Resolutions of De Concini-Procesi ideals indexed by hooks

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Abstract

We find a minimal generating set for the De Concini-Procesi ideals indexed by hooks, and study their minimal free resolutions as well as their Hilbert series and regularity.

1 Introduction

In their study of the cohomology ring of the flag variety, De Concini and Procesi [DP], defined for any partition μ of n, an ideal $\mathcal{I}_\mu$ of the polynomial ring $R = \mathbb{Q}[x_1, \dots, x_n]$. In particular, they showed that the cohomology ring of the variety of the flags fixed by a unipotent matrix of shape μ may be presented as the graded quotient of the polynomial ring R by the ideal $\mathcal{I}_\mu$. The space $R/\mathcal{I}_\mu$ is actually an interesting graded representation of the symmetric group S_n, and it has been studied from different points of view by several authors. Garsia and Procesi [GP], studied its graded character and showed that it can be expressed in terms of Kostka-Foulkes polynomials. These polynomials appear in the expansion of the classical Hall-Littlewood polynomials in the basis of Schur functions [M2, Chapter III], and were conjectured to have positive integer coefficients. The result of Garsia and Procesi mentioned above, gave an elegant proof of this positivity conjecture. N. Bergeron and Garsia [BG] showed that as symmetric group representations, the $R/\mathcal{I}_\mu$ are isomorphic to certain spaces of harmonic polynomials. Aval and N. Bergeron in [AB], and Tanisaki in [T] gave different sets of generators for the ideal $\mathcal{I}_\mu$. Finally, another important feature of the S_n-modules $R/\mathcal{I}_\mu$ is that they led Garsia and Haiman [GH] to the definition of the doubly graded modules that appears in the famous $n!$ conjecture, recently solved by Haiman [H1].

Despite the spaces $R/\mathcal{I}_\mu$ having been extensively studied from the point of view of representation theory and combinatorics, no commutative algebra investigation of these objects has been done so far. The goal of this paper is to begin that study. One of the strongest tools for finding numerical informations about an ideal in a polynomial ring is finding its minimal free resolution. The resolution in particular produces all the numerical invariants that are described by the Hilbert function of the ideal. Finding an exact description of the resolution for a general ideal is usually a difficult task, there is a lot of research and numerous conjectures on this problem. However, when the partition μ indexing the De Concini-Procesi ideal $\mathcal{I}_\mu$ is a hook, we are able to produce a minimal generating set for $\mathcal{I}_\mu$, that we break into two parts. We show that one part forms an ideal with linear quotients, whose resolutions are described by Herzog and Takayama [HT]. We then show that the second part forms a regular sequence over the first part, and hence the resolution of this part is also well understood. Below we describe this construction in detail, and compute the Poincaré series associated to

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such an ideal (i.e. the generating function encoding the ranks of the free modules appearing in a minimal free resolution of the ideal). We also give a description of the Hilbert series of $R/\mathcal{I}_\mu$.

This paper is organized as follows. In Section 2, we give the basic definitions of partitions and the language used in the paper. We introduce De Concini-Procesi ideals, and compute a new generating set for them in the case of hooks; we show later in Section 4 that this generating set is minimal. Section 3 contains a review of resolutions, Cohen-Macaulay rings, and the other commutative algebra tools that we use in the paper. In Section 4 we study the resolutions of such ideals, and conclude with the formula of the corresponding bigraded Poincaré series. Finally, in sections 5 and 6, we compute the regularity and build the Hilbert series of the module $R/\mathcal{I}_\mu$.

We hope that our exposition will appeal to readers not only in commutative algebra, but also in combinatorics and invariant theory. This is why, throughout the paper, we review the background material we need from each field to make the concepts accessible to a wider audience.

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2 De Concini-Procesi Ideals

In this section, we introduce a family of ideals of the polynomial ring $R = k[x_1, \dots, x_n]$ indexed by partitions of n. These ideals were first introduced by De Concini-Procesi [DP], as ideals of the polynomial ring with rational coefficients. For our purpose k may be an arbitrary field of characteristic 0. Let us start with some definitions and notation about partitions, that will be used in the rest of this paper.

2.1 Partitions

We let $\mathbb{P} = \{1, 2, \dots\}$, and $\mathbb{N} = \mathbb{P} \cup \{0\}$. The cardinality of a set S is denoted by $|S|$. We define a *partition* of $n \in \mathbb{N}$ to be a finite sequence $\mu = (\mu_1, \dots, \mu_k) \in \mathbb{N}^k$, such that $\sum_{i=1}^k \mu_i = n$ and $\mu_1 \geq \dots \geq \mu_k$. If μ is a partition of n we write $\mu \vdash n$. The nonzero terms μ_i are called *parts* of μ. The number of parts of μ is called the *length* of μ, denoted by $\ell(\mu)$.

The *Young diagram* of a partition $(\mu_1, \dots, \mu_k) \vdash n$, is the diagram with μ_i squares in the i^{th}-row. We use the symbol μ for both a partition and its associated Young diagram. For example, the diagram of $\mu = (5, 4, 2, 1)$ is illustrated in Figure 1.

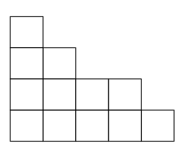
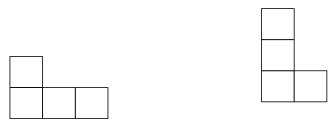


Figure 1: The partition $\mu = (5, 4, 2, 1)$

For a partition $\mu = (\mu_1, \dots, \mu_k)$ denote the *conjugate* partition $\mu' := (\mu'_1, \dots, \mu'_h)$, where for each $i \geq 1$, μ'_i is the number of parts of μ that are bigger than or equal to i. The diagram of μ' is obtained by flipping the diagram of μ across the diagonal.

Partitions of the form $\mu = (a)$ and $\mu = (1^b) = (\overbrace{1, \dots, 1}^{b})$, with $a, b \in \mathbb{P}$ are called *one-row* and *one-column* partitions, respectively. More generally, a partition is said to be a *hook* if it is of the form $\mu = (a+1, 1^b)$, with $a, b \in \mathbb{N}$.

Sometimes, it will be useful to denote hook partitions using a different notation. The hook $\mu = (a+1, 1^b)$ in *Frobenius's notation* [M2, page 3] will be denoted by $\mu = (a \mid b)$. Note that its conjugate is $\mu' = (b \mid a)$. See Figure 2 for an example.



$\mu = (2 \mid 1)$ $\qquad$ $\mu' = (1 \mid 2)$

Figure 2: Frobenius notation

2.2 De Concini-Procesi ideals

From now on, we shall assume that a partition of n has n terms. So we will add enough zero terms to any partition until we have the right number of terms. Let $\mu = (\mu_1, \dots, \mu_n)$ be a partition of n, and $\mu' = (\mu'_1 \dots, \mu'_n)$ its conjugate partition. For any $1 \le k \le n$, we define

$$\delta_k(\mu) := \mu'_n + \mu'_{n-1} + \dots + \mu'_{n-k+1}.$$

Recall that for any $1 \le r \le n$, the *elementary symmetric polynomial* [St2] is defined by

$$e_r(x_1, \dots, x_n) := \sum_{1 \le i_1 < \dots < i_r \le n} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

Given a subset $S \subseteq \{x_1, \dots, x_n\}$, let $e_r(S)$ be the r^{th} elementary symmetric polynomial in the variables in S. Clearly, every $e_r(S)$ is a homogeneous polynomial in R of degree r.

