the integral of the density function from $VaR_{.999}(X)$ to ∞. With $= 1$, and $VaR_{.999}X) = -log(.001)$.
Note that

$$\int_{u=p)}^{1} (log(1-u)du = \int_{v=0}^{1-p} log(v)dv = vlog(v) - v|_{v=0}^{1-p}$$
$$= (1-p)log(1-p) - (1-p)$$

This uses $lim_{v \to 0} vlog(v) = 0$, which can be shown using l'Hopital's rule.
For this example, $VaR_u(X) = -log(1-u)$, and so

$$TVaR_p(X) = \frac{1}{1-p}[(1-p) - (1-p)log(1-p)] = 1 - log(1-p)$$

then plug in $p = .999$.
Another solution is to use the result that when X is continuous, $TVaR_p(X) = VaR_p(X) + e(\pi_p)$, which is derived in section 3.5.4 of the text. Then use the memoryless property of the exponential distribution, $e(\pi_p) = E(X)$, which is 1 this example.