We are now ready to introduce the ideals originally defined by De Concini and Procesi [DP]. We use a different and simpler set of generators which was defined by Tanisaki [T].

Definition 2.1 (De Concini-Procesi ideal). We let $\mathcal{C}_\mu$ denote the collection of partial elementary symmetric polynomials

$$\mathcal{C}_\mu = \{e_r(S) \mid S \subseteq \{x_1, \dots, x_n\},\ |S| = k \ge 1,\ k \ge r > k - \delta_k(\mu)\}. \tag{1}$$

The *De Concini-Procesi ideal* $\mathcal{I}_\mu$ is the homogeneous ideal generated by the elements of $\mathcal{C}_\mu$, in symbols,

$$\mathcal{I}_\mu := (\mathcal{C}_\mu).$$

Note that $\delta_n(\mu) = n$, for any partition μ of n. Hence when we set $k = n$ in (1), we obtain that $\mathcal{I}_\mu$ contains the ideal generated by the elementary symmetric polynomials in all the variables $x_1, \dots, x_n$.

Example 2.2. Let $\mu = (3, 1, 0, 0) \vdash 4$ and $\mu' = (2, 1, 1, 0)$ be the partitions appearing in Figure 2. Then $(\delta_1(\mu), \dots, \delta_4(\mu)) = (0, 1, 2, 4)$. Hence

$$(1 - \delta_1(\mu), \dots, 4 - \delta_4(\mu)) = (1, 1, 1, 0),$$

and the collection $\mathcal{C}_\mu$ consists of the following elements. For $k = 1$ there is no admissible $e_r(S)$. For $k = 2$ we get the set of monomials:

$$x_1x_2,\ x_1x_3,\ x_1x_4,\ x_2x_3,\ x_2x_4,\ x_3x_4.$$

For $k = 3$:

$$x_1x_2 + x_1x_3 + x_2x_3,\ x_1x_2 + x_1x_4 + x_2x_4,\ x_1x_3 + x_1x_4 + x_3x_4,\ x_2x_3 + x_2x_4 + x_3x_4,$$

$$x_1x_2x_3,\ x_1x_2x_4,\ x_1x_3x_4,\ x_2x_3x_4.$$

Finally for $k = 4$, as already noted, we get the complete set of the elementary symmetric functions $e_r(x_1, x_2, x_3, x_4)$, for $1 \le r \le 4$.

When the indexing partition μ is a hook, the ideal $\mathcal{I}_\mu$ can be split in two parts. We have the following result.

Proposition 2.3 (Generators of De Concini-Procesi ideals indexed by hooks). *Let $\mu = (a \mid b) \vdash n$ be a hook. Then the De Concini-Procesi ideal associated to μ in the polynomial ring $k[x_1, \ldots, x_n]$ is*

$$\mathcal{I}_\mu = \mathcal{J}_\mu + \mathcal{E}_\mu,$$

where

$$\mathcal{J}_\mu = (x_{i_1} \cdots x_{i_{b+1}} \mid 1 \le i_1 < \ldots < i_{b+1} \le n) \tag{2}$$

is the ideal generated by all square-free monomials in $x_1, \ldots, x_n$ of degree $b + 1$, and

$$\mathcal{E}_\mu = (e_i(x_1, \ldots, x_n) \mid 1 \le i \le b) \tag{3}$$

is the ideal generated by all elementary symmetric polynomials of degree $\le b$ in the variables $x_1, \ldots, x_n$.

Proof. The partition $\mu = (a \mid b)$ is of size $n = a + b + 1$. We can write

$$\mu' = (b \mid a) = (b + 1, \underbrace{1, \ldots, 1}_{a}, \underbrace{0, \ldots, 0}_{b}).$$

Then we have

$$(\delta_1(\mu), \delta_2(\mu), \ldots, \delta_n(\mu)) = (\underbrace{0, \ldots, 0}_{b}, 1, 2, \ldots, a, n),$$

and so

$$(1 - \delta_1(\mu), 2 - \delta_2(\mu), \ldots, n - \delta_n(\mu)) = (1, 2, 3, \ldots, b, \underbrace{b, \ldots, b}_{a}, 0).$$

The definition of $\mathcal{C}_\mu$ in (1) implies that no k, with $1 \le k \le b$, contributes a generator to the ideal $\mathcal{I}_\mu$.

The first index making a nontrivial contribution to the set $\mathcal{C}_\mu$ is $k = b + 1$, which adds to $\mathcal{C}_\mu$ all $e_{b+1}(S)$, with $|S| = b + 1$, or in other words all the square-free monomials of degree $b + 1$ in the variables $x_1, \ldots, x_n$. We denote by $\mathcal{J}_\mu$ the ideal generated by these square-free monomials.

Now all the indices k, with $b + 2 \le k \le n - 1$ add to $\mathcal{C}_\mu$ elements of the form $e_r(S)$, with $k \ge r \ge b + 1$, and $|S| = k$. Each such $e_r(S)$ is a homogeneous polynomial of degree r, which we can write as the sum of square-free monomials of degree r. Since $r \geq b+1$, and all square-free monomials of degree $b+1$ or more are already in $\mathcal{I}_\mu$, such $e_r(S)$ do not contribute any new generators to $\mathcal{I}_\mu$.

Finally, for $k = n$ we obtain all the elementary symmetric polynomials in all the variables. For the same reasons as above, the only new contributions are

$$e_1(x_1, \ldots, x_n), e_2(x_1, \ldots, x_n), \ldots, e_b(x_1, \ldots, x_n).$$

We denote the ideal generated by these elementary symmetric polynomials by $\mathcal{E}_\mu$. We conclude that $\mathcal{I}_\mu = \mathcal{J}_\mu + \mathcal{E}_\mu$ $\square$

Example 2.4. Let $\mu = (2 \mid 1) \vdash 4$. It follows from the computations in Example 2.2, that the ideal $\mathcal{I}_\mu$ splits into two parts

$$\mathcal{I}_\mu = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4) + (x_1 + x_2 + x_3 + x_4).$$

The first part is generated by all monomials of degree 2 in the variables x_1, x_2, x_3, x_4, and the second is generated by $e_1(x_1, x_2, x_3, x_4)$, the elementary symmetric polynomial of degree 1.

3 Commutative algebra tools

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k of characteristic 0, with the standard grading $\deg x_i = 1$, for all i. Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the (irrelevant) homogeneous maximal ideal of R. We are usually interested in the quotient $S = R/I$ where I is an ideal of R generated by homogeneous polynomials. In this situation, S inherits the grading and the irrelevant maximal ideal from R via the quotient map. Much of what we discuss below will be in this context, but applies more generally to local rings.

3.1 Resolutions

Resolutions provide us with an effective method to study a finitely generated module M via a sequence of free modules mapping to it. Among the many applications, the ranks of these free modules, also known as "Betti numbers", are numerical invariants of M that make it possible to compute the Hilbert function of M directly. There is a large amount of literature focusing on different aspects of resolutions, studying them using homological, geometrical, or combinatorial tools. We refer the interested reader to [E2], [He] or [BH] to learn more about current research in this field. Eisenbud's book [E2] in particular contains a beautiful exposition on the history of the subject.

Definition 3.1 (Minimal free resolution). A *free resolution* of R/I is an exact complex $\mathbb{F}$

$$0 \longrightarrow \cdots \xrightarrow{\delta_{i+1}} F_i \xrightarrow{\delta_i} F_{i-1} \xrightarrow{\delta_{i-1}} \cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} R \xrightarrow{\delta_0} R/I \longrightarrow 0.$$

of free R-modules F_i ($F_0 = R$). The resolution is *minimal* if $\delta_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ for $i > 0$.

It is worth noting that the difference between the resolution of the ideal I (as an R-module), and the resolution of the quotient R/I is just the one free module F_0: given the above resolution for R/I, the resolution for I is the following:

$$0 \longrightarrow \cdots \xrightarrow{\delta_{i+1}} F_i \xrightarrow{\delta_i} F_{i-1} \xrightarrow{\delta_{i-1}} \cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} I \longrightarrow 0.$$

In this paper, we will always be considering the resolution of R/I.

It follows from the Hilbert Syzygy Theorem [E2, Theorem 1.1] that the length of a minimal free resolution of R/I is finite; i.e. $F_i = 0$ for $i > n$, where n is the number of variables in R (the resolution could stop even earlier, there are formulas to compute the length of a resolution). A minimal free resolution of R/I is unique up to isomorphism [E1, Theorem 20.2].

If each F_i is a free module of rank β_i, the resolution of R/I is

$$0 \longrightarrow R^{\beta_m} \xrightarrow{\delta_m} R^{\beta_{m-1}} \xrightarrow{\delta_{m-1}} \cdots \xrightarrow{\delta_2} R^{\beta_1} \xrightarrow{\delta_1} R \xrightarrow{\delta_0} R/I \longrightarrow 0. \tag{4}$$

The β_i are called the *Betti numbers* of R/I; these are independent of which minimal resolution one considers.

In the case where I is a homogeneous ideal, and therefore R/I is graded, we define the *graded Betti numbers* of R/I. This is done by making the maps δ_i homogeneous, so that they take a degree j element of F_i to a degree j element of F_{i-1}. To serve this purpose the degree of each generator of F_i is adjusted. So we can write the free module $F_i = R^{\beta_i}$ as

$$R^{\beta_i} = \bigoplus_j R(-j)^{\beta_{i,j}}$$

where for a given integer a, $R(a)$ is the same as R but with a new grading:

$$R(a)_d = R_{a+d}.$$

So the resolution shown in (4) becomes

$$0 \longrightarrow \bigoplus_j R(-j)^{\beta_{m,j}} \xrightarrow{\delta_m} \bigoplus_j R(-j)^{\beta_{m-1,j}} \xrightarrow{\delta_{m-1}} \cdots \xrightarrow{\delta_2} \bigoplus_j R(-j)^{\beta_{1,j}} \xrightarrow{\delta_1} R \xrightarrow{\delta_0} R/I \longrightarrow 0. \tag{5}$$

This is called the *graded minimal free resolution* of R/I, and the $\beta_{i,j}$ are the *graded Betti numbers* of R/I. Clearly, $\sum_j \beta_{i,j} = \beta_i$.

Definition 3.2 (Bigraded Poincaré series). The *bigraded Poincaré series* of an ideal I is the generating function for the graded Betti numbers of I:

$$P_{R/I}(q,t) = \sum_{i,j} \beta_{i,j} q^i t^j.$$

Definition 3.3 (Linear resolution). The graded resolution described in (5) is a *linear resolution*, if for some u, $\beta_{i,j} = 0$ unless $j = u + i - 1$. In other words, R/I has a linear resolution if for some u, it has a graded minimal free resolution of the form

$$\begin{gathered} 0 \longrightarrow R(-(u+m-1))^{\beta_{m,u+m-1}} \xrightarrow{\delta_m} R(-(u+m-2))^{\beta_{m-1,u+m-2}} \xrightarrow{\delta_{m-1}} \\ \cdots \xrightarrow{\delta_3} R(-(u+1))^{\beta_{2,u+1}} \xrightarrow{\delta_2} R(-(u))^{\beta_{1,u}} \xrightarrow{\delta_1} R \xrightarrow{\delta_0} R/I \longrightarrow 0. \end{gathered}$$

In this case, all the generators of the ideal I have degree equal to u.

Discussion 3.4 (Resolutions using mapping cones). The mapping cone technique provides a way to build a free resolution of an ideal by adding generators one at a time. A resolution obtained using mapping cones is not in general minimal. However, we will be focusing only on the special case of multiplication by a nonzerodivisor, in which case we obtain a minimal free resolution. For a more general or detailed description, see [Sc], [HT], or [E2].

Suppose that I is an ideal in the polynomial ring R, and $e \in \mathfrak{m}$ is a nonzerodivisor in R/I (i.e. e is a regular element mod I; see Definition 3.8). The goal is to build a minimal free resolution of $R/I + (e)$ starting from a minimal free resolution of R/I. Consider the short exact sequence

$$0 \longrightarrow R/I : (e) \xrightarrow{.e} R/I \longrightarrow R/I + (e) \longrightarrow 0$$

where $I : (e)$ is the quotient ideal consisting of all elements $x \in R$ such that $xe \in I$. Since e is a nonzerodivisor in R/I, we have $I : (e) = I$, and so our short exact sequence turns into

$$0 \longrightarrow R/I \xrightarrow{.e} R/I \longrightarrow R/I + (e) \longrightarrow 0.$$

Suppose we have a minimal free resolution of R/I

$$0 \longrightarrow \cdots \xrightarrow{\delta_{i+1}} A_i \xrightarrow{\delta_i} A_{i-1} \xrightarrow{\delta_{i-1}} \cdots \xrightarrow{\delta_2} A_1 \xrightarrow{\delta_1} R \xrightarrow{\delta_0} R/I \longrightarrow 0. \tag{6}$$

Then we can obtain the following minimal free resolution of $R/I + (e)$

$$0 \longrightarrow \cdots \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} R \xrightarrow{d_0} R/I + (e) \longrightarrow 0 \tag{7}$$

where for each $i > 0$, as a free R-module

$$F_i = A_i \oplus A_{i-1} \text{ and } d_i(x, y) = (ey + \delta_i(x), -\delta_{i-1}(y)).$$

The resolution is minimal because for $(x, y) \in A_i \oplus A_{i-1}$ and $e \in \mathfrak{m}$, we have

$$ey \in \mathfrak{m}A_{i-1},\ \delta_i(x) \in \mathfrak{m}A_{i-1},\ \delta_{i-1}(y) \in \mathfrak{m}A_{i-2} \Longrightarrow d_i(x, y) \in \mathfrak{m}F_{i-1}.$$

We now focus on the grading of each F_i. Suppose that the element $e \in R$ is homogeneous of degree m, and for each i, each of the free modules A_i in (6) are of the form

$$A_i = \bigoplus_j R(-j)^{\beta_{i,j}}$$

where the $\beta_{i,j}$ are the graded Betti numbers. We would like to compute the graded Betti numbers of $R/I + (e)$.

Lemma 3.5. *Let I be an ideal of the polynomial ring R, and $e \in \mathfrak{m}$ a homogeneous element of degree m which is a nonzerodivisor in R/I. Consider the minimal free resolutions (6) of R/I, and (7) of $R/I + (e)$ obtained by mapping cones. For each $i > 0$ we have*

$$F_i = \bigoplus_j R(-j)^{\beta_{i,j}} \ \oplus\ \bigoplus_j R(-j-m)^{\beta_{i-1,j}}.$$

Proof. We prove this by induction on i. In the case where $i = 1$, we have the homogeneous map

$$d_1 : A_1 \oplus R \longrightarrow R$$

where $d_1(x,y) = ey + \delta_1(x)$. In particular, if $x \in A_1$ is a homogeneous element of degree t, then $d_1(x,0) = \delta_1(x)$ is also a degree t homogeneous element of R. If $y \in R$ is a homogeneous element of degree t, then $d_1(0,y) = ey$ has degree $t+m$. In order to make d_1 a homogeneous (degree 0) map, we shift the grading of the component R of F_i by m, so that

$$F_1 = \bigoplus_j R(-j)^{\beta_{1,j}} \oplus R(-m).$$

Suppose our claim holds for all indices less than i, and we have the homogeneous map

$$d_i : F_i = A_i \oplus A_{i-1} \longrightarrow F_{i-1} = \bigoplus_j R(-j)^{\beta_{i-1,j}} \ \oplus \ \bigoplus_j R(-j-m)^{\beta_{i-2,j}}.$$

We use the same argument as we did in the case of $i = 1$. If $x \in A_i$ is a homogeneous element of degree t, then $d_i(x,0) = \delta_i(x)$ is also a degree t homogeneous element of A_{i-1}. If $y \in A_{i-1}$ is a homogeneous element of degree t, then $d_i(0,y) = (ey, -\delta_{i-1}(y))$ has to be a homogeneous element of F_{i-1} of degree t. By definition, this is already true for the component $\delta_{i-1}(y)$, but ey has degree $m+t$. So in order to make d_i a homogeneous (degree 0) map, we have to shift the grading of each component of F_i that comes from A_{i-1} by m, so that

$$F_i = \bigoplus_j R(-j)^{\beta_{i,j}} \ \oplus \ \bigoplus_j R(-j-m)^{\beta_{i-1,j}}.$$

□

Corollary 3.6. *Let I be an ideal of the polynomial ring R and $e \in \mathfrak{m}$ be a homogeneous element of degree m which is a nonzerodivisor in R/I. Then*

$$P_{R/I+(e)}(q,t) = (1+qt^m)P_{R/I}(q,t).$$

Proof. By Lemma 3.5, if for a fixed i, $A_i = \bigoplus_{j=0}^{b_i} R(-j)^{\beta_{i,j}}$ then

$$F_i = \bigoplus_{j=0}^{b_i} R(-j)^{\beta_{i,j}} \ \oplus \ \bigoplus_{j=0}^{b_{i-1}} R(-j-m)^{\beta_{i-1,j}}.$$

So we have

$$\begin{aligned} P_{R/I+(e)}(q,t) &= 1 + \sum_{i\geq 1}\left(\sum_{j=0}^{b_i}\beta_{i,j}t^j + \sum_{j=0}^{b_{i-1}}\beta_{i-1,j}t^{j+m}\right)q^i \\ &= \sum_{i\geq 0}\sum_{j=0}^{b_i}\beta_{i,j}t^jq^i + t^m\sum_{i\geq 0}\sum_{j=0}^{b_i}\beta_{i,j}t^jq^{i+1} \\ &= (1+qt^m)\sum_{i\geq 0}\sum_{j=0}^{b_i}\beta_{i,j}t^jq^i \\ &= (1+qt^m)P_{R/I}(q,t). \end{aligned}$$

□

3.2 Krull dimension, Cohen-Macaulay rings, minimal primes

A minimal prime ideal (with respect to inclusion) containing I is called a *minimal prime* of I. Given any ideal I of R, the *Krull dimension* or *dimension* of the quotient ring R/I is equal to the length r of the maximal chain of prime ideals containing I

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$$

(here, $\mathfrak{p}_0$ is a minimal prime of I). The *height* of a prime ideal $\mathfrak{p}$ is the maximal length of a chain of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_r,$$

and the height of a general ideal I is the smallest height of its minimal primes.

Example 3.7. If $I = (xy, xz) \subset R = k[x, y, z]$, then the minimal primes of I are (x) and (y, z). In this case $\text{height } I = 1$ (as $(x) \supset (0)$ is a maximal chain) and $\dim R/I = 2$.

Definition 3.8 (Regular sequence, depth, Cohen-Macaulay). Let R is a polynomial ring with standard grading, and I a homogeneous ideal of R. Consider $S = R/I$ with homogeneous maximal ideal $\mathfrak{m}$. A sequence $y_1, \dots, y_m$ of elements in $\mathfrak{m}$ is a *regular sequence* of S if

(i) $(y_1, \dots, y_m)S \neq S$,

(ii) y_1 is a nonzerodivisor in S,

(iii) y_i is a nonzerodivisor in $S/(y_1, \dots, y_{i-1})$.

The length of a maximal regular sequence in S is called the *depth* of S.

In general, the depth of S is less than or equal to the dimension of S, but in the case equality is obtained, i.e. $\text{depth }(S) = \dim (S)$, the ring is *Cohen-Macaulay*.

For more on dimension theory and on the theory of Cohen-Macaulay rings, see [E1], Appendix A of [BH], or [V].

We will need the definition of the dual of a square-free monomial ideal. This is the same as Alexander dual, but we state the (equivalent) definition in a slightly different language (see [F] for more). Recall that a *monomial ideal* is an ideal generated by monomials, and a *square-free* monomial ideal is an ideal generated by square-free monomials in the variables $x_1, \dots, x_n$.

Definition 3.9 (dual of an ideal). Let I be a square-free monomial ideal in the polynomial ring $k[x_1, \dots, x_n]$. Then $I^\vee$ is a square-free monomial ideal, where each generator of $I^\vee$ is the product of the variables appearing in the generating set of a minimal prime of I.

Note that if I is a monomial ideal, its minimal primes are generated by single variables.

Example 3.10. If $I = (xy, xz, yzw) \subset k[x, y, z, w]$, then the minimal primes of I are (x, y), (x, z), (x, w) and (y, z). So $I^\vee = (xy, xz, xw, yz)$.

Recall that if I and J are two ideals of R, their *quotient* is the ideal defined as

$$I : J = \{x \in R \mid xJ \subseteq I\}.$$

Definition 3.11 (linear quotients). If $I \subset k[x_1, \dots, x_n]$ is a monomial ideal and $G(I)$ is its unique minimal set of monomial generators, then I is said to have *linear quotients* if there is an ordering $M_1, \dots, M_q$ on the elements of $G(I)$ such that for every $i = 2, \dots, q$, the quotient ideal

$$(M_1, \dots, M_{i-1}) : M_i$$

is generated by a subset of the variables $x_1, \dots, x_n$.

Lemma 3.12. *Let I be an ideal in the polynomial ring $R = k[x_1, \ldots, x_n]$ generated by all square-free monomials of a fixed degree m. Then*

1. *I has linear quotients;*
2. *R/I has a linear resolution;*
3. *R/I is Cohen-Macaulay.*

Proof. We can order the generating monomials of I lexicographically as $M_1, \ldots, M_q$. Take such a monomial $M_i = x_{j_1} \ldots x_{j_m}$, written so that $j_1 < j_2 < \ldots < j_m$. Since $(M_1, \ldots, M_{i-1})$ is a monomial ideal, and M_i is also a monomial, the quotient ideal $(M_1, \ldots, M_{i-1}) : M_i$ is generated by monomials. Observe that

1. If $s < j_t$ for some $j_t \in \{j_1, \ldots, j_m\}$ and $s \notin \{j_1, \ldots, j_m\}$, then $x_s \in (M_1, \ldots, M_{i-1}) : M_i$. This is because the monomial $\frac{x_s M_i}{x_{j_t}}$ is a degree m monomial that is lexicographically smaller than M_i, that is, $\frac{x_s M_i}{x_{j_t}} \in \{M_1, \ldots, M_{i-1}\}$.
2. If u is a monomial in $(M_1, \ldots, M_{i-1}) : M_i$, then $M_l \mid uM_i$ for some $l < i$. Since $M_l <_{lex} M_i$, there exists x_s, such that $x_s \mid M_l$, $x_s \nmid M_i$ and $s < j_t$ for some $j_t \in \{j_1, \ldots, j_m\}$.

It follows that $x_s \mid u$, and $(M_1, \ldots, M_{i-1}) : M_i$ is generated by the set of variables x_s, with $s < j_m$ and $s \notin \{j_1, \ldots, j_m\}$ as described in part 1. This proves that I has linear quotients.

Since the generators of I all have the same degree, and since I has linear quotients, it also follows that R/I has linear resolution (Lemma 5.2 of [F]).

Now we focus on the structure of $I^\vee$. Since every generator of I has exactly m variables, each such generator misses exactly $n - m$ variables from the set $\{x_1, \ldots, x_n\}$. So if A is any $n-m+1$-subset of $\{x_1, \ldots, x_n\}$, A must contain at least one variable from each of the M_i. Also no proper subset of A will have this property (i.e. A is the minimal set with such a property). So A is a generating set for a minimal prime of I. Since all minimal primes of I are generated by subsets of $\{x_1, \ldots, x_n\}$, it follows that $I^\vee$ is generated by all square-free monomials of degree $n - m + 1$.

So we have shown that $I^\vee$ satisfies the hypotheses of our lemma, and hence it satisfies parts 1 and 2. In particular, $R/I^\vee$ has linear resolution, and so by Theorem 3 of [ER], equivalently, R/I is Cohen-Macaulay.

□

4 Resolutions of De Concini-Procesi ideals of hooks

In this section we study the minimal free resolutions of the De Concini-Procesi ideal $\mathcal{I}_\mu$ of a hook $\mu = (a \mid b)$. We have seen that $\mathcal{I}_\mu$ is the sum of two ideals

$$\mathcal{I}_\mu = \mathcal{J}_\mu + \mathcal{E}_\mu$$

where $\mathcal{J}_\mu$ is generated by monomials, and $\mathcal{E}_\mu$ is generated by elementary symmetric functions. Below we show how we can recover the resolution of $\mathcal{I}_\mu$ using the resolutions of each one of the summands.

Since $\mathcal{J}_\mu$ is generated by all square-free monomials of $R = k[x_1, \ldots, x_n]$ that have degree $b+1$, by Lemma 3.12, $\mathcal{J}_\mu$ is a Cohen-Macaulay ideal with linear resolutions and linear quotients. On the other hand, it is easy to see that all the minimal primes of $\mathcal{J}_\mu$ have uniform height $n - b$. This is because every generator of $\mathcal{J}_\mu$ is a product of exactly $b+1$ variables in the set $\{x_1,\dots,x_n\}$, and so a minimal subset of $\{x_1,\dots,x_n\}$ that shares at least one variable with each one of these generators must have $n-b$ elements. Such an ideal will have height equal to $n-b$, and so it follows that $\dim R/\mathcal{J}_\mu = b$.

We have thus shown that

Corollary 4.1. *For a hook $\mu = (a \,|\, b)$, the ideal $\mathcal{J}_\mu$ of R has linear quotients, linear resolution, and $R/\mathcal{J}_\mu$ is Cohen-Macaulay of (Krull) dimension b.*

Next, we focus on the ideal $\mathcal{E}_\mu$, which is generated by the first b elementary symmetric functions.

Proposition 4.2. *For a hook $\mu = (a \,|\, b)$, the set of generators*

$$e_1(x_1,\dots,x_n),\dots,e_b(x_1,\dots,x_n)$$

of $\mathcal{E}_\mu$ form a regular sequence over the quotient ring $R/\mathcal{J}_\mu$.

Proof. Let $S = R/\mathcal{J}_\mu$. We know by Corollary 4.1 that S is a Cohen-Macaulay ring, and $\dim S = b$. To show that $e_1(x_1,\dots,x_n),\dots,e_b(x_1,\dots,x_n)$ forms a regular sequence in S, by Theorem 2.1.2 of [BH], it is enough to show that $\dim S/\mathcal{E}_\mu = 0$. We prove this by induction. Let $\underline{\mu} = ((a-1) \,|\, b)$ and $\overline{\mu} = (a \,|\, (b-1))$ be two hooks consisting of $n-1$ squares. Notice that

1. $\mathcal{J}_\mu = x_n \mathcal{J}_{\overline{\mu}} + \mathcal{J}_{\underline{\mu}}$.

   To see this, split the generating set of $\mathcal{J}_\mu$ into two sets: $\underline{G}$ consists of all those generators that do not contain the variable x_n, and $\overline{G}$ is the rest. So $\underline{G}$ consists of all square-free monomials of degree $b+1$ in the variables $x_1,\dots,x_{n-1}$, which is by definition the generating set of $\mathcal{J}_{\underline{\mu}}$.

   Similarly, if we factor out the variable x_n from each monomial in $\overline{G}$, we will be left with all square-free monomials of degree b in the variables $x_1,\dots,x_{n-1}$, which by definition generate the ideal $\mathcal{J}_{\overline{\mu}}$.

2. Since every term in $e_i(x_1,\dots,x_n)$ is a square-free monomial of degree i, we can partition all such monomials into those that contain x_n and those that don't. It is then easy to see that for every i,
$$e_i(x_1,\dots,x_n) = e_i(x_1,\dots,x_{n-1}) + x_n e_{i-1}(x_1,\dots,x_{n-1}).$$

It follows that
$$\frac{S}{\mathcal{E}_\mu + (x_n)} = \frac{k[x_1,\dots,x_n]}{\mathcal{J}_\mu + \mathcal{E}_\mu + (x_n)} \cong \frac{k[x_1,\dots,x_{n-1}]}{\mathcal{J}_{\underline{\mu}} + \mathcal{E}_{\underline{\mu}}}$$
and so by the induction hypothesis,
$$\dim \frac{S}{\mathcal{E}_\mu + (x_n)} = 0. \tag{8}$$
It follows that $\dim S/\mathcal{E}_\mu = 0$ or 1.

Now suppose that $\dim S/\mathcal{E}_\mu = \dim R/(\mathcal{J}_\mu + \mathcal{E}_\mu) = 1.$

So it follows that there is a prime ideal $\mathfrak{p}$ of R such that
$$\mathcal{J}_\mu + \mathcal{E}_\mu \subseteq \mathfrak{p} \subset \mathfrak{m}.$$

Since $\mathfrak{p}$ is a prime ideal and every monomial generator of $\mathcal{J}_\mu$ belongs to $\mathfrak{p}$, at least one variable of R has to be in $\mathfrak{p}$; say, $x_n \in \mathfrak{p}$ (the equality (8) that we shall use holds if one replaces x_n with any other variable in R). But then
$$\mathcal{J}_\mu + \mathcal{E}_\mu + (x_n) \subseteq \mathfrak{p} \subset \mathfrak{m}$$

but this contradicts the fact that

$$\dim \frac{k[x_1,\ldots,x_n]}{\mathcal{J}_\mu + \mathcal{E}_\mu + (x_n)} = 0.$$

□

We are now ready to state our central claim.

Theorem 4.3 (Main theorem). *Let $\mu = (a \,|\, b)$ be a hook. Then the bigraded Poincaré series for the ideal $\mathcal{I}_\mu$ is the following*

$$P_{R/\mathcal{I}_\mu}(q,t) = \prod_{k=1}^{b}(1+qt^k) \cdot \Big(1 + qt^{b+1}\sum_{i=0}^{a}\binom{b+i}{b}(1+q\,t)^i\Big). \tag{9}$$

Proof. As usual, let $\mathcal{I}_\mu = \mathcal{J}_\mu + \mathcal{E}_\mu$.

Step 1. The ideal $\mathcal{J}_\mu$ has linear quotients (Corollary 4.1). It follows from Corollary 1.6 of [HT] that, if $G(\mathcal{J}_\mu)$ indicates the generating set for $\mathcal{J}_\mu$, the bigraded Poincaré series of $\mathcal{J}_\mu$ is the following:

$$P_{R/\mathcal{J}_\mu}(q,t) = 1 + \sum_{M\in G(\mathcal{J}_\mu)} (1+qt)^{|\mathrm{set}(M)|} qt^{\deg(M)} \tag{10}$$

where, if we order the elements of $G(\mathcal{J}_\mu)$ lexicographically as $M_1,\ldots,M_q$, then for $i = 1,\ldots,m$

$$\mathrm{set}(M_i) = \{j \in \{1,\ldots,n\} \mid x_j \in (M_1,\ldots,M_{i-1}) : M_i\}.$$

In our case, as the degree of each of the monomials generating $\mathcal{J}_\mu$ is $b+1$, Equation (10) turns into

$$\begin{aligned} P_{R/\mathcal{J}_\mu}(q,t) &= 1 + \sum_{M\in G(\mathcal{J}_\mu)} (1+qt)^{|\mathrm{set}(M)|} qt^{b+1} \\ &= 1 + qt^{b+1} \sum_{M\in G(\mathcal{J}_\mu)} (1+qt)^{|\mathrm{set}(M)|}. \end{aligned} \tag{11}$$

So now we focus on $|\mathrm{set}(M_j)|$ for $M_j \in \{M_1,\ldots,M_q\}$. Suppose $M_j = x_{i_1}\cdots x_{i_{b+1}}$, where $i_1 < \ldots < i_{b+1}$. Then each $M \in \{M_1,\ldots,M_{j-1}\}$ is of the form $M = x_{u_1}\cdots x_{u_{b+1}}$, with $u_1 < \ldots < u_{b+1}$, and M is lexicographically smaller than M_j. So the relationship between the indices is such that

$$u_1 < i_1 \qquad \text{or} \qquad \text{if } u_1 = i_1,\ldots,u_l = i_l \text{ then } u_{l+1} < i_{l+1}.$$

So by an argument identical to that in the proof of Lemma 3.12

$$\begin{aligned} \mathrm{set}(M_j) &= \{u \le n \mid x_u \in (M_1,\ldots,M_{j-1}) : M_j\} \\ &= \{u \le i_{b+1} \mid x_u \nmid M_j\}. \end{aligned}$$

We can now conclude that

$$|\mathrm{set}(M_j)| = i_{b+1} - (b+1). \tag{12}$$

We have shown that, if M is any degree $b+1$ square-free monomial with highest index u (that is, $x_u \mid M$ and $x_v \nmid M$ for $v > u$), then $|\mathrm{set}(M)| = u - (b+1)$. So to compute the sum in (11), all we have to do is count the number of square-free degree $b+1$ monomials with highest index u, for any given u. This number is clearly $\binom{u-1}{b}$. So for a given i, the number of degree $b+1$ square-free monomials M with $|\mathrm{set}(M)| = i$ is exactly $\binom{b+i}{b}$.

Therefore

$$\begin{aligned} P_{R/\mathcal{J}_\mu}(q,t) &= 1 + qt^{b+1} \sum_{i=0}^{n-b-1} \binom{b+i}{b} (1+qt)^i \\ &= 1 + qt^{b+1} \sum_{i=0}^{a} \binom{b+i}{b} (1+qt)^i \end{aligned} \tag{13}$$

since by Equation (12), i can reach at most $n-b-1$, which by definition is equal to a.

Step 2. Since $\mathcal{E}_\mu$ is generated by a regular sequence over $R/\mathcal{J}_\mu$ (Proposition 4.2), we can use a mapping cone construction to find its minimal graded resolution (see Discussion 3.4). We do this by adding the generators of $\mathcal{E}_\mu$, one at a time, to $\mathcal{J}_\mu$, and applying Corollary 3.6. As the generators $e_1(x_1,\dots,x_n),\dots,e_b(x_1,\dots,x_n)$ of $\mathcal{E}_\mu$ have degrees $1,\dots,b$, respectively, each time we add a $e_i(x_1,\dots,x_n)$, the Poincaré series gets multiplied by a factor of $(1+qt^i)$, and hence

$$\begin{aligned} P_{R/\mathcal{I}_\mu}(q,t) &= \prod_{k=1}^{b} (1+qt^k) \cdot P_{R/\mathcal{J}_\mu}(q,t) \\ &= \prod_{k=1}^{b} (1+qt^k) \cdot \left(1 + qt^{b+1} \sum_{i=0}^{a} \binom{b+i}{b} (1+q\,t)^i\right) \quad (\text{from } (13)). \end{aligned}$$

□

Corollary 4.4 (The set of generators of $\mathcal{I}_\mu$ is minimal). *Let $\mu = (a \mid b)$ be a hook. The generating set for $\mathcal{I}_\mu$ described in Proposition 2.3 is minimal.*

Proof. The number of generators of $\mathcal{I}_\mu$ is by definition $\binom{n}{b+1} + b$. On the other hand, the minimal number of generators of $\mathcal{I}_\mu$ is the first Betti number β_1 of $R/\mathcal{I}_\mu$, which is the coefficient of q in the Poincaré series $P_{R/\mathcal{I}_\mu}(q,1)$. It is easy to see by Theorem 4.3 that this coefficient is

$$b + 1 + \sum_{i=1}^{a} \binom{b+i}{b}.$$

So all we have to show is that

$$\binom{n}{b+1} + b = b + 1 + \sum_{i=1}^{a} \binom{b+i}{b}$$

which is equivalent to showing that

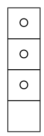
$$\binom{n}{b+1} = \sum_{i=0}^{n-b-1} \binom{b+i}{b}.$$

This last equation follows easily from induction on n. □

4.1 Combinatorial interpretations

From Theorem 4.3 it can be seen that the Poincaré series of $\mathcal{I}_\mu$ can be defined recursively. For simplicity, for any hook $\mu = (a \mid b)$ we denote by $P_{(a|b)}(q,t)$ the bigraded Poincaré series $P_{R/\mathcal{I}_{(a|b)}}(q,t)$, and by $P_{(a|b)}(q)$ the nongraded Poincaré series $P_{R/\mathcal{I}_{(a|b)}}(q,1)$.

We start with the vertical partition $(0 \mid b)$. In this case the ideal $\mathcal{I}_\mu$ is generated only by the elementary symmetric functions; the quotient of $\mathcal{I}_\mu$ is the coinvariant algebra, a well-known representation of the symmetric group (see e.g., [Hu]). The nongraded Poincaré series in this case is

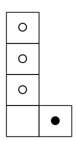


$$P_{(0|b)}(q) = (1+q)^{b+1}.$$

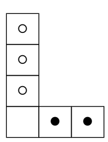
Using Equation (9), by subtracting $P_{(a-1|b)}(q)$ from $P_{(a|b)}(q)$, we find that the nongraded Poincaré polynomial of μ satisfies the following recurrence:

$$P_{(a|b)}(q) = P_{(a-1|b)}(q) + \binom{a+b}{b} q(1+q)^{a+b}.$$

This recursion allows us to compute the nongraded Poincaré polynomial of $(a \mid b)$ by adding one cell at a time to the first row of the vertical partition $(0 \mid b)$



$$P_{(1|b)}(q) = (1+q)^{b+1}\Big(1 + \binom{b+1}{b} q\Big)$$



$$P_{(2|b)}(q) = (1+q)^{b+1}\Big(1 + \binom{b+1}{b} q + \binom{b+2}{b} q(1+q)\Big),$$

until we reach the hook $(a \mid b)$, which gives us

$$P_{(a|b)}(q) = (1+q)^{b+1}\Big(1 + q\sum_{i=1}^{a} \binom{b+i}{b} (1+q)^{i-1}\Big).$$

The graded Poincaré polynomial satisfies a similar recurrence:

$$P_{(0|b)}(q,t) = \prod_{k=1}^{b+1}(1+qt^k)$$

$$P_{(a|b)}(q,t) = P_{(a-1|b)}(q,t) + \prod_{k=1}^{b}(1+qt^k)\ \cdot\ qt^{b+1}\binom{b+a}{a}(1+qt)^a.$$

Once again, like the nongraded case, one can use this recurrence to build $P_{(a|b)}(q,t)$ starting from $P_{(0|b)}(q,t)$.

Now we turn to the question of a combinatorial interpretation of the coefficients $\beta_{i,j}$ of $P_{(a|b)}(q,t)$. In the case of the vertical partition $(0 \mid b)$ such an interpretation is given by Cauchy's t-binomial theorem, which states that

$$\prod_{k=1}^{n}(1+q\,t^k) = \sum_{k=0}^{n} q^k\,t^{\frac{k(k+1)}{2}} \binom{n}{k}_t.$$

Here

$$\binom{n}{k}_t := \frac{[n]_t!}{[k]_t!\,[n-k]_t!}$$

are the t-binomial coefficients which have many interesting combinatorial interpretations ([St1]), and

$$[j]_t! := [1]_t\,[2]_t\,\cdots[j]_t \;\text{ with }\; [j]_t := 1+t+\ldots+t^{j-1}. \tag{14}$$

The following question begs to be answered: in general, is it possible to find a combinatorial interpretation for the graded Betti numbers of the De Concini-Procesi ideals?

5 Regularity of Hooks

Definition 5.1 (Castelnuovo-Mumford regularity). Let I be an ideal of a $R = k[x_1,\ldots,x_n]$. The *Castelnuovo-Mumford regularity* or simply *regularity* of R/I, denoted by $\operatorname{reg}(R/I)$ is defined as the maximum value of of $j-i$ where the graded Betti number $\beta_{i,j} \neq 0$ in a minimal free resolution of R/I.

Corollary 5.2 (Regularity of hooks). *Let $\mu = (a \mid b)$ be a hook. Then* $\operatorname{reg}(R/I) = b(b+1)/2$.

Proof. The graded Betti numbers $\beta_{i,j}$ appear as the coefficients of the Poincaré series

$$P_{R/\mathcal{I}_\mu}(q,t) = \underbrace{\prod_{k=1}^{b}(1+qt^k)}_{\text{Factor 1}} \cdot \underbrace{\left(1+qt^{b+1}\sum_{i=0}^{a}\binom{b+i}{b}(1+q\,t)^i\right)}_{\text{Factor 2}}.$$

So the question is to find the term $q^i t^j$ in this polynomial, where the coefficient $\beta_{i,j}$ is nonzero and $j-i$ is maximum. The terms with nonzero coefficients in each factor are of the following forms:

Factor 1:	$q^m t^{b_1+\ldots+b_m}$	where $1 \le b_1 < \ldots < b_m \le b,\ 0 \le m \le b$,
Factor 2:	$q^{e+1}t^{e+b+1}$	where $0 \le e \le a$.

To show that $\operatorname{reg}(R/I) = \dfrac{b(b+1)}{2}$, we need to show that this bound is achieved by the possible choices of $j-i$, and is the maximum possible bound.

Consider the terms in Factor 1. We have

$$\begin{aligned}
b_1+\ldots+b_m-m &\le \Big((b-(m-1))+(b-(m-2))+\ldots+b\Big)-m \\
&= \Big((1+2+\ldots+b)-(1+2+\ldots+(b-m))\Big)-m \\
&= \frac{b(b+1)}{2} - \frac{(b-m)(b-m+1)}{2} - m \\
&\le \frac{b(b+1)}{2}.
\end{aligned}$$

Similarly, for terms in Factor 2, since $b \geq 0$, we have

$$e+b+1-(e+1) = b \leq \frac{b(b+1)}{2}.$$

For the product of a term in Factor 1 and a term in Factor 2 we have

$$\begin{aligned}
& b_1+\ldots+b_m+e+b+1-(m+e+1) \\
& = b_1+\ldots+b_m+b-m \\
& \leq \frac{b(b+1)}{2} - \frac{(b-m)(b-m+1)}{2} + b - m \quad \text{same argument as above} \\
& = \frac{b(b+1)}{2} - (b-m)\left(\frac{b-m+1}{2} - 1\right) \\
& = \frac{b(b+1)}{2} - (b-m)\left(\frac{b-m-1}{2}\right) \\
& \leq \frac{b(b+1)}{2} \qquad \text{since } m \leq b.
\end{aligned}$$

The bound is achieved if $m = b$, so that $b_1 = 1, \ldots, b_m = b$, and for any e, so that we have the term with nonzero coefficient

$$q^{e+b+1} t^{(1+\ldots+b)+e+b+1} = q^{e+b+1} t^{\frac{b(b+1)}{2}+e+b+1}$$

which clearly has the property that $j - i = \dfrac{b(b+1)}{2}$, as desired. $\square$

6 The Hilbert series of hooks

The goal of this section is to find the Hilbert series of $R/\mathcal{I}_\mu$ when μ is a hook partition, namely, the series

$$h_{R/\mathcal{I}_\mu}(q) = \sum_{s=0}^{\infty} \dim{}_k (R/\mathcal{I}_\mu)_s q^s,$$

where as usual $\dim{}_k$ means dimension as a vector space over k. This has been done in the general case of a partition $\mu = (\mu_1, \ldots, \mu_n)$ of n by Garsia and Procesi. In [GP], they provide an explicit basis for $R/\mathcal{I}_\mu$ as a $\mathbb{Q}$-module, from which it follows that

$$\dim_{\mathbb{Q}}(R/\mathcal{I}_\mu) = \binom{n}{\mu_1, \ldots, \mu_n}. \tag{15}$$

and

$$h_{R/\mathcal{I}_\mu}(q) = \sum_{\lambda \vdash n} f^\lambda \, K_{\lambda\mu}(1/q) \, q^{n(\mu)}. \tag{16}$$

Here, f^λ and $n(\mu)$ are two well-known parameters associated with partitions ([St2]), and $K_{\lambda\mu}(q)$ are the Kostka-Foulkes polynomials we referred to in the introduction ([LS]). The computation of $K_{\lambda\mu}(q)$ is somewhat complicated. This motivates us to use the results of this paper to give a new description of the Hilbert series in the case of hooks.

Let $\mu = (a \,|\, b)$ be a hook partition of n, and consider the ideal $\mathcal{I}_\mu = \mathcal{J}_\mu + \mathcal{E}_\mu$. Since $R/\mathcal{J}_\mu$ is a Cohen-Macaulay ring (Corollary 4.1), and the generators $e_1(x_1, \ldots, x_n), \ldots, e_b(x_1, \ldots, x_n)$ of $\mathcal{E}_\mu$ form a regular sequence over $R/\mathcal{J}_\mu$ (Proposition 4.2), it follows that (see [V] Theorem 4.2.5)

$$h_{R/\mathcal{I}_\mu}(q) = \prod_{i=1}^{b}(1-q^i) h_{R/\mathcal{J}_\mu}(q). \tag{17}$$

So we focus on finding $h_{R/\mathcal{J}_\mu}(q)$. Recall that $\mathcal{J}_\mu$ is generated by all square-free monomials of degree $b+1$ with variables in $\{x_1, \ldots, x_n\}$. So each graded piece $(R/\mathcal{J}_\mu)_s$ is generated by all monomials of degree s, involving c of the n variables with $c \leq b$. There are $\binom{n}{c}$ ways of choosing c variables from $\{x_1, \ldots, x_n\}$. Choose such a monomial, without loss of generality,

$$x_1^{a_1} \ldots x_c^{a_c}.$$

We need to choose the positive integers $a_1, \ldots, a_c$ such that $a_1 + \ldots + a_c = s$.

This is classically equivalent to inserting $c-1$ bars between the sequence of integers $1, \ldots, s$, as below:

$$1, \ldots, a_1 \,|\, a_1 + 1, \ldots, a_1 + a_2 \,|\, \ldots \,|\, (a_1 + \ldots + a_{c-1}) + 1, \ldots, s.$$

What we are doing here is choosing $c-1$ of the $s-1$ available slots, and there are $\binom{s-1}{c-1}$ ways of doing that. So we have

$$h_{R/\mathcal{J}_\mu}(q) = 1 + \sum_{s=1}^{\infty} \sum_{c=1}^{b} \binom{n}{c} \binom{s-1}{c-1} q^s.$$

Therefore, by Equation (17)

$$\begin{aligned} h_{R/\mathcal{I}_\mu}(q) &= \prod_{i=1}^{b}(1-q^i) \left(1 + \sum_{s=1}^{\infty} \sum_{c=1}^{b} \binom{n}{c} \binom{s-1}{c-1} q^s\right) \\ &= \prod_{i=1}^{b}(1-q^i) \left(1 + \sum_{c=1}^{b} \binom{n}{c} \sum_{s=1}^{\infty} \binom{s-1}{c-1} q^s\right). \end{aligned}$$

On the other hand

$$\begin{aligned} \sum_{s=1}^{\infty} \binom{s-1}{c-1} q^s &= \sum_{s=c}^{\infty} \binom{s-1}{c-1} q^s \\ &= q^c \sum_{s=c}^{\infty} \binom{(s-c)+(c-1)}{c-1} q^{s-c} \\ &= q^c \sum_{s=c}^{\infty} \binom{(s-c)+(c-1)}{s-c} q^{s-c} \qquad \text{because} \binom{i+j}{i} = \binom{i+j}{j} \\ &= \frac{q^c}{(1-q)^c} \qquad \text{because} \sum_{j=0}^{\infty} \binom{i+j}{j} q^j = \frac{1}{(1-q)^{i+1}} \ \text{([W])}. \end{aligned}$$

So

$$\begin{aligned}
h_{R/\mathcal{I}_\mu}(q) &= \prod_{i=1}^{b}(1-q^i)\left(1+\sum_{c=1}^{b}\binom{n}{c}\frac{q^c}{(1-q)^c}\right)\\
&= \frac{(1-q)^b}{(1-q)^b}\prod_{i=1}^{b}(1-q^i)\left(1+\sum_{c=1}^{b}\binom{n}{c}\frac{q^c}{(1-q)^c}\right)\\
&= \prod_{i=1}^{b}\left(\frac{1-q^i}{1-q}\right)\sum_{c=0}^{b}\binom{n}{c}q^c(1-q)^{b-c}\\
&= [b]_q!\,\sum_{c=0}^{b}\binom{n}{c}q^c(1-q)^{b-c}
\end{aligned}$$

where $[b]_q!$ is described in (14) above. So we have proved that

Proposition 6.1. *Let* $\mu=(a\mid b)$ *be a hook partition of* n. *Then*

$$h_{R/\mathcal{I}_\mu}(q) = [b]_q!\,\sum_{c=0}^{b}\binom{n}{c}q^c(1-q)^{b-c}. \tag{18}$$

Note that if we set $q=1$ in (18), we find that

$$\dim_k(R/\mathcal{I}_\mu) = \frac{n!}{(a+1)!} = \frac{n!}{\mu_1!}$$

as was expected by Formula (15).

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